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Dynamic response of a pre-stressed bi-layered plate-strip subjected to an arbitrary inclined time-harmonic force

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ABSTRACT. Within the scope of the piecewise homogeneous body model with utilizing of the three dimensional linearized theory of elastic waves in initially stressed bodies the dynamical stress field problem in a bi-layered plate-strip with initial stress under the action of an arbitrary inclined time-harmonic force resting on a rigid foundation is investigated. The concrete materials such as a pair of Aluminum and Steel are selected. It is assumed that there exists a complete contact interaction between the layers. The mathematical modeling of the problem under consideration is carved out, and the governing system of the partial differential equations of motion is approximately solved by employing Finite Element Method. The numerical results related to the influence of certain parameters on the dynamic response of the plate-strip are presented.

1. INTRODUCTION

Recently, multi-layered materials have considerably under dense study by a great number of researchers since they are extensively encountered in daily life. Especially, the investigations on the wave propagations in elastic bodies have been studied. For example, the case such that propagation of shock waves in solids, dynamic stress concentrations, and wave propagations in inhomogeneous and anisotropic materials is encouraged in many problems. To investigate the corresponding problems, many systematic theories have been developed due the fact that different situations require different approaches.

Note that the influence of the initial stresses cannot be investigated within the classical linear theory of elasticity due to the fact that the influence is non-linear. According to the well-known mechanical consideration, when the amplitudes of the deformations subjected to the pre-stressed body are significantly smaller than the magnitudes of the initial deformations, the corresponding investigations can be made within the scope of the three-dimensional linearized theory of elastic waves in initially stressed bodies (TLTEWISB). The mentioned theory has been developed within the scope of elastodynamics. For more details, the monographs [6, 7] can be investigated. In addition, it is assumed that the pre-stressed state (or initial stress-state) is exactly homogeneous and static in this theory.

According to the fundamental principle of TLTEWISB and its other version, certain interesting problems have been investigated. Emiroglu et. all develop an approach to investigating the Lamb problem for a half-space covered with a layer subject to a normal point force changing harmonically with time [5]. Akbarov and Guler investigate the stress field in a half-plane covered by the pre-stretched layer under the action of the inclined linearly located time-harmonic forces [1]. Cilli and Ozturk present an analysis for the propagation of torsional waves in multilayered compound cylinders [3]. Kepceler investigates the torsional wave propagation in the bi-material compounded cylinder with an imperfect interface in the absence of initial stresses [9]. Zamanov and Agasiyev study the problem

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on the propagation of Lamb waves in a three-layer plate made from compressible materials with finite initial deformations [12]. Wen-tao et al. consider influence of identical applied initial pressures on the radial surfaces of a hollow cylinder composed of materials with first power hypo-elastic constitutive model [11]. Ipek investigates the influence of the interface imperfect bonding on the flexural wave dispersion in the bi-layered hollow circular cylinder [8]. In addition, the forced vibration of a pre-stressed bi-layered plate-strip with finite length under a time-harmonic force is widely studied under the different assumptions by employing the finite element method (FEM) in [2,4].

It is evident from the numerical results presented in [4] that the influence of certain problem parameter on the frequency response of a pre-stressed bi-layered plate-strip under the action of an arbitrary inclined time-harmonic force resting on a rigid foundation has not been investigated so far, and there is a lack of mathematical modeling to present fundamental insights for characterizing frequency response of the bi-layered plate-strip for the concrete materials. To address the issue, the mathematical modeling under consideration is constituted within the scope of the piecewise homogeneous body model with utilizing of TLTEWISB, and it is numerically solved by employing the FEM. Note that the numerical investigations presented in this paper can be also considered as expansion of that in [4].

2. Statement of problem

Consider a bi-layered plate-strip with length $2a$ and thickness h $(= h_1 + h_2)$, where h_1 (h_2) denotes the thickness of the upper (lower) layer. For convenience, two homogeneous transversely isotropic materials are selected. The Cartesian coordinates denoted by x_i are assumed to be associated with the initial state and in the natural state coincide with the Lagrange coordinates.

