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Some sequence spaces and completeness of normed spaces

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ABSTRACT. In this paper we introduce new sequence spaces obtained by series in normed spaces and Cesáro summability method. We prove that completeness and barrelledness of a normed space can be characterized by means of these sequence spaces. Also we establish some inclusion relationships associated with the aforementioned sequence spaces.

1. INTRODUCTION

We denote the space of all real sequences $x = (x_k)$ by $\mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of positive integers. Any vector subspace of $\mathbb{R}^{\mathbb{N}}$ is called a sequence space. We write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences $x = (x_k)$, respectively, normed by $\|x\|_\infty = \sup_k \mid x_k \mid$. Also by bs, cs and ℓ_1, we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.

Let X be a real Banach space. A series $\sum_i x_i$ is called weakly unconditionally Cauchy series ($wuCs$) if $\left(\sum_{i=1}^n x_{\pi(i)}\right)_{n\in\mathbb{N}}$ is a weakly Cauchy for every permutation π of $\mathbb{N}$. It is known that $\sum_i x_i$ is a $wuCs$ if and only if $\sum_{i=1}^\infty |f(x_i)| < \infty$ for every $f \in X^*$, where X^* is the dual space of X. By $bs(X)$, $\ell_1(X)$, $cs(X)$, $wcs(X)$ and $wuCs(X)$, we denote the X-valued sequence spaces of all bounded, absolutely convergent, convergent, weakly convergent and weakly unconditionally Cauchy series, respectively.

It is well known that [3, 6, 7]:

1. The sequence $x = (x_k) \in wuCs(X)$ if and only if $(a_k x_k) \in cs(X)$ for every $a = (a_k) \in c_0$.
2. Let X be a normed space. The sequence $x = (x_i) \in wuCs(X)$ if and only if the set
$$S = \left\{ \sum_{i=1}^n a_i x_i : |a_i| \le 1,\ i = 1, 2, \ldots, n;\ n \in \mathbb{N} \right\} \tag{1.1}$$
is bounded.

Let $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n,\ k \in \mathbb{N}$. Then we write $Ax = \big((Ax)_n\big)$, the A-transform of $x \in w$, if $(Ax)_n = \sum_k a_{nk} x_k$ converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞. For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by
$$\lambda_A = \big\{ x = (x_k) \in w : Ax \in \lambda \big\},$$
which is a sequence space.

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The Cesàro matrix C with *Cesàro mean of order one*, which is a well–known method of summability and is defined by the matrix $C = (c_{nk})$ as follows;

$$c_{nk} = \begin{cases} \frac{1}{n}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

The C-transform of a sequence $a = (a_k)$ is the sequence $\tau(a) = (\tau_n(a))$ defined by

$$\tau_n(a) = \frac{1}{n} \sum_{k=1}^{n} a_k \text{ for all } n \in \mathbb{N}.$$

The set of all sequences whose C-transforms are in the spaces ℓ_∞ and c_0 were defined by Shiue in [15], Ng and Lee in [12], and Şengönül and Başar in [17], respectively. Some other works about the study of Cesáro sequence spaces are [4, 5, 9, 10, 11, 13, 16, 18].

For a sequence $x = (x_k)$ in a normed space X, the spaces $S(x)$ and $S_w(x)$ were defined by the set of all sequences $a = (a_i) \in \ell_\infty$ such that $(a_i x_i) \in cs(X)$ and $(a_i x_i) \in wcs(X)$, respectively and by means of these spaces, conditionally and weakly unconditionally Cauchy series were characterized in [1]. Using these spaces, Pérez-Fernández et al. obtained new characterizations of completeness and barrelledness of a normed space via the behaviour of its weakly and *-weakly unconditionally Cauchy series in [14]. In [2], the space $BS(x)$, $LS(x)$ and $LS_w(x)$ were defined by the set of all sequences $a = (a_i) \in \mathbb{R}^{\mathbb{N}}$ such that $(a_i x_i) \in bs(X)$, $(a_i x_i) \in cs(X)$ and $(a_i x_i) \in wcs(X)$, respectively and some properties of these spaces were studied. In [8], the spaces $S(x)$ and $S_w(x)$ were extended to the spaces $SC(x)$ and $SC_w(x)$.

