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A note on $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of an infinite product

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ABSTRACT. In this paper, we define $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of an infinite product by Cauchy conditions and prove the relation between these two notions.

1. INTRODUCTION

In [5], Fast introduced the definition of statistical convergence using the natural density of a set. For a subset K of natural numbers $\mathbb{N}$, it is defined by $\delta(K) = \lim_n \frac{1}{n}|K_n|$, where $K_n = \{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. It is a practical tool to study the problems related to convergence of numerical sequences by means of the concept of density. A sequence (x_k) of reel numbers is said to be statistical convergent to x provided that for every $\varepsilon > 0$ the natural density of the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}$ is equal to zero. Thus, this definition is equivalent to meet the condition $(C_1\chi_{K_\varepsilon})_n \to 0$ as $n \to \infty$, where C_1 is the Cesaro mean of order one and χ_{K_ε} is the characteristic function of K_ε. Also by using a nonnegative regular summability matrix instead of C_1 statistical convergence was generalized by some authors. In [15], Schoenberg gave some basic properties of statistical convergence and also studied the concept as a summability method. Later on it was further investigated and linked with the summability theory by Connor [2], Freedman and Sember [6], Fridy [7] and many others.

As a generalization of statistical convergence, ideal convergence was defined by the aid of ideal $\mathcal{I}$ which is a family of subsets of natural numbers $\mathbb{N}$ and studied by several authors. Kostyrko et al. [9] studied ideal convergence in metric spaces. In connection with definition of $\mathcal{I}$-convergence, the authors also introduced another type of convergence which is closely related to $\mathcal{I}$-convergence and called as $\mathcal{I}^*$-convergence. They prove that $\mathcal{I}^*$-convergence implies $\mathcal{I}$-convergence and for the converse implication ideal has an additional property called (AP) condition. Later, $\mathcal{I}$ and $\mathcal{I}^*$-convergence were extended to topological spaces and studied by Lahiri and Das [10].

In [4], $\mathcal{I}$-Cauchy condition was introduced for a sequence in any metric space and the author proved that $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy condition are equivalent in complete metric spaces. Later, Nabiev et al. [11] introduced $\mathcal{I}^*$-Cauchy sequences in a linear metric space. They prove that if a sequence is $\mathcal{I}$-convergent, then it satisfies $\mathcal{I}$-Cauchy condition. Further, they showed that if a sequence is $\mathcal{I}^*$-Cauchy, then it is $\mathcal{I}$-Cauchy and the converse is true if the ideal $\mathcal{I}$ satisfies the condition (AP).

For more papers about ideal convergence of sequences, one can see [3]-[18].

Before continuing, we recall some concepts used throughout this paper.

Let X be a nonempty set. A class $\mathcal{I} \subseteq 2^X$ of subsets of X is said to be an ideal in X provided that the following conditions hold:

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(1) $\emptyset \in \mathcal{I}$,
(2) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$,
(3) $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ in X is called proper (or non-trivial) if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$.
A proper ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Definition 1.1. [1] An ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is a P-ideal if for every sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ there is a set $A_\infty \in \mathcal{I}$ such that $A_n \backslash A_\infty$ is finite for every n.

Definition 1.2. [9] An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that the symmetric difference $A_n \triangle B_n$ is finite for every $n \in \mathbb{N}$ and $B = \cup_{n=1}^{\infty} B_n \in \mathcal{I}$.

We need the following lemma to prove one of our main results. That is, equivalently we will use a P-ideal instead of the satisfaction of the condition (AP) .

Lemma 1.1. [1] *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. The following conditions are equivalent:*

(1) $\mathcal{I}$ is a P-ideal.
(2) $\mathcal{I}$ satisfies the condition (AP).

Let X be a nonempty set. A class $\mathcal{F} \subseteq 2^X$ of subsets of X is said to be a filter in X provided that the following conditions hold:

(1) $\emptyset \notin \mathcal{F}$,
(2) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$,
(3) $A \in \mathcal{F}, A \subset B$ imply $B \in \mathcal{F}$.

If $\mathcal{I}$ is a proper ideal in X, then

$$\mathcal{F}(\mathcal{I}) = \{X \backslash A : A \in \mathcal{I}\}$$

is a filter in X.

