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Stability of nonlinear Volterra-Fredholm integro differential equation: A fixed point approach

YUNUS ATALAN and VATAN KARAKAYA

ABSTRACT. The objective of the present work is to analyze stability in the sense of Hyers-Ulam and Hyers-Ulam-Rassias for nonlinear Volterra Fredholm integro-differential equation by using fixed point approach.

1. INTRODUCTION

The beginning of the stability theory is based on the following problem that Ulam presented in [25]:
Let θ_1 be a group and let θ_2 be a metric group with the metric σ. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $u : \theta_1 \to \theta_2$ satisfies the inequality

$$\sigma(u(xy), u(x)u(y)) < \delta$$

for all $x, y \in \theta_1$, then there exists a homomorphism $P : \theta_1 \to \theta_2$ such that $\sigma(u(x), P(x)) < \varepsilon$ for all $x \in \theta_1$?
In 1941, Hyers [9] answered Ulam's question for the approximately additive functions in Banach spaces. In 1978, Rassias [19] generalized Hyers's result by taking Cauchy difference as unbounded. Since then, a lot of researchers have studied Hyers-Ulam stabilitiy for a wide range of equations and have obtained several important results (see [3], [4], [6], [10], [18]). The use of differential and integral equations instead of functional equations in Ulam's problem has created a new field of study which has a rich literature (for more detail see [1], [7], [8], [12], [20], [23]). The first authors who investigated Hyers-Ulam stability of a differential equation are Alsina and Ger (see [2]). Miura, Miyajima and Takahasi (see [16], [17], [24]) extended the result of Alsina and Ger to the Hyers-Ulam stability of the first order linear differential equation. Furthermore S.-M. Jung ([13], [14], [15]) showed the stability of linear differential equations by developing the results of Takahasi, Takagi and Miura. I. A. Rus showed the stability of differential and integral equations using Gronwall lemma and weakly Picard operator technique (see [21], [22]).

The objective of the present work is to analyze stability in the sense of Hyers-Ulam and Hyers-Ulam-Rassias for the following nonlinear Volterra Fredholm integro-differential equation (VFIDE) by using fixed point approach:

$$\begin{cases} x^{'}(t) = f(t, x(t)) + \lambda_1 \int\limits_a^t k_1(t, s, x(s))ds + \lambda_2 \int\limits_a^b k_2(t, s, x(s))ds \\ x(0) = \alpha, \qquad\qquad t, s \in I = [a, b] \end{cases} \tag{1.1}$$

where $\alpha \in \mathbb{R}$ and given function $f : I \times \mathbb{R} \to \mathbb{R}$ and the kernels $k_1, k_2 : I^2 \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous functions satisfying the following Lipschitz conditions: there

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Corresponding author: Yunus Atalan; yunus_atalan@hotmail.com

exist $L_f, L_{k_1}, L_{k_2} \geq 0$ such that

$$|f(t,x) - f(t,y)| \leq L_f\,|x-y| \tag{1.2}$$

$$|k_1(t,s,x) - k_1(t,s,y)| \leq L_{k_1}\,|x-y| \tag{1.3}$$

$$|k_2(t,s,x) - k_2(t,s,y)| \leq L_{k_2}\,|x-y| \tag{1.4}$$

for $t,s \in I$ and for $x,y \in \mathbb{R}$.

If $k_1(t,s,x(s)) = 0$ in (1.1), the equation reduces to a nonlinear Volterra integro-differential equation; if $k_2(t,s,x(s)) = 0$, it becomes a nonlinear Fredholm integro-differential equation. It is clear that, if $k_1(t,s,x(s)) = k_2(t,s,x(s)) = 0$ in (1.1) then the equation is transformed into an ordinary differential equation.

Definition 1.1. The equation (1.1) is said to be stable in the sense of Hyers-Ulam if, for all $\varepsilon > 0$ and all continuously differentiable function $x(t)$ satisfying the inequality

$$\left| x^{'}(t) - f(t,x(t)) - \lambda_1 \int\limits_a^t k_1(t,s,x(s))ds - \lambda_2 \int\limits_a^b k_2(t,s,x(s))ds \right| \leq \varepsilon,\ \forall t \in I,$$

there exists a solution $x_0(t)$ of the equation (1.1) and a constant $C > 0$ with

$$|x(t) - x_0(t)| \leq C\varepsilon,$$

for all t, where C is independet of $x(t)$ and $x_0(t)$. If the above inequality is also valid when $\varepsilon = \phi(t)$, where $\phi : I \to (0,\infty)$ is independet of $x(t)$ and $x_0(t)$, then it is said that the equation (1.1) has Hyers-Ulam Rassias stability.

