CREAT. MATH. INFORM.
Volume **26** (2017), No. 2,
Pages 211 - 219

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2017.02.10

Product version of reciprocal Gutman indices of composite graphs

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ABSTRACT. In this paper, we present the upper bounds for the product version of reciprocal Gutman indices of the tensor product, join and strong product of two connected graphs in terms of other graph invariants including the Harary index and Zagreb indices.

1. INTRODUCTION

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. For two simple graphs G and H their *tensor product*, denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Note that if G and H are connected graphs, then $G \times H$ is connected only if at least one of the graph is nonbipartite. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$. The *join* $G + H$ of graphs G and H is obtained from the disjoint union of the graphs G and H, where each vertex of G is adjacent to each vertex of H.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index; for other related topological indices see [22]. For more details, see [2, 3, 4, 5, 18].

Let G be a connected graph. Then $Wiener\ index$ of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,\, v \,\in\, V(G)} d_G(u, v)$ with the summation going over all pairs of distinct vertices of G. Similarly, the $Harary\ index$ of G is defined as $H(G) = \frac{1}{2} \sum\limits_{u,\, v \,\in\, V(G)} \frac{1}{d_G(u,v)}$. Gutman et al. [11, 12] were introduced the *product version of Wiener index* which is defined as $W^*(G) = \prod\limits_{\{u,v\} \subseteq V(G)} d_G(u, v)$.

Dobrynin and Kochetova [7] and Gutman [10] independently proposed a vertex-degree-weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as $DD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v))d_G(u, v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. Hua and Zhang [13] introduced

Received: 04.07.2016. In revised form: 31.03.2017. Accepted: 07.04.2017
2010 *Mathematics Subject Classification.* 05C12, 05C76.
Key words and phrases. *Degree distance, Gutman index, product graphs.*

a new graph invariant named reciprocal degree distance, which can be seen as a degree-weight version of Harary index, that is, $H_A(G) = \frac{1}{2} \sum\limits_{u,v\in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$. Hua and Zhang [13] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex and edge-connectivity. Similarly, the*Gutman index* is defined as $DD_*(G) = \frac{1}{2} \sum\limits_{u,v\in V(G)} d_G(u)d_G(v)d_G(u,v)$. In Su et.al. [19] introduce the *reciprocal Gutman index* of graph, which can be seen as a product -degree-weight version of Harray index $H_M(G) = \frac{1}{2} \sum\limits_{u,v\in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)}$. In this sequence, the product version of *reciprocal degree distance* and *reciprocal Gutman index* are defined as $H_A^*(G) = \prod\limits_{\{u,v\}\subseteq V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)}$ and $H_M^*(G) = \prod\limits_{\{u,v\}\subseteq V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)}$, respectively.

The *first Zagreb index* and *second Zagerb index* are defined as $M_1(G) = \sum\limits_{u\in V(G)} d_G(u)^2 = \sum\limits_{uv\in E(G)} (d_G(u)+d_G(v))$ and $M_2(G) = \sum\limits_{uv\in E(G)} d_G(u)d_G(v)$. Similarly, the *first Zagreb coindex* and *second Zagerb coindex* are defined as $\overline{M}_1(G) = \sum\limits_{uv\notin E(G)} (d_G(u) + d_G(v))$ and $\overline{M}_2(G) = \sum\limits_{uv\notin E(G)} d_G(u)d_G(v)$. The Zagreb indices are found to have appilications in QSPR and QSAR studies as well, see [8]. Various topological indices on different operations of graphs have been studied various authors, see [1, 20, 21, 6, 15, 16, 17, 14]. In this paper, we present the upper bounds for the product version of reciprocal Gutman index of the tensor produt, join and strong product of two connected graphs in terms of other graph invariants including the Harary index and Zagreb indices.

2. TENSOR PRODUCT

In this section, we compute the product version of the reciprocal Gutman index of $G \times K_r$.