The considered body is being under the influence of an arbitrary inclined (in being at both the normal and tangential directions) time-harmonic lineal load applied to the free surface as shown in Fig. 1 and resting on a rigid foundation. But, note that the length of the plate in the direction of Ox_3 axis is infinite, and it is assumed that the time-harmonic force extends to infinity in this direction which is inclined to the $x_2 = 0$ plane. According to all the foregoing assumptions, the plane deformation state arises in the Ox_1x_2 plane. Consequently, all the numerical investigations for the present case are presented in the Ox_1x_2 plane. The corresponding quantities related to the upper and lower layers are denoted by the superscripts "(1)" and "(2)" respectively, and the subscript "0" to the initial state. According to Fig. 1, the considered plate-strip occupies the domain $B = B_1 \cup B_2$, where

$$\begin{aligned} B_1 &= \{(x_1, x_2) : -a \leqslant x_1 \leqslant a,\ -h_1 \leqslant x_2 \leqslant 0\}, \\ B_2 &= \{(x_1, x_2) : -a \leqslant x_1 \leqslant a,\ -h \leqslant x_2 \leqslant -h_1\}. \end{aligned} \tag{2.1}$$

Before compounding each layer with one another and with rigid foundation, each layer is separately subjected to uniaxial uniformly distributed normal mechanical force. These initial stresses are determined by utilizing the linear theory of elasticity as

$$\sigma_{11}{}^{0,(m)} = q^{(m)} \text{ and } \sigma_{ij}{}^{0,(m)} = 0 \text{ for all } ij \neq 11, \tag{2.2}$$

where $m = 1, 2$ and $q^{(m)}$ is the known constant for each layer.

According to the plane-strain state within the scope of TLTEWISB based on the piecewise homogeneous body model, the general forms of the governing field equations under consideration are expressed as follows [6,7]:

$$\sigma_{ij,j}{}^{(m)} + \left(\sigma_{kj}{}^{0,(m)} u_{i,k}{}^{(m)}\right)_{,j} = \rho^{(m)} \ddot{u}_i^{(m)}, \tag{2.3}$$

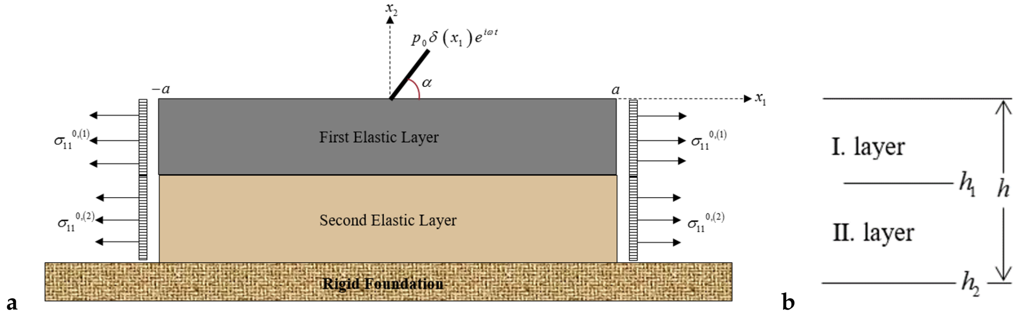


FIGURE 1. **a** Geometry of problem. **b** Scheme of heights of layers.

where $i, j, k = 1, 2$, $\rho^{(m)}$ is the mass density of the mth layer, ${u_i}^{(m)}$ are the mechanical displacements of the plate in the direction of x_i, and ${\sigma_{ij}}^{(m)}$ are the components of the stress tensor. The dot over the quantities is time differentiation and the subscripts followed by the comma indicate the space-coordinate differentiation. Here and below, the repeated index in the subscript is summed with respect to that index.

The mechanical and geometrical relations under consideration can be written as

$$\sigma_{ij}^{(m)} = \lambda^{(m)}\varepsilon_{\ell\ell}^{(m)}\delta_{ij} + 2\mu^{(m)}\varepsilon_{ij}^{(m)}, \quad \varepsilon_{ij}^{(m)} = \frac{1}{2}\left(u_{i,j}^{(m)} + u_{j,i}^{(m)}\right), \tag{2.4}$$

where $\lambda^{(m)}$ and $\mu^{(m)}$ are the Lamé constants, δ_{ij} is the Kronecker delta, and ε_{ij} are the components of the strain tensor.