2. BASIC CONCEPTS

In this section we give the definiton of the generalized metric space and one of the fundamental results of fixed point theory which extremely important in obtaining our main results:

Definition 2.2. [11] Let $\sigma : B \times B \to [0,+\infty]$ be a function. If σ satisfies the following conditions, then it is called a generalized metric on B :

(D1) $\sigma(b_1,b_2) = 0$ if and only if $b_1 = b_2$,
(D2) $\sigma(b_1,b_2)$= $\sigma(b_2,b_1)$ for all $b_1,b_2 \in B$,
(D3) $\sigma(b_1,b_3) \leq \sigma(b_1,b_2) + \sigma(b_2,b_3)$ for all $b_1,b_2,b_3 \in B$.

Theorem 2.1. [5] *Let (B,σ) be a generalized complete metric space. Suppose that $\Gamma : B \to B$ a strictly contractive operator with the Lipschitz constant $\delta < 1$. If there is a nonnegative integer m such that $\sigma(\Gamma^{m+1}u, \Gamma^m u) < \infty$ for some $u \in B$, then the followings are true:*

(a) *the sequence $\{\Gamma^n u\}$ converges to a fixed point p of Γ;*

(b) *p is the unique fixed point of Γ in*

$$V = \{v \in B : \sigma(\Gamma^m u, v) < \infty\};$$

(c) *If $v \in V$, then*

$$\sigma(v,p) \leq \frac{1}{1-\delta}\sigma(\Gamma v, v).$$

3. HYERS-ULAM STABILITY

Now, we show the Hyers-Ulam stability of the nonlinear VFIDE (1.1) under some appropriate conditions:

Theorem 3.2. *Let a and b be real numbers such that $a < b$ and set $I = [a,b]$. Let $\lambda_1, \lambda_2, L_f, L_{k_1}$ and L_{k_2} be positive constants with $0 < L_f(b-a) + \lambda_1 L_{k_1}\frac{(b-a)^2}{2} + \lambda_2 L_{k_2}(b-a)^2 < 1$. Suppose that $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition (1.2) and the kernels $k_1, k_2 : I^2 \times \mathbb{R} \to \mathbb{R}$ are two continuous functions which satisfy Lipschitz condition (1.3) and (1.4), respectively. If for $\varepsilon \geq 0$ and $\forall t \in I$, continuously differentiable function $x : I \to \mathbb{R}$ satisfies*

$$\left| x^{'}(t) - f(t,x(t)) - \lambda_1 \int_a^t k_1(t,s,x(s))ds - \lambda_2 \int_a^b k_2(t,s,x(s))ds \right| \leq \varepsilon, \tag{3.5}$$

then there is only one continuous function $x_0 : I \to \mathbb{R}$ such that

$$\begin{aligned} x_0(t) &= \alpha + \int_a^t f(u,x(u))du + \lambda_1 \int_a^t \int_a^u k_1(u,s,x(s))dsdu \\ &\quad + \lambda_2 \int_a^t \int_a^b k_2(u,s,x(s))dsdu \end{aligned} \tag{3.6}$$

and

$$|x(t) - x_0(t)| \leq \frac{(b-a)\varepsilon}{1 - [L_f(b-a) + \lambda_1 L_{k_1}\frac{(b-a)^2}{2} + \lambda_2 L_{k_2}(b-a)^2]}$$

Proof. Let $B := C(I,\mathbb{R})$ be the set of all continuous functions from I to $\mathbb{R}$. For $v, w \in B$, we set

$$d(v,w) : \inf\{C \in [0,\infty] : |v(t) - w(t)| \leq C, \forall t \in I\}.$$

It can easily be seen that (B,d) is a complete generalized metric space (see [11]).