The proof of the following lemma follows easily from the properties and structure of $G \times K_r$. The lemma is used in the proof of the main theorem of this section.

Lemma 2.1. *Let G be a connected graph on $n \geq 2$ vertices. For any pair of vertices x_{ij}, $x_{kp} \in V(G \times K_r)$, $r \geq 3$, $i, k \in \{1, 2, \ldots, n\}$ $j, p \in \{1, 2, \ldots, r\}$. Then*
(i) If $u_iu_k \in E(G)$, then

$$d_{G\times K_r}(x_{ij}, x_{kp}) = \begin{cases} 1, & \text{if } j \neq p, \\ 2, & \text{if } j = p \text{ and } u_iu_k \text{ is on a triangle of } G, \\ 3, & \text{if } j = p \text{ and } u_iu_k \text{ is not on a triangle of } G. \end{cases}$$

(ii) If $u_iu_k \notin E(G)$, then $d_{G\times K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.
(iii) $d_{G\times K_r}(x_{ij}, x_{ip}) = 2$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \times K_r$. We only prove the case when $u_iu_k \notin E(G)$, $i \neq k$ and $j = p$. The proofs for other cases are similar.

We may assume $j = 1$. Let $P = u_iu_{s_1}u_{s_2} \ldots u_{s_p}u_k$ be the shortest path of length $p + 1$ between u_i and u_k in G. From P we have a (x_{i1}, x_{k1})-path $P_1 = x_{i1}x_{s_12} \ldots x_{s_{p-1}2}x_{s_p3}x_{k1}$ if the length of P is odd, and $P_1 = x_{i1}x_{s_12} \ldots x_{s_{p-1}2}x_{s_p2}x_{k1}$ if the length of P is even.

Obviously, the length of P_1 is $p+1$, and thus $d_{G\times K_r}(x_{i1},x_{k1}) \le p+1 \le d_G(u_i,u_k)$. If there were a (x_{i1},x_{k1})-path in $G\times K_r$ that is shorter than $p+1$ then it is easy to find a (u_i,u_k)-path in G that is also shorter than $p+1$ in contrast to $d_G(u_i,u_k)=p+1$.

Remark 2.1. *(Arithmetic Geometric Inequality) Let $a_1,a_2,\dots,a_n$ be non negative n numbers. Then $\sqrt[n]{a_1a_2\dots a_n} \le \frac{a_1+a_2+\dots+a_n}{n}$.*

Theorem 2.1. *Let G be a connected graph with $n\ge 2$ vertices and m edges. Then $H_M^*(G\times K_r) \le \frac{(r-1)^{8nr}}{4n^{3nr}}\Big[H_M(G)M_1(G)(H_M(G)-\frac{M_2(G)}{2}-t)\Big]^{nr}$, where $r\ge 3$ and $t=\sum_{u_iu_k\in E_2}\frac{d_G(u_i)d_G(u_k)}{6}$.*

Proof. Set $V(G)=\{u_1,u_2,\dots,u_n\}$ and $V(K_r)=\{v_1,v_2,\dots,v_r\}$. Let x_{ij} denote the vertex (u_i,v_j) of $G\times K_r$. The degree of the vertex x_{ij} in $G\times K_r$ is $d_G(u_i)d_{K_r}(v_j)$, that is $d_{G\times K_r}(x_{ij})=(r-1)d_G(u_i)$. By the definition of H_M^*

$$
\begin{aligned}
H_M^*(G\times K_r) &= \frac{1}{2}\prod_{x_{ij},\,x_{kp}\,\in V(G\times K_r)}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij},x_{kp})}\\
&= \frac{1}{2}\prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\ne p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij},x_{ip})}\times\prod_{\substack{i,\,k=0\\ i\ne k}}^{n-1}\prod_{j=0}^{r-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}\\
&\quad\times\prod_{\substack{i,\,k=0\\ i\ne k}}^{n-1}\prod_{\substack{j,\,p=0\\ j\ne p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij},x_{kp})}. \qquad (2.1)
\end{aligned}
$$

We shall calculate the sums of (2.1) are separately.