Definition 1.3. [9] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to a real number ξ $(\mathcal{I} - \lim x_n = \xi)$ if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

Definition 1.4. [9] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}^*$-convergent to a real number ξ $(\mathcal{I}^* - \lim x_n = \xi)$ if there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < ...\}$ such that

$$\lim_{k\to\infty} x_{m_k} = \xi \quad \left(\lim_{n\to\infty, n\in M} x_n = \xi\right).$$

Definition 1.5. [11] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-Cauchy if for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

Definition 1.6. [11] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}^*$-Cauchy if there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < ...\}$ such that the subsequence (x_{m_k}) is an ordinary Cauchy sequence; that is,

$$\lim_{k,p\to\infty} |x_{m_k} - x_{m_p}| = 0.$$

2. Main results

In this section, we define $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of an infinite product by the Cauchy conditions. Afterwards, $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$.

Definition 2.7. An infinite product $\prod_{n=1}^{\infty} u_n$ is said to be $\mathcal{I}$-convergent if it satisfies $\mathcal{I}$-Cauchy condition; that is, for every $\varepsilon > 0$ there exist $N = N(\varepsilon) \in \mathbb{N}$ and $A_\varepsilon \in \mathcal{I}$ such that

$$\left| \prod_{i \in \{q+1,\dots,p\} \setminus A_\varepsilon} u_i - 1 \right| < \varepsilon$$

for all $p > q > N$.

Definition 2.8. An infinite product $\prod_{n=1}^{\infty} u_n$ is said to be $\mathcal{I}^*$-convergent if it satisfies $\mathcal{I}^*$-Cauchy condition; that is, there exits $A \in \mathcal{I}$ such that for every $\varepsilon > 0$ there is a $N = N(\varepsilon) \in \mathbb{N} \setminus A$ satisfying

$$\left| \prod_{i \in \{q+1,\dots,p\} \setminus A} u_i - 1 \right| < \varepsilon$$

for all $p > q > N$.

Theorem 2.1. *If an infinite product $\prod_{n=1}^{\infty} u_n$ satisfies $\mathcal{I}^*$-Cauchy condition, then it also satisfies $\mathcal{I}$-Cauchy condition.*

Proof. Assume that $\mathcal{I}^*$-Cauchy condition holds for $\prod_{n=1}^{\infty} u_n$. Then there exits $A \in \mathcal{I}$ such that $\prod\limits_{n \in \mathbb{N} \setminus A} u_n$ satisfies Cauchy condition. Given $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N} \setminus A$ such that

$$\left| \prod_{i \in \{q+1,\dots,q+j\} \setminus A} u_i - 1 \right| < \varepsilon$$

for all $q > N$ and $j \geq 1$. Let $A_\varepsilon = A \cup \{1, 2, \dots, N\}$. Hence we have $A_\varepsilon \in \mathcal{I}$ and

$$\left| \prod_{i \in \{q+1,\dots,q+j\} \setminus A_\varepsilon} u_i - 1 \right| < \varepsilon$$

for all $q > N$ and $j \geq 1$ which mean that $\prod_{n=1}^{\infty} u_n$ satisfies $\mathcal{I}$-Cauchy condition. $\square$

Theorem 2.2. *Let $\mathcal{I}$ be a P-ideal. If an infinite product $\prod_{n=1}^{\infty} u_n$ satisfies $\mathcal{I}$-Cauchy condition, then it satisfies $\mathcal{I}^*$-Cauchy condition.*

Proof. Let $\mathcal{I}$ be a P-ideal and $\prod_{n=1}^{\infty} u_n$ satisfy $\mathcal{I}$-Cauchy condition. There exist $m_j \in \mathbb{N}$ and $A_j \in \mathcal{I}$ such that

$$\left| \prod_{i \in \{l+1,\dots,k\} \setminus A_j} u_i - 1 \right| < \frac{1}{j}$$

for all $j \in \mathbb{N}$ and $k > l > m_j$. Since $\mathcal{I}$ is a P-ideal, there exists $A_\infty \in \mathcal{I}$ such that $A_j \setminus A_\infty$ is finite. We can write $A_j \setminus A_\infty \subseteq \{1, 2, \dots, p_j\}$, where $p_j \in A_j \setminus A_\infty$. If $k, l \in A_j$, then $k, l \in A_\infty$ for $k > l > p_j$. Hence we have

$$\left| \prod_{i \in \{l+1,\dots,k\} \setminus A_\infty} u_i - 1 \right| < \frac{1}{j}$$

for $k > l > \max\{m_j, p_j\}$. This means $\prod_{n\in\mathbb{N}\setminus A_\infty} u_n$ satisfies Cauchy condition, where $A_\infty \in \mathcal{I}$. □

Theorem 2.3. *If $\prod_{n=1}^{\infty} u_n$ is $\mathcal{I}$-convergent, then (u_n) is $\mathcal{I}$-convergent to* 1.