Now the boundary-contact conditions for the present problem are investigated. For the analysis presented here, the case where there exists complete contact interaction between the layers is considered. Hence, the contact conditions

$$\sigma_{i2}^{(1)}\Big|_{x_2=-h_1} = \sigma_{i2}^{(2)}\Big|_{x_2=-h_1} \quad \text{and} \quad u_i^{(1)}\Big|_{x_2=-h_1} = u_i^{(2)}\Big|_{x_2=-h_1} \tag{2.5}$$

are given.

At the same time, on the surfaces of the plate-strip, the boundary-contact conditions

$$\sigma_{21}^{(1)}\Big|_{x_2=0} = -p_0\delta(x_1)\,\mathrm{e}^{\mathrm{i}\omega t}\cos\alpha, \; \sigma_{22}^{(1)}\Big|_{x_2=0} = -p_0\delta(x_1)\,\mathrm{e}^{\mathrm{i}\omega t}\sin\alpha, \tag{2.6}$$

$$\left(\sigma_0^{(m)}u_{j,1}^{(m)} + \sigma_{1j}^{(m)}\right)\Big|_{x_1=\pm a} = 0, \tag{2.7}$$

are written.

In addition, since the considered body is resting on a rigid foundation, the contact conditions

$$u_j^{(2)}\Big|_{x_2=-h} = 0 \tag{2.8}$$

can be given.

This completes presentation of the governing field equations and the corresponding boundary-contact conditions for the plate-strip shown in Fig. 1.

3. Solution procedure

An analytical solution of the problem cannot be obtained by the fact that the equations of motion and boundary-contact conditions are quite complex. Hence, the solution to this problem is obtained by employing the FEM.

First of all, the external force applied to the plate-strip is assumed to be time-harmonic, with frequency ω, as $p_o\delta(x_1)e^{i\omega t}$. Thus, all the corresponding dependent variables can be written in the form

$$\{\sigma_{ij},u_i,\varepsilon_{ij}\}^{(m)}(x_1,x_2,t)=\{\bar{\sigma}_{ij},\bar{u}_i,\bar{\varepsilon}_{ij}\}^{(m)}(x_1,x_2)e^{i\omega t}, \tag{3.9}$$

where the superposed bar represents the amplitude of the corresponding quantities. The dimensionless coordinate system is also introduced as

$$\hat{x}_1=\frac{x_1}{h}\ \text{ and }\ \hat{x}_2=\frac{x_2}{h}. \tag{3.10}$$

Substituting the expression in (3.9) into the foregoing equations and conditions after the coordinate transformation in (3.10), the same equations and boundary-contact conditions are directly obtained for the amplitude of the sought values by replacing the terms $\partial^2 u_j{}^{(m)}/\partial t^2$ and $p_o\delta(x_1)e^{i\omega t}$ with $-\omega^2 u_j{}^{(m)}$ and $p_o\delta(x_1)$, respectively.

To obtain FEM modeling of the last boundary-contact problem, the functional

$$\begin{aligned} J\left(\mathbf{u}^{(\mathbf{m})}\right)&=\tfrac{1}{2}\int\limits_B\left[T_{ij}{}^{(m)}u_{j,i}{}^{(m)}-\left(\Omega^{(m)}\right)^2\left\{\left(u_1{}^{(m)}\right)^2+\left(u_2{}^{(m)}\right)^2\right\}\right]dB\\ &+\int\limits_{-a/h}^{a/h}\tfrac{p_o\delta(x_1)}{\mu^{(1)}}u_2{}^{(1)}\Big|_{x_2=0}dx_1 \end{aligned} \tag{3.11}$$

is proposed, where

$$T_{ij}{}^{(m)}=\sigma_{ij}{}^{(m)}+\eta^{(m)}u_{j,n}, \tag{3.12}$$

$$\Omega^{(m)}=\omega h\sqrt{\frac{\rho^{(m)}}{\mu^{(m)}}}\ \text{ and }\ \eta^{(m)}=\frac{q^{(m)}}{\mu^{(m)}}. \tag{3.13}$$

In Eq. (3.13), $\Omega^{(m)}$ denotes the dimensionless frequency of the plate-strip, and $\eta^{(m)}$ is the initial stress parameter of the mth layer.