First we compute $\prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\ne p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij},x_{ip})}$.

$$
\begin{aligned}
\prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\ne p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij},x_{ip})} &= \prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\ne p}}^{r-1}\frac{(r-1)^2d_G^2(u_i)}{2}, \text{ by Lemma 2.1}\\
&\le \left[\frac{\frac{1}{2}\sum_{i=0}^{n-1}\sum_{\substack{j,\,p=0\\ j\ne p}}^{r-1}\frac{(r-1)^2d_G^2(u_i)}{2}}{nr}\right]^{nr}, \text{ by Remark 2.1}\\
&= \left[\frac{r(r-1)^3M_1(G)}{4nr}\right]^{nr}\\
&= \left[\frac{(r-1)^3M_1(G)}{4n}\right]^{nr}. \qquad (2.2)
\end{aligned}
$$

Next we compute $\prod_{j=0}^{r-1}\prod_{\substack{i,\,k=0\\ i\ne k}}^{n-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}$. By Remark 2.1, we have

$$
\begin{aligned}
\prod_{j=0}^{r-1}\prod_{\substack{i,\,k=0\\ i\ne k}}^{n-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})} &\le \left[\frac{\frac{1}{2}\sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\ne k}}^{n-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}}{nr}\right]^{nr}\\
&= \left[\frac{\frac{1}{2}\sum_{j=0}^{r-1}S}{nr}\right]^{nr}. \qquad (2.3)
\end{aligned}
$$

First we obtain the sum S. For that we define $E_1 = \{uv \in E(G) \,|\, uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2 = E(G) - E_1$.

$$
\begin{aligned}
S &= \Big(\sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \notin E(G)}}^{n-1} + \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_1}}^{n-1} + \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_2}}^{n-1} \Big)\Big(\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})} \Big) \\
&= \left(\sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \notin E(G)}}^{n-1} \frac{(r-1)^2 d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)} + \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_1}}^{n-1} \frac{(r-1)^2 d_G(u_i)d_G(u_k)}{2} \right. \\
&\quad \left. + \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_2}}^{n-1} \frac{(r-1)^2 d_G(u_i)d_G(u_k)}{3} \right), \text{ by Lemma 2.1} \\
&= (r-1)^2 \left\{ \left(\sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \notin E(G)}}^{n-1} \frac{d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)} + \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_1}}^{n-1} \frac{d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)} \right. \right. \\
&\quad \left. \left. + \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_2}}^{n-1} \frac{d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)} \right) - \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_1}}^{n-1} \frac{d_G(u_i)d_G(u_k)}{2} - 2\sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_2}}^{n-1} \frac{d_G(u_i)d_G(u_k)}{3} \right\} \\
&= (r-1)^2 \left\{ 2H_M(G) - \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E(G)}}^{n-1} \frac{d_G(u_i)d_G(u_k)}{2} - \sum_{\substack{i,k=0 \\ i\neq k \\ u_iu_k \in E_2}}^{n-1} \frac{d_G(u_i)d_G(u_k)}{6} \right\} \\
&= (r-1)^2 \Big(2H_M(G) - M_2(G) - \sum_{u_iu_k \in E_2} \frac{d_G(u_i)d_G(u_k)}{3} \Big). \qquad (2.4)
\end{aligned}
$$

Now summing (2.4) over $j = 0, 1, \ldots, r-1$, we get,

$$
\sum_{j=0}^{r-1} S = r(r-1)^2 \Big(2H_M(G) - M_2(G) - \sum_{u_iu_k \in E_2} \frac{d_G(u_i)d_G(u_k)}{3} \Big). \qquad (2.5)
$$