For $\forall t \in I$, $\Gamma : B \to B$ be defined as follows,

$$\begin{aligned} (\Gamma v)(t) &= \alpha + \int_a^t f(\varsigma, v(\varsigma))d\varsigma + \lambda_1 \int_a^t \int_a^u k_1(t,\varsigma,v(\varsigma))d\varsigma du \\ &\quad + \lambda_2 \int_a^t \int_a^b k_2(t,\varsigma,v(\varsigma))d\varsigma du \end{aligned} \tag{3.7}$$

We shall show Γ is strictly contractive on the space B. For any $v, w \in B$, let $C(v,w) \in [0,\infty]$ be an arbitrary constant such that $d(v,w) \leq C(v,w)$. From (3.7), we get

$$|v(t) - w(t)| \leq C(v,w), \forall t \in I.$$

For any $t \in I$, we have

$$\begin{aligned}
|(\Gamma v)(t) - (\Gamma w)(t)| &\leq \int_a^t |f(\varsigma, v(\varsigma)) - f(\varsigma, w(\varsigma))|\, d\varsigma \\
&\quad + \lambda_1 \int_a^t \int_a^u |k_1(t, \varsigma, v(\varsigma) - k_1(t, \varsigma, w(\varsigma))|\, d\varsigma du \\
&\quad + \lambda_2 \int_a^t \int_a^b |k_2(t, \varsigma, v(\varsigma) - k_2(t, \varsigma, w(\varsigma))|\, d\varsigma du \\
&\leq L_f \int_a^t |v(\varsigma) - w(\varsigma)|\, d\varsigma + \lambda_1 L_{k_1} \int_a^t \int_a^u |v(\varsigma) - w(\varsigma)|\, d\varsigma du \quad (3.8) \\
&\quad + \lambda_2 L_{k_2} \int_a^t \int_a^b |v(\varsigma) - w(\varsigma)|\, d\varsigma du \\
&\leq L_f C(v, w)(t-a) \\
&\quad + \lambda_1 L_{k_1} C(v, w) \frac{(t-a)^2}{2} \\
&\quad + \lambda_2 L_{k_2} C(v, w)(b-a)(t-a)
\end{aligned}$$

Since $a \leq t \leq b$, from (3.8) we obtain

$$\begin{aligned}
|(\Gamma v)(t) - (\Gamma w)(t)| &\leq L_f C(v, w)(b-a) \\
&\quad + \lambda_1 L_{k_1} C(v, w) \frac{(b-a)^2}{2} \\
&\quad + \lambda_2 L_{k_2} C(v, w)(b-a)^2,
\end{aligned}$$

that is,

$$d(\Gamma v, \Gamma w) \leq C(v, w)[L_f(b-a) + \lambda_1 L_{k_1} \frac{(b-a)^2}{2} + \lambda_2 L_{k_2}(b-a)^2].$$

We conclude that

$$d(\Gamma v, \Gamma w) \leq [L_f(b-a) + \lambda_1 L_{k_1} \frac{(b-a)^2}{2} + \lambda_2 L_{k_2}(b-a)^2] d(v, w),$$

for any $v, w \in B$. Since by assumption, we have

$$L_f(b-a) + \lambda_1 L_{k_1} \frac{(b-a)^2}{2} + \lambda_2 L_{k_2}(b-a)^2 < 1,$$

then Γ is strictly contractive. Let w_0 be an arbitrary element in B. Then there exists a constant $C \in (0, \infty)$ for all $t \in I$ such that

$$\begin{aligned}
|(\Gamma w_0)(t) - w_0(t)| &= \Bigg| \alpha + \int_a^t f(\varsigma, w_0(\varsigma)) d\varsigma + \lambda_1 \int_a^t \int_a^u k_1(t, \varsigma, w_0(\varsigma)) d\varsigma du \\
&\quad + \lambda_2 \int_a^t \int_a^b k_2(t, \varsigma, w_0 v(\varsigma)) d\varsigma du - w_0(t) \Bigg| \\
&\leq C.
\end{aligned}$$

We deduce that

$$d(\Gamma w_0, w_0) < \infty.$$

Then by Theorem 2.1, there exists a continuous function $x_0 : I \to \mathbb{R}$ such that $(\Gamma^n w_0)$ converges to x_0 and $\Gamma x_0 = x_0$, that is x_0 is a solution to the equation (VFIDE).