In this paper we define some new sets of real sequences obtained by series in a normed space and Cesáro summability method. We give some characterizations related to completeness and barrelledness of a normed space and some inclusion relations associated with these sequence spaces.

2. Characterizations of completeness and barrelledness

In this section, we define the sets $BSC(x)$, $LSC(x)$, $LSC_w(x)$, $LSC_0(x)$ and $LSC_{w^*}(x)$. Also, we give some characterizations the completeness and barrelledness of a normed space X by means of the spaces $LSC(x)$, $LSC_w(x)$ and $LSC_{w^*}(x)$.

For a sequence $x = (x_k)$ in a normed space X, the sets $BSC(x)$, $LSC(x)$, $LSC_w(x)$ and $LSC_0(x)$ are defined by

$$BSC(x) = \left\{a = (a_i) \in \mathbb{R}^{\mathbb{N}} : (\tau_i(a) x_i) \in bs(X)\right\},$$

$$LSC(x) = \left\{a = (a_i) \in \mathbb{R}^{\mathbb{N}} : (\tau_i(a) x_i) \in cs(X)\right\},$$

$$LSC_w(x) = \left\{a = (a_i) \in \mathbb{R}^{\mathbb{N}} : (\tau_i(a) x_i) \in wcs(X)\right\},$$

$$LSC_0(x) = \left\{a = (a_i) \in \mathbb{R}^{\mathbb{N}} : (\tau_i(a) x_i) \in w^* cs(X^{**})\right\},$$

where

$$\tau_i(a) = \frac{1}{i} \Big(\sum_{j=1}^{i} a_j \Big) \ (i \in \mathbb{N}).$$

The sets $BSC(x)$, $LSC(x)$, $LSC_w(x)$ and $LSC_0(x)$ are linear spaces with the co-ordinatewise addition and scalar multiplication which are the normed spaces with the norm

$$\|a\|_{BSC} = \sup_n \left\| \sum_{i=1}^{n} \tau_i(a) x_i \right\|. \tag{2.2}$$

It is obvious that the inclusions $LSC(x) \subset LSC_w(x) \subset LSC_0(x) \subset BSC(x)$ are hold.

Theorem 2.1. *Let X be a normed space and $x = (x_k)$ be a sequence in X. Then $BSC(x)$ is a Banach space with the norm (2.2).*

Proof. Let $a = (a^m)$ be a Cauchy sequence in $BSC(x)$. There exists $\epsilon > 0$ and $m_0 \in \mathbb{N}$ such that for $p, q > m_0$

$$\|a^p - a^q\|_{BSC} < \epsilon \tag{2.3}$$

and thus

$$\|\tau_j(a^p - a^q)x_j\| = \left\|\sum_{i=1}^{j} \tau_i(a^p - a^q)x_i - \sum_{i=1}^{j-1} \tau_i(a^p - a^q)x_i\right\| \leq 2\|a^p - a^q\|_{BSC} < \epsilon.$$

This means that $(\tau_j(a^m))$ is a Cauchy sequence in $\mathbb{R}^{\mathbb{N}}$ for $j \in \mathbb{N}$. We suppose that $\tau_j(a^m) \to \tau_j(a^0) \in \mathbb{R}$ for every $j \in \mathbb{N}$.

Now, we will show that $a^0 \in BSC(x)$. From (2.3) if we take limit as $q \to \infty$, then

$$\left\|\sum_{i=1}^{n} \tau_i(a^p - a^0)x_i\right\| \leq \epsilon \tag{2.4}$$

for every $n \in \mathbb{N}$. Since $a^p \in BSC(x)$ for each $p \in \mathbb{N}$ there exists $M_p > 0$ such that

$$\|a^p\|_{BSC} \leq M_p. \tag{2.5}$$

From (2.4) and (2.5) for $\epsilon > 0$ and $p > m_0$ we have the inequality

$$\|a^0\|_{BSC} = \sup_n \left\|\sum_{i=1}^{n} \tau_i(a^0)x_i\right\| \leq \sup_n \left\|\sum_{i=1}^{n} \tau_i(a^0 - a^p)x_i\right\| + \sup_n \left\|\sum_{i=1}^{n} \tau_i(a^p)x_i\right\| \leq \epsilon + M_p.$$

□

Theorem 2.2. *Let X be a normed space and $x = (x_k)$ be a sequence in X. Then $LSC_0(x)$ is a Banach space with the norm (2.2).*