The validity of the functional presented in (3.11) can be shown as follows: Considering the notations in (3.12), using the famous Gauss's theorem and computing the statements $\delta J\left(\mathbf{u}^{(\mathbf{m})}\right)=0$, which is the first variation of the functional in (3.11), the equations of motion and the corresponding boundary-contact conditions under consideration can be obtained. So, the desired proof is completed.

To do the FEM modeling of the considered problem, the virtual work principle and the standard Rayleigh-Ritz method are considered [13]. According to the method, the domain B is divided into a finite number of sub-domains whose structures are nine-node smooth rectangular elements. The number of these finite elements is selected from the requirements that the boundary conditions must be satisfied with very high accuracy and numerical results obtained must converge. After certain mathematical arrangements, a system of algebraic equations

$$\left(\mathrm{K}\text{ - }\omega^2\mathrm{M}\right)\tilde{\mathbf{x}}=\mathbf{F} \tag{3.14}$$

is attained, where, $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix, $\tilde{\mathbf{x}}$ is the column vector of un-known nodal displacements, and $\mathbf{F}$ is the force vector. To reduce the size of the present paper the explicit forms of the above-stated matrices and vectors are not given here. Note that their explicit forms are directly derived from Eq. (3.11) by using the considered procedure. So, with the above-stated the FEM modeling of the problem being considered is exhausted.

4. NUMERICAL RESULTS

Introduce the notation $e = E^{(1)}/E^{(2)}$, where $E^{(m)}$ is the Young modulus of mth layer. In this study, certain concrete materials are selected, and their mechanical properties are given in Table 1. All the numerical results are presented under the case where $h/2a = 0.2$, $h_1 = h_2$, $\Omega = \Omega^{(1)} = \Omega^{(2)} = 0$ and $\eta = \eta^{(1)} = \eta^{(2)} = 0$ unless otherwise.The investigated figures are presented at the interface plane between the layers and on the bottom surface between the plate-strip and rigid foundation, and the letters a and b in figures show the graphs plotted at the points $(-h_1/h, 0)$ and $(-1, 0)$, respectively.

TABLE 1. Mechanical constants of selected materials

	Abbreviation	$E \times (GPa)$	ν	$\rho \times (kgm^{-3})$
Aluminum	Al	70	0.35	2712
Steel	St	205	0.29	7860
Nickel	Ni	210	0.31	8890
Titanium	Ti	110	0.33	4540

In order to prove the validity of the used programs, the case where $e = 1$, $\nu^{(1)} = \nu^{(2)} = 0.33$ and $\alpha = \pi/2$ is considered. The problem which for the plate with infinite length was considered by Uflyand [10], and note that the mentioned problem was solved by employing Fourier integral transformation method. It should be noted that, as $h/2a \to 0$, the geometry of the considered plate-strip begins to resemble that in [10]. In this case, the numerical results given by the present FEM algorithm must converge to the corresponding ones given in [10]. Fig. 2 prove this prediction. Consequently, the validity and trustiness of the algorithm and programs has been shown.

The one of the main goals of the paper is to present the consideration of the frequency response of the bi-layered plate-strip, especially the influence of the initial stress parameter η on this response. The each graph in Fig. 3 displays the dependence between the stress $\sigma_{22}h/p_0$ and Ω for various values of η in the case where $\alpha = \pi/2$. The used materials are selected as a pair of Al+St. The numerical results indicate that there exist certain locations where the parametric resonance of $\sigma_{22}h/p_0$ occurs for certain values of the initial stress parameter η. It follows from the investigations of the graphs that there exist locations where $\sigma_{22}h/p_0$ reach the extrema for the certain values of Ω. As known, these values are called as the "resonance" values and denoted by Ω_*. An increase in the values of the parameter η causes to decrease the values of these parametric resonance. It is concluded that the influence of the initial stress parameter η on the frequency response of the stress $\sigma_{22}h/p_0$ is considerable not only in the quantitative sense, but also in the qualitative sense.