Because of d is a metric, $x_0 : I \to \mathbb{R}$ is the unique continuous function such that

$$x_0(t) = \alpha + \int_a^t f(u, x(u))du + \lambda_1 \int_a^t \int_a^u k_1(u, s, x(s))dsdu + \lambda_2 \int_a^t \int_a^b k_2(u, s, x(s))dsdu.$$

By assumption (3.5), for $\forall t \in I$ we get

$$-\varepsilon \leq x'(t) - f(t, x(t)) - \lambda_1 \int_a^t k_1(t, s, x(s))ds - \lambda_2 \int_a^b k_2(t, s, x(s))ds \leq \varepsilon,$$

If each term of the above inequality is integrated, then

$$\begin{aligned} &\Bigg| x(t) - \alpha - \int_a^t f(u, x(u))du - \lambda_1 \int_a^t \int_a^u k_1(u, s, x(s))dsdu \\ &\qquad -\lambda_2 \int_a^t \int_a^b k_2(u, s, x(s))dsdu \Bigg| \\ \leq\ & \varepsilon(b-a). \end{aligned}$$

That is we obtain

$$d(x, \Gamma x) \leq \varepsilon(b-a). \tag{3.9}$$

By using Theorem 2.1 (c) and (3.9), we conclude that

$$\begin{aligned} d(x, x_0) &\leq \frac{1}{1 - [L_f(b-a) + \lambda_1 L_{k_1} \frac{(b-a)^2}{2} + \lambda_2 L_{k_2}(b-a)^2} d(x, \Gamma x) \\ &\leq \frac{(b-a)}{1 - [L_f(b-a) + \lambda_1 L_{k_1} \frac{(b-a)^2}{2} + \lambda_2 L_{k_2}(b-a)^2]} \varepsilon. \end{aligned}$$

□

4. Hyers-Ulam-Rassias stability

Finally, we show the Hyers-Ulam-Rassias stability of the nonlinear VFIDE (1.1).

Theorem 4.3. *Let a and b be real numbers such that $a < b$ and set $I = [a, b]$. Let $V, \lambda_1, \lambda_2, L_f, L_{k_1}$ and L_{k_2} be positive constants with $0 < L_f V + \lambda_1 L_{k_1} V^2 + \lambda_2 L_{k_2} V^2 < 1$. Let $\phi : I \to (0, \infty)$ be a continuous function which takes minumum value at b such that*

$$\int_a^t \phi(\varsigma)d\varsigma \leq V\phi(t) \tag{4.10}$$

for each $t \in I$. Suppose that $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition (1.2) and the kernels $k_1, k_2 : I^2 \times \mathbb{R} \to \mathbb{R}$ are two continuous functions which satisfy Lipschitz condition (1.3) and (1.4), respectively. If for each $t \in I$, a continuously differentiable function $x : I \to \mathbb{R}$ satisfies

$$\left| x'(t) - f(t, x(t)) - \lambda_1 \int_a^t k_1(t, s, x(s))ds - \lambda_2 \int_a^b k_2(t, s, x(s))ds \right| \leq \phi(t), \tag{4.11}$$

then there is only one continuous function $x_0 : I \to \mathbb{R}$ satisfying (3.6) and

$$|x(t) - x_0(t)| \leq \frac{V}{1 - [L_f V + \lambda_1 L_{k_1} V^2 + \lambda_2 L_{k_2} V^2]}\phi(t)$$

Proof. Let $B := C(I, \mathbb{R})$ be the set of all continuous functions from I to $\mathbb{R}$. For $v, w \in B$, we set

$$d(v,w) : \inf\{C \in [0,\infty] : |v(t) - w(t)| \leq C\phi(t), \forall t \in I\}.$$

It can easily be seen that (B, d) is a complete generalized metric space (see [11]). For $\forall t \in I$, $\Gamma : B \to B$ be defined as follows

$$\begin{aligned} (\Gamma v)(t) &= \alpha + \int_a^t f(\varsigma, v(\varsigma))d\varsigma + \lambda_1 \int_a^t \int_a^u k_1(t,\varsigma,v(\varsigma))d\varsigma du \\ &\quad + \lambda_2 \int_a^t \int_a^b k_2(t,\varsigma,v(\varsigma))d\varsigma du \end{aligned} \tag{4.12}$$