Proof. Let (a^m) be a Cauchy sequence in $LSC_0(x)$. Since $LSC_0(x) \subset BSC(x)$ and $BSC(x)$ is complete by Theorem 2.1, there exists a sequence $a^0 \in BSC(x)$ such that $a^m \to a^0$. For $x^* \in X^*$ there exists $(y_m^{**}) \subset X^{**}$ and $n_0 \in \mathbb{N}$ such that for $\epsilon > 0$ and $n > n_0$

$$\left|\sum_{i=1}^{n} \tau_i(a^m)x^*(x_i) - x^*(y_m^{**})\right| < \frac{\epsilon}{3} \tag{2.6}$$

for $m \in \mathbb{N}$. On the other hand, since (a^m) be a Cauchy sequence, there exists $\epsilon > 0$ and $m_0 \in \mathbb{N}$ such that for $p, q > m_0$

$$\|a^p - a^q\|_{BSC} < \frac{\epsilon}{3}. \tag{2.7}$$

We can choose $x^* \in S_{X^*}$ (the unit sphere of X^*). From (2.6) for $\epsilon > 0$ and $p, q > m_0$

$$\|y_p^{**} - y_q^{**}\| = |x^*(y_p^{**} - y_q^{**})| \leq \frac{2\epsilon}{3} + \|a^p - a^q\|_{BSC} < \epsilon. \tag{2.8}$$

Hence (y_m^{**}) is a Cauchy sequence in X^{**}. Thus there exists $y_0^{**} \in X^{**}$ such that $\lim_m y_m^{**} = y_0^{**}$. If we take limit as $q \to \infty$ from (2.7) and (2.8), then

$$\|a^p - a^0\|_{BSC} < \frac{\epsilon}{3} \quad \text{and} \quad \|y_p^{**} - y_0^{**}\| < \epsilon,$$

and also using (2.6), for $n > n_0$ we get

$$\left|\sum_{i=1}^{n} \tau_i(a^0)x^*(x_i) - x^*(y_0^{**})\right| \leq \left|\sum_{i=1}^{n} \tau_i(a^0)x^*(x_i) - \sum_{i=1}^{n} \tau_i(a^p)x^*(x_i)\right|$$

$$+\left|\sum_{i=1}^{n}\tau_i(a^p)x^*(x_i)-x^*(y_p^{**})\right|+|x^*(y_p^{**})-x^*(y_0^{**})|<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\epsilon=\frac{5\epsilon}{3},$$

which shows that $a^0\in LSC_0(x)$. □

Theorem 2.3. *The normed space X is a Banach space if and only if $LSC(x)$ is a Banach space for every sequence $x=(x_k)$ in X with the norm (2.2).*

Proof. Let $x=(x_k)$ be a sequence in X and (a^m) be a Cauchy sequence in $LSC(x)$ such that $a^m\to a^0\in BSC(x)$. Since the sequence (a^m) is in $LSC(x)$, there exists $(y_m)\subset X$ and $n_0\in\mathbb{N}$ such that for $\epsilon>0$ and $n>n_0$

$$\left\|\sum_{i=1}^{n}\tau_i(a^m)x_i-y_m\right\|<\frac{\epsilon}{3} \tag{2.9}$$

for $m\in\mathbb{N}$. Since (a^m) be a Cauchy sequence, there exists $\epsilon>0$ and $m_0\in\mathbb{N}$ such that for $p,q>m_0$

$$\|a^p-a^q\|_{BSC}<\frac{\epsilon}{3}. \tag{2.10}$$

Then from (2.9) for $\epsilon>0$ and $p,q>m_0$

$$\|y_p-y_q\|\le\frac{2\epsilon}{3}+\|a^p-a^q\|_{BSC}<\epsilon. \tag{2.11}$$

Therefore (y_m) is a Cauchy sequence in X, and by the completeness of X there exists $y_0\in X$ such that $\lim_m y_m=y_0$. If we take limit as $q\to\infty$ from (2.10) and (2.11), then

$$\|a^p-a^0\|_{BSC}<\frac{\epsilon}{3}\quad\text{and}\quad\|y_p-y_0\|<\epsilon,$$

and also using (2.9), for $n>n_0$ we have that

$$\left\|\sum_{i=1}^{n}\tau_i(a^0)x_i-y_0\right\|\le\left\|\sum_{i=1}^{n}\tau_i(a^0-a^p)x_i\right\|+\left\|\sum_{i=1}^{n}\tau_i(a^p)x_i-y_p\right\|+\|y_p-y_0\|<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\epsilon=\frac{5\epsilon}{3},$$

which means that $a^0\in LSC(x)$.