Hence

$$
\begin{aligned}
&\prod_{j=0}^{r-1} \prod_{\substack{i,k=0 \\ i\neq k}}^{n-1} \frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})} \\
&\leq \left[\frac{\frac{r(r-1)^2}{2}\Big(2H_M(G) - M_2(G) - \sum_{u_iu_k \in E_2} \frac{d_G(u_i)d_G(u_k)}{3} \Big)}{nr} \right]^{nr} \\
&= \left[\frac{(r-1)^2 \Big(H_M(G) - \frac{M_2(G)}{2} - \sum_{u_iu_k \in E_2} \frac{d_G(u_i)d_G(u_k)}{6} \Big)}{n} \right]^{nr}. \qquad (2.6)
\end{aligned}
$$

Next we compute $\prod_{\substack{i,k=0 \\ i\neq k}}^{n-1} \prod_{\substack{j,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij},x_{kp})}$.

$$\prod_{\substack{i,k=0\\i\neq k}}^{n-1}\prod_{\substack{j,p=0,\\j\neq p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})d_{G\times K_r}(x_{kp}))}{d_{G\times K_r}(x_{ij},x_{kp}}\ \le\ \left[\frac{\frac{1}{2}\sum\limits_{\substack{i,k=0\\i\neq k}}^{n-1}\sum\limits_{\substack{j,p=0,\\j\neq p}}^{r-1}\frac{(r-1)^2d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)}}{nr}\right]^{nr},$$

by Lemma 2.1 and Remark 2.1

$$= \left[\frac{r(r-1)^3H_M(G)}{nr}\right]^{nr}$$

$$= \left[\frac{(r-1)^3H_M(G)}{n}\right]^{nr}. \tag{2.7}$$

Using (2.1) and the sums in (2.2),(2.6) and (2.7), respectively, we obtain the required result. □

Using Theorem 2.1, we have the following corollaries.

Corollary 2.1. *Let G be a connected graph on $n \geq 2$ vertices with m edges. If each edge of G is on a C_3, then $H^*_M(G\times K_r) \leq \frac{(r-1)^{8nr}}{4n^{3nr}}\Big[H_M(G)M_1(G)(H_M(G)-\frac{M_2(G)}{2})\Big]^{nr}$, where $r \geq 3$.*

For a triangle free graph $\sum\limits_{u_iu_k\in E_2} d_G(u_i)d_G(u_k) = M_2(G)$.

Corollary 2.2. *If G is a connected triangle free graph on $n \geq 2$ vertices and m edges, then $H^*_M(G\times K_r) \leq \frac{(r-1)^{8nr}}{4n^{3nr}}\Big[H_M(G)M_1(G)(H_M(G)-\frac{2M_2(G)}{3})\Big]^{nr}$, where $r \geq 3$.*

By direct calculations we obtain expressions for the values of the Harary indices of K_n and C_n. $H(K_n) = \frac{n(n-1)}{2}$ and $H(C_n) = n\Big(\sum\limits_{i=1}^{\frac{n}{2}}\frac{1}{i}\Big) - 1$ when n is even, and $n\Big(\sum\limits_{i=1}^{\frac{n-1}{2}}\frac{1}{i}\Big)$ otherwise. Similarly, $H_M(K_n) = \frac{n(n-1)^3}{2}$, $H_A(K_n) = n(n-1)^2$ and $H_M(C_n) = H_A(C_n) = 4H(C_n)$.

Using Corollaries 2.1 and 2.2, we obtain the product version of reciprocal Gutman indices of the graphs $K_n\times K_r$ and $C_n\times K_r$.

Example 2.1. (i) $H^*_M(K_n\times K_r) \leq \frac{1}{4}\Big(\frac{(r-1)^8(n-1)^8}{8}\Big)^{nr}$.

$$(ii)\ H^*_M(C_n\times K_r) \leq \begin{cases}\frac{(r-1)^{24n}}{4}\Big(\frac{384(24-n)}{n}\Big)^{3n}, & \text{if } n=3,\\ \frac{(r-1)^{8nr}}{4n^{2nr}}\Big[64H(C_n)(H(C_n)-\frac{2n}{3})\Big]^{nr}, & \text{if } n>3.\end{cases}$$

3. Join

In this section, we compute the product version of reciprocal Gutman index of join of two graphs.

