CREAT. MATH. INFORM.
Volume **26** (2017), No. 3,
Pages 297 - 308

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2017.03.07

Fixed point theorems for expansive mappings in G_p-metric spaces

MELTEM KAYA and HASAN FURKAN

ABSTRACT. In the present paper, we adopt the concept of expansive mapping in the context of G_p-metric spaces in a similar manner expansive mapping in metric spaces. Furthermore, we obtain some results on fixed points of expansive type mappings. Also, we prove some common fixed point results for expansive mappings by using the notion of weak compatibility in G_p-metric space. Our results generalize some comparable results in metric spaces and partial metric spaces to G_p-metric spaces. Moreover, some examples are introduced in order to support our new results.

1. INTRODUCTION

Fixed point theory has been one of the most important research fields and one of the most rapidly developing fields in analysis during the last few decades. Since wide application potential of this theory, the study of fixed points of mappings has been at the center of strong research activity. In a large class of studies, the classical concept of a metric space has been generalized in different directions by partly chancing the conditions of the metric. Among this generalizations, we can mention the partial metric spaces and G-metric spaces.

The notion of partial metric space was described by Matthews [12] in 1994 as a generalization of metric spaces where self-distances are not necessarily zero. In 2005, Mustafa and Sims [13] identified a new structure of generalized metric spaces named G-metric space.

Recently, based on the two above notions, Zand and Nezhad [20] introduced a new generalized metric space as a generalization of both partial metric spaces and G-metric spaces by defining the notion of G_p-metric space. Following this remarkable research, Aydi et al. [2] established some fixed point results in G_p-metric spaces which are first fixed point results in G_p-metric spaces. Then, many fixed point results for mappings satisfying various contractive conditions have been presented in G_p-metric spaces. Some of these results are noted in [3, 4, 5, 14, 16, 18, 10, 15].

In 1984, Wang et al. [19] defined the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [6] defined expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Thereafter, the result of Daffer and Kaneko [6] was extended to compatible mappings by Rhoades [17]. In 2008, Kumar [11] generalized the results of Rhoades [17] to weakly compatible mappings in metric spaces. Recently, Huang et al. [7] defined expanding mappings in the framework of partial metric space in the similar way to expanding mappings in metric spaces and also extended a result of Daffer and Kaneko [6] for two mappings to the partial metric spaces. Also, Imdad

Received: 30.09.2016. In revised form: 31.03.2017. Accepted: 07.04.2017
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. *Fixed point theorems, G_p-metric space, expansive mappings, weakly compatible mappings.*
Corresponding author: Meltem Kaya; meltemkaya55@hotmail.com

et al. [8] proved a general common fixed point theorem for two pairs of occasionally weak compatible expansive mappings in symmetric spaces.

The purpose of this study is to define the notion of expansive mapping in G_p-metric spaces and also to generalize some results of Wang et al. [19], Daffer and Kaneko [6], Huang et al. [7] and other comparable results by proving some fixed point theorems for expansive mappings ensuring the existence and uniqueness of a fixed point and common fixed point under certain conditions within the context of G_p-metric space.

2. Basic facts and definitions

In this section, we recall some fundamental definitions and useful results for the sake of completeness of this study.

Zand and Nezhad defined the concept of G_p-metric space by combining the notions of patrial metric space and G-metric space in the following manner.

Definition 2.1. [20] Let X be a nonempty set. A function $G_p : X \times X \times X \to [0, +\infty)$ is called a G_p-metric if the following conditions are satisfied:

G_{p_1}. $x = y = z$ if $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;
G_{p_2}. $0 \leq G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$;
G_{p_3}. $G_p(x, y, z) = G_p(x, z, y) = G_p(y, z, x) = \dots$, symmetry in all three variables;
G_{p_4}. $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ for any $x, y, z, a \in X$.

Then the pair (X, G_p) is called a G_p-metric space.

On the other hand, instead of G_{p_2}, Parvaneh, Roshan and Kadelburg used the following condition in [14]:

$G^*_{p_2}$. $0 \leq G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$.

Also, they stated an important remark as following:

Remark 2.1. [14] With G_{p_2} assumption, it is very easy to obtain that

$$G_p(x, x, y) = G_p(x, y, y)$$

holds for all $x, y \in X$, i.e., the respective space is symmetric. On the other hand, there are a lot of examples of asymmetric G-metric spaces. Hence, the conclusion stated in [20, 2] that each G-metric space is a G_p-metric space (satisfying G_{p_2}) does not hold. With the assumption $G^*_{p_2}$, this conclusion holds true.