If X is not complete then there exists a sequence $x=(x_k)\in\ell_1(X)\setminus cs(X)$. Let suppose that $\|x_i\|<\frac{1}{i2^i}$ for $i\in\mathbb{N}$. We denote the sequence $a^n\in\mathbb{R}^{\mathbb{N}}$ for $n\in\mathbb{N}$ by

$$a_i^n=\begin{cases}1, & \text{if } i\le n,\\ -n, & \text{if } i=n+1,\quad (i\in\mathbb{N}).\\ 0, & \text{if } i>n+1,\end{cases}$$

We have that $a^n\in LSC(x)$ for each $n\in\mathbb{N}$. If we consider $a^0\in\mathbb{R}^{\mathbb{N}}$ such that $a_i^0=1$ for all $i\in\mathbb{N}$, then $a^0\in BSC(x)\setminus LSC(x)$ and $\lim_n a^n=a^0$. Hence $LSC(x)$ is not complete. □

Theorem 2.4. *The normed space X is a Banach space if and only if $LSC_w(x)$ is a Banach space for every sequence $x=(x_i)$ in X with the norm (2.2).*

Proof. Because of this is easily obtained in the similar way used in proving Theorem 2.2 and Theorem 2.3, we omit the detailed proof. □

For a sequence $f=(f_i)$ in X^*, we define the set

$$LSC_{w^*}(f)=\left\{a=(a_i)\in\mathbb{R}^{\mathbb{N}}:\sum\tau_i(a)f_i\text{ weak}^*\text{ convergent in }X^*\right\}.$$

It is clear that the inclusions $LSC(f)\subset LSC_w(f)\subset LSC_{w^*}(f)$ and $SC_{w^*}(f)=LSC_{w^*}(f)\cap(l_\infty)_C$ are hold.

Since it may be proved in the similar way used in proving Theorems 2.2 and 2.3, we give the following theorem without proof.

Theorem 2.5. *If $f = (f_i)$ is a sequence in X^*, then $LSC_{w^*}(f) \cap BSC(f)$ is a Banach space.*

Theorem 2.6. *Let X be a normed space. Then X is a barrelled space if and only if $LSC_{w^*}(f) \subset BSC(f)$ for every series $f = (f_i)$ in X^*.*

Proof. The necessary condition is obtained from the barrelledness of the space X.

If X is not a barrelled space, then there exists a weak* bounded set $M \subset X^*$ which is unbounded. Therefore there exists $(g_k) \subset M$ and $K_x > 0$ such that

$$\|g_k\| > k^2 \quad \text{and} \quad \sup_k |g_k(x)| < K_x$$

for every $x \in X$. We consider the sequence $(h_k) \subset X^*$ defined by

$$h_k = \begin{cases} g_1, & \text{if } k = 1, \\ \frac{1}{k} g_k - \frac{1}{k-1} g_{k-1}, & \text{if } k > 1. \end{cases}$$

If we take the sequence $e = (1, 1, 1, ...)$, $e \in LSC_{w^*}(h) \setminus BSC(h)$. □

3. Some inclusion relations

In [8], we were defined the spaces $SC(x)$ and $SC_w(x)$ by

$$\begin{aligned} SC(x) &= \left\{ a = (a_i) \in (\ell_\infty)_C : \sum_i \tau_i(a) x_i \text{ converges in } X \right\}, \\ SC_w(x) &= \left\{ a = (a_i) \in (\ell_\infty)_C : \sum_i \tau_i(a) x_i \text{ weakly converges in } X \right\}. \end{aligned}$$

It is obvious that the inclusions $SC(x) = LSC(x) \cap (l_\infty)_C$ and $SC_w(x) = LSC_w(x) \cap (l_\infty)_C$ are hold. Now, we give some inclusion relations between $LSC(x)$, $SC(x)$, $SC_w(x)$, $LSC_w(x)$ and $(c_0)_C$.