We shall show Γ is strictly contractive on the space B. For any $v, w \in B$, let $C(v,w) \in [0,\infty]$ be an arbitrary constant such that $d(v,w) \leq C(v,w)$. From (4.12), we get

$$|v(t) - w(t)| \leq C(v,w)\phi(t), \forall t \in I.$$

For any $t \in I$, we have

$$\begin{aligned} |(\Gamma v)(t) - (\Gamma w)(t)| &\leq \int_a^t |f(\varsigma, v(\varsigma)) - f(\varsigma, w(\varsigma))| \, d\varsigma \\ &\quad + \lambda_1 \int_a^t \int_a^u |k_1(t,\varsigma,v(\varsigma) - k_1(t,\varsigma,w(\varsigma))| \, d\varsigma du \\ &\quad + \lambda_2 \int_a^t \int_a^b |k_2(t,\varsigma,v(\varsigma) - k_2(t,\varsigma,w(\varsigma))| \, d\varsigma du \\ &\leq L_f \int_a^t |v(\varsigma) - w(\varsigma)| \, d\varsigma + \lambda_1 L_{k_1} \int_a^t \int_a^u |v(\varsigma) - w(\varsigma)| \, d\varsigma du \\ &\quad + \lambda_2 L_{k_2} \int_a^t \int_a^b |v(\varsigma) - w(\varsigma)| \, d\varsigma du \\ &\leq L_f C(v,w) \int_a^t \phi(\varsigma)d\varsigma + \lambda_1 L_{k_1} C(v,w) \int_a^t \int_a^u \phi(\varsigma)d\varsigma du \\ &\quad + \lambda_2 L_{k_2} C(v,w) \int_a^t \int_a^b \phi(\varsigma)d\varsigma du \\ &\leq L_f C(v,w) V\phi(t) + \lambda_1 L_{k_1} C(v,w) V^2 \phi(t) \\ &\quad + \lambda_2 L_{k_2} C(v,w) V^2 \phi(t) \\ &= C(v,w)\phi(t)[L_f V + \lambda_1 L_{k_1} V^2 + \lambda_2 L_{k_2} V^2] \end{aligned}$$

that is,

$$d(\Gamma v, \Gamma w) \leq C(v,w)\phi(t)[L_f V + \lambda_1 L_{k_1} V^2 + \lambda_2 L_{k_2} V^2].$$

We conclude that

$$d(\Gamma v, \Gamma w) \leq [L_f V + \lambda_1 L_{k_1} V^2 + \lambda_2 L_{k_2} V^2] d(v,w),$$

for any $v, w \in B$. Since by assumption, we have

$$[L_f V + \lambda_1 L_{k_1} V^2 + \lambda_2 L_{k_2} V^2] < 1,$$

then Γ is strictly contractive. Let w_0 be an arbitrary element in B. Then there is a constant $C \in (0, \infty)$ such that

$$\begin{aligned} |(\Gamma w_0)(t) - w_0(t)| &= \left| \alpha + \int_a^t f(\varsigma, w_0(\varsigma)) d\varsigma + \lambda_1 \int_a^t \int_a^u k_1(t, \varsigma, w_0(\varsigma)) d\varsigma du \right. \\ &\quad \left. + \lambda_2 \int_a^t \int_a^b k_2(t, \varsigma, w_0 v(\varsigma)) d\varsigma du - w_0(t) \right| \\ &\leq C\phi(t), \text{ for all } t \in I. \end{aligned}$$

We observe that

$$d(\Gamma w_0, w_0) < \infty.$$

Then by Theorem 2.1, there is a continuous function $x_0 : I \to \mathbb{R}$ such that $(\Gamma^n w_0)$ converges to x_0 and $\Gamma x_0 = x_0$, that is x_0 is a solution to the equation (VFIDE). Because of d is a metric $x_0 : I \to \mathbb{R}$ is the unique continuous function which is satisfies (3.6). By assumption (4.11), for $\forall t \in I$, we get

$$-\phi(t) \leq x'(t) - f(t, x(t)) - \lambda_1 \int_a^t k_1(t, s, x(s)) ds - \lambda_2 \int_a^b k_2(t, s, x(s)) ds \leq \phi(t),$$

If each term of the above inequality is integrated, then

$$\begin{aligned} &\left| x(t) - \alpha - \int_a^t f(u, x(u)) du - \lambda_1 \int_a^t \int_a^u k_1(u, s, x(s)) ds du \right. \\ &\left. - \lambda_2 \int_a^t \int_a^b k_2(u, s, x(s)) ds du \right| \\ \leq &\int_a^t \phi(\varsigma) d\varsigma. \end{aligned}$$