Now, the influence of the initial stress parameter η on the dependence between $\sigma_{22}h/p_0$ and the angle α is now considered for a pair of Ti+Ni or Ni+Ti at the points $(0, -1/2)$ and $(0, -1)$. Note that the case where the plate-strip is subjected to only initial stretching was investigated in [4]. The comparison of the influence of the initial stretching and compressing on the dynamic behavior of the plate-strip is presented here. The comparison of the numerical results in Figs. 4 and 5 and those given in [4] indicates that the influence of the initial stretching parameter on the dynamic response of the stress $\sigma_{22}h/p_0$ exhibits a behavior unlike that for the initial compressing parameter. The absolute values of the stress $\sigma_{22}h/p_0$ decrease with the initial compressing parameter η. The stress $\sigma_{22}h/p_0$ depend linearly on the initial stress (stretching or compressing) parameter. It can be shown that the influence of the choice of materials of plates and the values of $\sigma_{22}h/p_0$ increase with the angle α. It follows from the graphs in Figs. 4 and 5 that the absolute values of the normal stress $\sigma_{22}h/p_0$ decreases with the selection of the materials. It means that the influence of the initial compressing parameter on dynamic behavior of the stress $\sigma_{22}h/p_0$ decreases with the selection of a pair of Ti+Ni instead of a pair of Ni+Ti. Consequently, the mentioned influence damps with increasing the ratio of e.

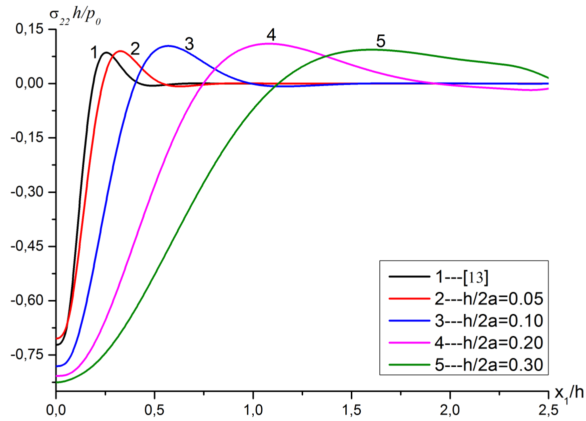
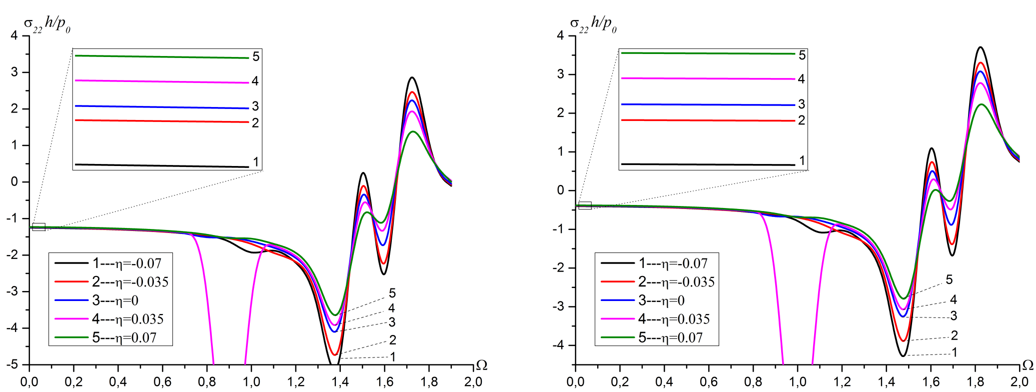


FIGURE 2. The variation of $\sigma_{22}h/p_0$ versus the line x_1/h for various thickness ratios under the same assumption in [10]