Theorem 3.7. *Let X be a normed space and $x = (x_i)$ be a sequence in X. If $\inf_i \|x_i\| > 0$, then $LSC(x) = SC(x)$.*

Proof. From the definitions $SC(x)$ and $LSC(x)$, the inclusion $SC(x) \subset LSC(x)$ is obvious.

Let $a = (a_i) \in LSC(x)$. Then

$$\|\tau_n(a) x_n\| = \left\| \sum_{i=1}^{n} \tau_i(a) x_i - \sum_{i=1}^{n-1} \tau_i(a) x_i \right\| \to 0, n \to \infty. \tag{3.12}$$

Thus $\tau_n(a) \to 0$, and hence $(a_i) \in (c_0)_C$. This shows that $(a_i) \in SC(x)$. □

Theorem 3.8. *If X is a Banach space and $x = (x_i)$ be a sequence in X, then $\inf_i \|x_i\| > 0$ if and only if $LSC(x) = SC(x)$.*

Proof. Necessity follows immediately from Theorem 3.7.

If $\inf_i \|x_i\| = 0$, then there exists a strictly increasing sequence (m_j) in $\mathbb{N}$ such that $\|x_{m_j}\| < \dfrac{1}{j^3}$. We define the sequence $a = (a_i)$ by

$$\tau_i(a) = \begin{cases} j, & \text{if } i = m_j, \\ 0, & \text{if } i \neq m_j. \end{cases}$$

It is obvious that $(a_i) \notin SC(x)$. Since the series $\sum_{i=1}^{\infty} \tau_i(a) x_i$ is convergent by Cauchy criterion, we have $(a_i) \in LSC(x)$. □

Theorem 3.9. *Let X be a normed space and $x = (x_i)$ be a sequence in X. If $\inf_i \|x_i\| > 0$, then $SC(x) \subset (c_0)_C$.*

Proof. If $(a_i) \in SC(x)$,

$$\|\tau_n(a)x_n\| = \left\| \sum_{i=1}^{n} \tau_i(a)x_i - \sum_{i=1}^{n-1} \tau_i(a)x_i \right\| \to 0, n \to \infty.$$

Thus $\tau_n(a) \to 0$, and hence $(a_i) \in (c_0)_C$. □

Theorem 3.10. *Let X be a Banach space and $x = (x_i)$ be a sequence in X. Then $\inf_i \|x_i\| > 0$ if and only if $SC(x) \subset (c_0)_C$.*

Proof. Necessity follows immediately from Theorem 3.9.

To prove the sufficiency it is enough to show $SC(x) \setminus (c_0)_C \neq \emptyset$. Let $\inf_i \|x_i\| = 0$. Then there exists a strictly increasing sequence (m_j) in $\mathbb{N}$ such that $\|x_{m_j}\| < \frac{1}{j^2}$. Let $a = (a_i)$ be the sequence defined by

$$\tau_i(a) = \begin{cases} 1, & \text{if } i = m_j, \\ 0, & \text{if } i \neq m_j. \end{cases}$$

It can be easily seen that $a \notin (c_0)_C$. Since $(\tau_i(a)x_i) \in cs(X)$ by Cauchy criterion, $a \in SC(x)$. □

Corollary 3.1. *Let X be a Banach space. Then the sequence $x = (x_i) \in wuCs(X)$ and $\inf_i \|x_i\| > 0$ if and only if $SC(x) = (c_0)_C$.*

Proof. Necessity. The inclusion $SC(x) \subset (c_0)_C$ is obtained from Theorem 3.10.

Let x be a sequence in $wuCs(X)$ and $b = (b_i) \in (c_0)_C$. Then the series $\sum_i \tau_i(b)x_i$ is convergent. Thus $b = (b_i) \in SC(x)$, and hence the inclusion $(c_0)_C \subset SC(x)$ is valid.

Sufficiency. Let the sequence $x = (x_k)$ is in X. Since $SC(x) \subset (c_0)_C$, the inequality $\inf_i \|x_i\| > 0$ follows from Theorem 3.10. Also, by the inclusion $(c_0)_C \subset SC(x)$ we have $x \in wuCs(X)$. □

Remark 3.1. Theorem 3.7, Theorem 3.8, Theorem 3.9, Theorem 3.10 and Corollary 3.1 are also valid if we take the spaces $SC_w(x)$ and $LSC_w(x)$ instead of $SC(x)$ and $LSC(x)$, respectively.

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