Theorem 3.2. *Let G_1 and G_2 be graphs with n and m vertices p and q edges, respectively. Then $H^*_M(G_1+G_2) \leq \frac{1}{2^{2nm}nm^{5nm}}\Big[(M_2(G_1)+mM_1(G_1)+m^2p)(M_2(G_2)+nM_1(G_2)+n^2q)(\overline{M}_2(G_1)+m\overline{M}_1(G_1)+m^2(\frac{n(n-1)-2p}{2}))(\overline{M}_2(G_2)+n\overline{M}_1(G_2)+n^2(\frac{m(m-1)-2q}{2}))(4pq+2mnq+2mnp+m^2n^2)\Big]^{nm}$.*

Proof. Set $V(G_1) = \{u_1,u_2,\ldots,u_n\}$ and $V(G_2) = \{v_1,v_2,\ldots,v_m\}$. By definition of the join of two graphs, one can see that,

$$d_{G_1+G_2}(x)=\begin{cases} d_{G_1}(x)+|V(G_2)|\,, & \text{if } x\in V(G_1)\\ d_{G_2}(x)+|V(G_1)|\,, & \text{if } x\in V(G_2)\end{cases}$$

and $d_{G_1+G_2}(u,v)=\begin{cases} 0, & \text{if } u=v\\ 1, & \text{if } uv\in E(G_1) \text{ or } uv\in E(G_2) \text{ or } (u\in V(G_1) \text{ and } v\in V(G_2))\\ 2, & \text{otherwise.}\end{cases}$

Therefore,

$$\begin{aligned}
H^*_M(G_1+G_2) &= \prod_{\{u,v\}\subseteq V(G_1+G_2)} \frac{d_{G_1+G_2}(u)d_{G_1+G_2}(v)}{d_{G_1+G_2}(u,v)}\\
&= \prod_{uv\in E(G_1)} (d_{G_1}(u)+m)(d_{G_1}(v)+m) \times \prod_{uv\notin E(G_1)} \frac{(d_{G_1}(u)+m)(d_{G_1}(v)+m)}{2}\\
&\times \prod_{uv\in E(G_2)} (d_{G_2}(u)+n)(d_{G_2}(v)+n) \times \prod_{uv\notin E(G_2)} \frac{(d_{G_2}(u)+n)(d_{G_2}(v)+n)}{2}\\
&\times \prod_{u\in V(G_1),\, v\in V(G_2)} (d_{G_1}(u)+m)(d_{G_2}(v)+n)\\
&\le \Big[\frac{\sum\limits_{uv\in E(G_1)} (d_{G_1}(u)+m)(d_{G_1}(v)+m)}{nm}\Big]^{nm} \Big[\frac{\sum\limits_{uv\notin E(G_1)} \frac{(d_{G_1}(u)+m)(d_{G_1}(v)+m)}{2}}{nm}\Big]^{nm}\\
&\Big[\frac{\sum\limits_{uv\in E(G_2)} (d_{G_2}(u)+n)(d_{G_2}(v)+n)}{nm}\Big]^{nm} \Big[\frac{\sum\limits_{uv\notin E(G_2)} \frac{(d_{G_2}(u)+n)(d_{G_2}(v)+n)}{2}}{nm}\Big]^{nm}\\
&\Big[\frac{\sum\limits_{u\in V(G_1),\, v\in V(G_2)} (d_{G_1}(u)+m)(d_{G_2}(v)+n)}{nr}\Big]^{nm}, \text{ by Remark 2.1}\\
&= \frac{1}{2^{2nm}nm^{5nm}}\Big[M_2(G_1)+mM_1(G_1)+m^2p\Big]^{nm}\Big[M_2(G_2)+nM_1(G_2)+n^2q\Big]^{nm}\\
&\Big[\overline{M}_2(G_1)+m\overline{M}_1(G_1)+m^2(\frac{n(n-1)-2p}{2})\Big]^{nm}\times
\end{aligned}$$

$$\Big[\overline{M}_2(G_2)+n\overline{M}_1(G_2)+n^2(\frac{m(m-1)-2q}{2})\Big]^{nm}\times\Big[4pq+2mnq+2mnp+m^2n^2\Big]^{nm}.$$