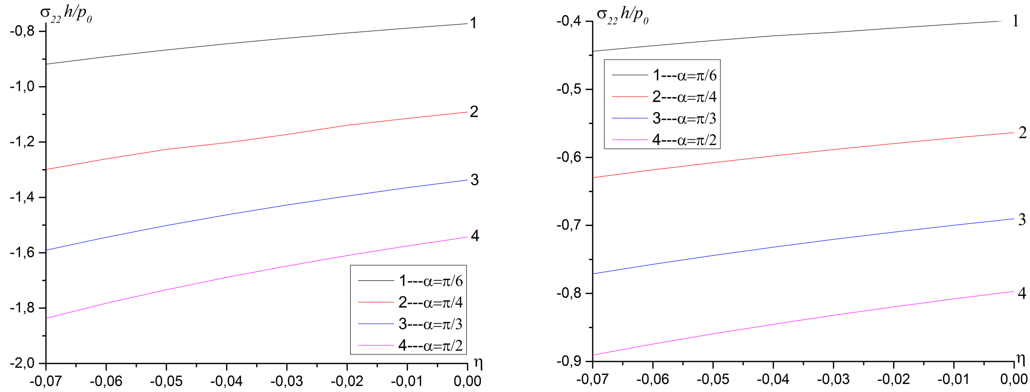
5. CONCLUSIONS

In the present paper, the forced vibration of the pre-stressed bi-layered plate-strip subjected to the action of the arbitrary inclined time-harmonic force resting on rigid foundation has been investigated within the scope of the piecewise homogeneous body model with utilizing of the three dimensional linearized theory of elastic waves in initially stressed bodies (TLTEWISB). The mathematical modeling of the considered problem is carved out and is numerically solved by employing Finite Element Method (FEM). The numerical results illustrating the influence of certain parameter on the dynamic behavior of the considered plate-strip are presented and discussed.



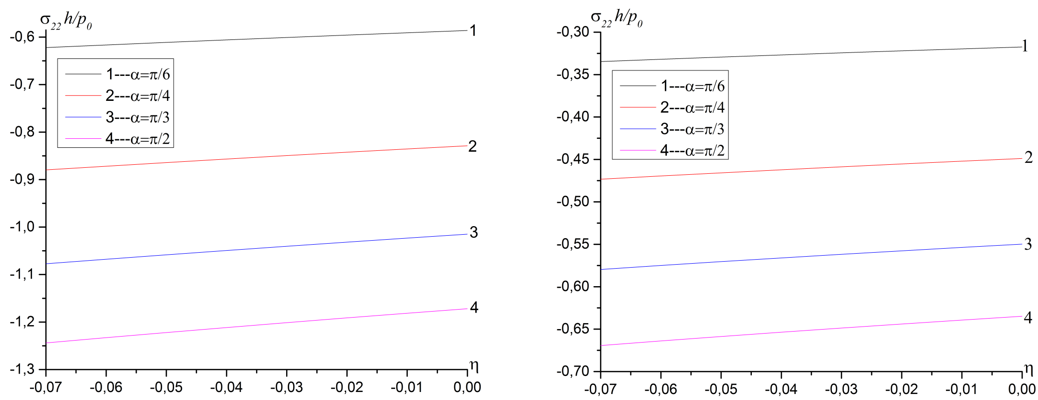
a b

Figure 3. The influence of the angle α on the dependence between $\sigma_{22}h/p_0$ and η for a pair of Ni+Ti; **a** at the interface, **b** on the bottom surface



a b

Figure 4. The influence of the angle α on the dependence between $\sigma_{22}h/p_0$ and η for a pair of Ni+Ti; **a** at the interface, **b** on the bottom surface



a b

Figure 5. The influence of the angle α on the dependence between $\sigma_{22}h/p_0$ and η for a pair of Ti+Ni; **a** at the interface, **b** on the bottom surface

According to all the numerical investigations, certain inferences of the important results can be drawn as follows:

i An increase in the values of the initial stress parameter η causes to vanish the resonance mode of the normal stress $\sigma_{22}h/p_0$;
ii the initial stretching parameter prevents the resonance of $\sigma_{22}h/p_0$, but the compressing parameter exceed the this resonance mode;
iii the influence of the initial compressing parameter on dynamic behavior of the normal stress $\sigma_{22}h/p_0$ decreases with the selection of the plates,
iv there exist certain locations where the parametric resonance of $\sigma_{22}h/p_0$ occurs for certain values of the initial stress parameter η.

The numerical results listed above have been presented for under two different cases (for example a pair of Al+St), but note that they also have a general validity in a qualitative sense. Moreover, these numerical results are encountered daily in the engineering practice under an impact treatment of metals which lie on the others.

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