From (4.10) and (4.12), for all $t \in I$, we have $|x(t) - (\Gamma x)(t)| \leq \int_a^t \phi(\varsigma) d\varsigma \leq V\phi(t)$, which implies that

$$d(x, \Gamma x) \leq V\phi(t). \tag{4.13}$$

By using Theorem 2.1 (c) and (4.13), we conclude that

$$\begin{aligned} d(x, x_0) &\leq \frac{1}{1 - [L_f V + \lambda_1 L_{k_1} V^2 + \lambda_2 L_{k_2} V^2]} d(x, \Gamma x) \\ &\leq \frac{V}{1 - [L_f V + \lambda_1 L_{k_1} V^2 + \lambda_2 L_{k_2} V^2]} \phi(t) . \end{aligned}$$

□

REFERENCES

[1] Akkouchi, M., *Hyers-Ulam-Rassias stability of nonlinear Volterra integral equations via a fixed point approach*. Acta Univ. Apulensis Math. Inform., **26** (2011), 257–266

[2] Alsina, C. and Ger, R., *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl., **4** (1998), 373–380

[3] Brzdek, J., *On the quotient stability of a family of functional equations*, Nonlinear Anal., **71** (2009), 4396-–4404

[4] Czerwik, S., *Functional Equations and Inequalities in Several Variables*, World Scientific (2002)

[5] Diaz, J. B. and Margolis, B., *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305–309

[6] Forti, G.-L., *Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations*, J. Math. Anal. Appl., **295** (2004), 127-–133

[7] Gachpazan, M. and Baghani, O., *Hyers-Ulam stability of Volterra integral equation*, J. Nonlinear Anal. Appl., **1** (2010), 19–25

[8] Gachpazan, M. and Baghani, O., *Hyers-Ulam stability of nonlinear integral equation*, Fixed Point Theory Appl., **2010** (2010), 1–6

[9] Hyers, D. H., *On the stability of the linear functional equation*, Proceedings of the National Academy of Sciences of the United States of America, **27** (1941), 222–224

[10] Hyers, D. H., Isac, G. and Rassias, Th. M., *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, (1998)

[11] Jung, S. M., *A fixed point approach to the stability of differential equations* $y' = F(x, y)$, Bull. Malays. Math. Sci. Soc., **33** (2010), 47–56

[12] Jung, S. M., *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, NY, USA, 2011

[13] Jung, S. M., *Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients*, J. Math. Anal. Appl., **320** (2006), 549-–561

[14] Jung, S. M.,*Hyers–Ulam stability of linear differential equation of the first order (III)*, J. Math. Anal. Appl., **311**, (2005), 139-–146

[15] Jung, S. M.,*Hyers–Ulam stability of linear differential equations of first order (II)*, Appl. Math. Lett., **19** (2006), 854-–858

[16] Miura, T., Miyajima, S. and Takahasi, S. E., *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl., **286** (2003), 136-–146

[17] Miura, T., Miyajima, S. and Takahasi, S. E., *Hyers-Ulam stability of linear differential operator with constant coefficients*, Math. Nachr., **258** (2003), 90-–96

[18] Popa, D., *Hyers-Ulam-Rassias stability of a linear recurrence*, J. Math. Anal. Appl., **309** (2005), 591—597

[19] Rassias, T. M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300

[20] Rus, I. A., *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math, **26** (2010), 103–107

[21] Rus, I. A., *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10** (2009), 305–320

[22] Rus, I. A., *Ulam stabilities of ordinary differential equations*, Stud. Univ. Babes-Bolyai Math, **54** (2009), 125–134

[23] Sevgina, S. and Sevlib, H., *Stability of a nonlinear Volterra integro-differential equation via a fixed point approach*, Journal of Nonlinear Sciences and Applications, **9** (2016), 200–207

[24] Takahasi, S. E., Miura, T., Miyajima, S., *On the Hyers-Ulam stability of the Banach space-valued differential equation* $y' = \lambda y$, Bull. Korean Math. Soc., **39** (2002), 309-–315

[25] Ulam, S. M., *A Collection of Mathematical Problems*, Interscience, New York, NY, USA, 1960

DEPARTMENT OF MATHEMATICS
AKSARAY UNIVERSITY
AKSARAY,TURKEY
Email address: yunus_atalan@hotmail.com

DEPARTMENT OF MATHEMATICAL ENGINEERING
YILDIZ TECHNICAL UNIVERSITY
DAVUTPASA CAMPUS, 34220, ISTANBUL, TURKEY
Email address: vkkaya@yahoo.com