$\square$

One can observe that $M_1(C_n)=4n,\ n\ge 3$, $M_1(P_1)=0$, $M_1(P_n)=4n-6,\ n>1$ and $M_1(K_n)=n(n-1)^2$. Similarly, $\overline{M_1}(K_n)=\overline{M_2}(K_n)=0$. Moreover $M_2(P_n)=4(n-2)$ and $M_2(C_n)=4n$. Using Theorem 3.2, we have the following corollaries.

Corollary 3.3. *Let G be graph on n vertices and p edges. Then*
$H^*_M(G+K_m)\le \frac{1}{2^{2nm}nm^{5nm}}\Big[(M_2(G)+mM_1(G)+m^2p)(\frac{m(m-1)(m^2+n^2+nm-2m-n)}{2})(\overline{M}_2(G)+m\overline{M}_1(G)+m^2(\frac{n(n-1)-2p}{2}))(2p+mn)(m^2+nm-m)\Big]^{nm}.$

Let $K_{n,m}$ be the bipartite graph with two partitions having n and m vertices. Note that $K_{n,m}=\overline{K}_n+\overline{K}_m$.

Corollary 3.4. $H^*_M(K_{n,m})=H^*_M(\overline{K}_n+\overline{K}_m)\le\Big(\frac{n^5m^5(n-1)(m-1)}{4}\Big)^{nm}.$

4. Strong product

In this section, we obtain the product version of reciprocal Gutman index of $G \boxtimes K_r$.

Theorem 4.3. *Let G be a connected graph with n vertices and m edges. Then* $H^*_M(G \boxtimes K_r) \le \frac{(r-1)^{2nr}}{2n^{3nr}}\Big[n(r-1)^2 + 4mr(r-1) + r^2M_1(G)\Big]^{nr}\Big[2r^2H_M(G) + 2r(r-1)H_A(G) + 2(r-1)^2H(G)\Big]^{2nr}$.

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G\boxtimes K_r$. The degree of the vertex x_{ij} in $G\boxtimes K_r$ is $d_G(u_i)+d_{K_r}(v_j)+d_G(u_i)d_{K_r}(v_j)$, that is $d_{G\boxtimes K_r}(x_{ij}) = rd_G(u_i) + (r-1)$. One can see that for any pair of vertices x_{ij}, $x_{kp} \in V(G \boxtimes K_r)$, $d_{G\boxtimes K_r}(x_{ij}, x_{ip}) = 1$ and $d_{G\boxtimes K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.

$$\begin{aligned}
H^*_M(G \boxtimes K_r) = &\prod_{x_{ij},\, x_{kp} \in V(G\boxtimes K_r)} \frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})}\\
= &\frac{1}{2}\prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij}, x_{ip})} \times \prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\prod_{j=0}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij}, x_{kj})}\\
&\times \prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\prod_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})}. \qquad (4.8)
\end{aligned}$$

We shall obtain sums of (4.8), separately.
First we calculate $\prod\limits_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij}, x_{ip})}$.