Proof. Since $\prod_{n=1}^{\infty} u_n$ satisfies $\mathcal{I}$-Cauchy condition, given $\varepsilon > 0$ there are $N \in \mathbb{N}$ and $A_\varepsilon \in \mathcal{I}$ such that $\left| \prod_{i\in\{q+1,\dots,p\}\setminus A_\varepsilon} u_i - 1 \right| < \varepsilon$ for $p > q > N$. Let $p = q+1$ and $p \notin A_\varepsilon$, where $q > N$. Then we have $|u_p - 1| < \varepsilon$. Therefore $p \notin U = \{n \in \mathbb{N} : |u_n - 1| \geq \varepsilon\}$. Obviously, $U \subseteq \{1,2,...,N\} \cup A_\varepsilon$. Since $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$, $\{1,2,...,N\} \in \mathcal{I}$ and so $\{1,2,...,N\} \cup A_\varepsilon \in \mathcal{I}$ which implies that U belongs to $\mathcal{I}$. Consequently, (u_n) is $\mathcal{I}$-convergent to 1. □

Now, we give some examples which show that the converse of the last theorem is not valid.

Example 2.1. Let $\mathcal{I}_\delta$ be the non-trivial admissible ideal consisting of all $A \subset \mathbb{N}$ with $\delta(A) = 0$. Then $\mathcal{I}_\delta$-convergence coincides with the statistical convergence. Define the sequence (u_n) as follows:

$$u_n = \begin{cases} 2, & \text{if } n \in P \\ 1, & \text{otherwise} \end{cases}$$

where P is the set of all prime numbers. From the prime number theorem (see [12]; p.217), we obtain that $\delta(\mathbb{N}\setminus P) = 1$. Also, since

$$\lim_{n\to\infty, n\in\mathbb{N}\setminus P} u_n = 1$$

holds, Lemma 1.1 in [14] implies that $\mathcal{I}_\delta\text{-}\lim u_n = 1$.

On the other hand, choosing $\varepsilon \leq 1$, for every $n \in \mathbb{N}$ and $A \in \mathcal{I}_\delta$ we can find a prime number $p = q+1$, where $p > q > n$. Hence, we have $|u_p - 1| \geq \varepsilon$ which means that $\prod_{n=1}^{\infty} u_n$ is not $\mathcal{I}_\delta$-convergent.

Example 2.2. Let define a sequence (u_n) as follows:

$$u_n = \begin{cases} x, & \text{if } n = k^2 \text{ for some } k \in \mathbb{N} \\ 1, & \text{if } n \neq k^2 \text{ for all } k \in \mathbb{N} \end{cases}$$

where x is a real number different from 1. Then this sequence is $\mathcal{I}_\delta$-convergent to 1. However, $\prod_{n=1}^{\infty} u_n$ is not $\mathcal{I}_\delta$-convergent. Indeed, let $\varepsilon_0 = |x-1|$. Since for every $n \in \mathbb{N}$ there exits $k \in \mathbb{N}$ such that $p = k^2 > q = k^2 - 1 > n$, for every $A \in \mathcal{I}$ we obtain

$$\left| \prod_{i\in\{q+1,\dots,p\}\setminus A} u_i - 1 \right| = |u_p - 1| = |x-1| \geq \varepsilon_0.$$

Example 2.3. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. The infinite product $\prod_{n=1}^{\infty} \frac{n}{n+1}$ is $\mathcal{I}$-convergent although it is not convergent in the usual sense. In fact, given any $\varepsilon > 0$, there exits $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{n_\varepsilon} < \varepsilon$. Also, we have $A_\varepsilon = \{q+1, ..., p-1\} \in \mathcal{I}$ for every $p > q > n_\varepsilon$. Thus, we conclude that

$$\left| \prod_{n\in\{q+1,\dots,p\}\setminus A_\varepsilon} \frac{n}{n+1} - 1 \right| = \left| \frac{p}{p+1} - 1 \right| = \frac{1}{p+1} < \varepsilon.$$

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