$$\begin{aligned}
&\prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij}, x_{ip})}\\
= &\prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\Big(rd_G(u_i) + (r-1)\Big)\Big(rd_G(u_i) + (r-1)\Big)\\
\le &\left[\frac{\frac{1}{2}\sum\limits_{i=0}^{n-1}\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\Big(rd_G(u_i) + (r-1)\Big)\Big(rd_G(u_i) + (r-1)\Big)}{nr}\right]^{nr},\\
&\text{by Remark 2.1}\\
= &\left[\frac{r(r-1)\Big(n(r-1)^2 + 4mr(r-1) + r^2M_1(G)\Big)}{2nr}\right]^{nr}\\
= &\left[\frac{(r-1)\Big(n(r-1)^2 + 4mr(r-1) + r^2M_1(G)\Big)}{2n}\right]^{nr}. \qquad (4.9)
\end{aligned}$$

Next we obtain $\prod\limits_{j=0}^{r-1}\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij},x_{kj})}$.

$$\begin{aligned}
& \prod_{j=0}^{r-1}\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij},x_{kj})}\\
= & \prod_{j=0}^{r-1}\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{(d_G(u_i)+(r-1)+(r-1)d_G(u_i))(d_G(u_k)+(r-1)+(r-1)d_G(u_k))}{d_G(u_i,u_k)}\\
\leq & \Bigg[\frac{\frac{r^2}{2}\sum\limits_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_G(u_i)d_G(u_k)}{d_G(u_i,u_k)}}{nr}+\frac{\frac{1}{2}\sum\limits_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{r(r-1)(d_G(u_i)+d_G(u_k))}{d_G(u_i,u_k)}}{nr}\\
& +\frac{\frac{1}{2}\sum\limits_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{(r-1)^2}{d_G(u_i,u_k)}}{nr}\Bigg]^{nr}, \text{by Remark 2.1}\\
= & \Bigg[\frac{r\Big(2r^2H_M(G)+2r(r-1)H_A(G)+2(r-1)^2H(G)\Big)}{2nr}\Bigg]^{nr}\\
= & \Bigg[\frac{\Big(r^2H_M(G)+r(r-1)H_A(G)+(r-1)^2H(G)\Big)}{n}\Bigg]^{nr}. \qquad (4.10)
\end{aligned}$$

Finally, we compute $\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\prod_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij},x_{kp})}$.

$$\begin{aligned}
& \prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\prod_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij},x_{kp})}\\
\leq & \Bigg[\frac{r(r-1)\Big(2r^2H_M(G)+2r(r-1)H_A(G)+2(r-1)^2H(G)\Big)}{2nr}\Bigg]^{nr},\\
& \text{by Remark 2.1}\\
= & \Bigg[\frac{(r-1)\Big(r^2H_M(G)+r(r-1)H_A(G)+(r-1)^2H(G)\Big)}{n}\Bigg]^{nr}. \qquad (4.11)
\end{aligned}$$

Using (4.9), (4.10) and (4.11) in (4.8), we obtain the required result. □

Using Theorem 4.3, we obtain the following corollary.

Corollary 4.5. $H_M^*(C_n\boxtimes K_r)\leq\Big(\frac{(r-1)}{n}\Big)^{2nr}(\frac{1}{2})^{nr}(9r^2-6r+1)^{3nr}\Big(H(C_n)\Big)^{2nr}$.

As an application we present formula for product version of reciprocal Gutman index of closed fence graph, $C_n\boxtimes K_2$.

Example 4.2. *By Corolarry 4.5, we have* $H_M^*(C_n\boxtimes K_2)\leq\begin{cases}\Big(\frac{(25)^3}{2n^2}\Big)^{2n}\Big(n\sum\limits_{i=1}^{\frac{n}{2}}\frac{1}{i}-1\Big)^{4n}, & \textit{if } n \textit{ is even}\\ \Big(\frac{(25)^3}{2}\Big)^{2n}\Big(\sum\limits_{i=1}^{\frac{n}{2}}\frac{1}{i}\Big)^{4n}, & \textit{if } n \textit{ is odd.}\end{cases}$

REFERENCES

[1] Alizadeh, Y., Iranmanesh, A. and Doslić, T., *Additively weighted Harary index of some composite graphs*, Discrete Math., **313** (2013), 26–34

[2] Berinde, Zoiţa-Mărioara, *Comparing the molecular graph degeneracy of Wiener, Harary, Balaban, Randić and ZEP topological indices*, Creat. Math.& Inform., **23** (2014), 165–174

[3] Berinde, Zoiţa-Mărioara, *QSTR mathematical models for the toxicity of aliphatic carboxylic acids on tetrahymena pyriformis*, Creat. Math.& Inform., **22** (2013) 151–160

[4] Berinde, Zoiţa-Mărioara, *A QSPR study of hydrophobicity of phenols and 2-(aryloxy-α-acetyl)-phenoxathiin derivatives using the topological index ZEP*, Creat. Math.& Inform., **22** (2013), 33–40

[5] Berinde, Zoiţa-Mărioara, *Modelling normal boiling points of alkanes by linear regression using the SD index*, Creat. Math.& Inform., **19** (2010), 135–139

[6] Doslić, T., *Vertex-weighted Wiener polynomials for composite graphs*, Ars Math. Contemp., **1** (2008), 66–80

[7] Dobrynin, A. A. and Kochetova, A. A., *Degree distance of a graph: a degree analogue of the Wiener index*, J. Chem. Inf. Comput. Sci., **34** (1994), 1082–1086

[8] Devillers, J., Balaban, A. T., Eds., *Topological indices and related descriptors in QSAR and QSPR*, Gordon and Breach, Amsterdam, The Netherlands, 1999

[9] Das, K. C., Zhou, B. and Trinajstić, N., *Bounds on Harary index*, J. Math. Chem., **46** (2009), 1369–1376

[10] Gutman, I., *Selected properties of the Schultz molecular topogical index*, J. Chem. Inf. Comput. Sci., **34** (1994), 1087–1089

[11] Gutman, I., Linert, W., Lukovits, I. and Tomović, Z., *On the multiplicative Wiener index and its possible chemical applications*, Monatshefte for Chemie, **131** (2000), 421–427

[12] Gutman, I., Linert, W., Lukovits, I. and Tomović, Z., *The multiplicative version of Wiener index*, J. Chem. Inf. Comput. Sci., **40** (2000), 113–116

[13] Hua, H. and Zhang, S., *On the reciprocal degree distance of graphs*, Discrete Appl. Math., **160** (2012), 1152–1163

[14] Kaladevi, V., Murugesan, R. and Pattabiraman, K., *Reverse degree distance of some graph operations*, Creat. Math. & Inform., (in press)

[15] Pattabiraman, K. and Paulraja, P., *On some topological indices of the tensor product of graphs*, Discrete Appl. Math., **160** (2012), 267–279

[16] Pattabiraman, K., Paulraja, P. *Wiener and vertex PI indices of the strong product of graphs*, Discuss. Math. Graph Thoery, **32** (2012), 749–769

[17] Pattabiraman, K. and Vijayaragavan, M., *On the reformulated reciprocal degree distance of graphs*, Creat. Math. & Inform., **25** (2016), No. 2, 205–213

[18] Pattabiraman, K., *F-indices and its coindices of some classes of graphs*, Creat. Math. & Inform., (in press)

[19] Su, G., Gutman, I., Xiong, L. and Xu, L., *Reciprocal product degree distance of graphs*, Manuscript

[20] Xu, K., Das, K. C., Hua, H., Diudea, M. V., *Maximal Harary index of unicyclic graphs with given matching number*, Stud. Univ. Babes-Bolyai Chem., **58** (2013), 71–86

[21] Xu, K., Wang, J. and Liu, H., *The Harary index of ordinary and generalized quasi-tree graphs*, J. Appl. Math. Comput., DOI 10.1007/s12190-013-0727-4

[22] Yousefi-Azari, H., Khalifeh, M. H. and Ashrafi, A. R., *Calculating the edge Wiener and edge Szeged indices of graphs*, J. Comput. Appl. Math., **235** (2011), 4866–4870

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