Some easy examples of G_p-metric space are given as follows:

Example 2.1. [20] Let $X = [0, \infty)$ and define $G_p(x, y, z) = \max\{x, y, z\}$, for all $x, y, z \in X$. Then (X, G_p) is a symmetric G_p-metric space. Also, one can show that (X, G_p) is not a G-metric space.

Example 2.2. [20] If (X, d) is an ordinary metric space, then (X, d) can define symmetric G_p-metrics on X by

i. $G_p(x, y, z) = d(x, y) + d(y, z) + d(x, z)$,
ii. $G_p(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$.

Example 2.3. [14] *Let* $X = \{0, 1, 2, 3\}$ *and*

$$\begin{aligned} A &= \{(1,0,0),(0,1,0),(0,0,1),(2,0,0),(0,2,0),(0,0,2),(3,0,0),(0,3,0),(0,0,3),\\ &\quad (1,2,2),(2,1,2),(2,2,1),(1,3,3),(3,1,3),(3,3,1),(2,3,3),(3,2,3),(3,3,2)\},\\ B &= \{(0,1,1),(1,0,1),(1,1,0),(0,2,2),(2,0,2),(2,2,0),(0,3,3),(3,0,3),(3,3,0),\\ &\quad (2,1,1),(1,2,1),(1,1,2),(3,1,1),(1,3,1),(1,1,3),(3,2,2),(2,3,2),(2,2,3)\}. \end{aligned}$$

Define $G_p : X \times X \times X \to [0, +\infty)$ *by*

$$G_p(x,y,z) = \begin{cases} 1, & \text{if } x = y = z \neq 2, \\ 0, & \text{if } x = y = z = 2, \\ 2, & \text{if } (x,y,z) \in A, \\ \frac{5}{2}, & \text{if } (x,y,z) \in B, \\ 3, & \text{if } x \neq y \neq z \neq x. \end{cases}$$

It is easy to see that (X, G_p) *is an asymmetric* G_p*-metric space.*

In the rest of this paper, we will use the definition of G_p-metric space given by Zand and Nezhad, that is, we will consider that (X, G_p) is a symmetric G_p-metric space.

Proposition 2.1. [20] *Let* (X, G_p) *be a* G_p*-metric space. Then for any* x, y, z *and* $a \in X$*, it follows that*

i. $G_p(x,y,z) \leq G_p(x,x,y) + G_p(x,x,z) - G_p(x,x,x)$;
ii. $G_p(x,y,y) \leq 2G_p(x,x,y) - G_p(x,x,x)$;
iii. $G_p(x,y,z) \leq G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) - 2G_p(a,a,a)$;
iv. $G_p(x,y,z) \leq G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$.

Proposition 2.2. [20] *Every* G_p*-metric space* (X, G_p) *defines a metric space* (X, d_{G_p}) *where*

$$d_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y)$$

for all $x, y \in X$.

Definition 2.2. [20] Let (X, G_p) be a G_p-metric space and $\{x_n\}$ be a sequence of points of X.

i. The sequence $\{x_n\}$ is called G_p-convergent to $x \in X$ if $\lim\limits_{m,n\to\infty} G_p(x, x_m, x_n) = G_p(x,x,x)$. A point $x \in X$ is said to be limit point of the sequence $\{x_n\}$;
ii. A sequence $\{x_n\}$ is said to be a G_p-Cauchy sequence if and only if $\lim\limits_{m,n\to\infty} G_p(x_n, x_m, x_m)$ exists (and is finite);
iii. A G_p-metric space (X, G_p) is said to be G_p-complete if and only if every G_p-Cauchy sequence $\{x_n\}$ in X is G_p-convergent to $x \in X$ such that $G_p(x,x,x) = \lim_{m,n\to\infty} G_p(x_n, x_m, x_m)$.

Proposition 2.3. [20] *Let* (X, G_p) *be a* G_p*-metric space. Then, for any sequence* $\{x_n\}$ *in* X *and a point* $x \in X$ *the followings are equivalent:*

i. $\{x_n\}$ *is* G_p*-convergent to* x;
ii. $G_p(x_n, x_n, x) \to G_p(x,x,x)$ *as* $n \to \infty$;
iii. $G_p(x_n, x, x) \to G_p(x,x,x)$ *as* $n \to \infty$.

Lemma 2.1. [14]

i. *A sequence* $\{x_n\}$ *is a* G_p*-Cauchy sequence in a* G_p*-metric space* (X, G_p) *if and only if it is a Cauchy sequence in the metric space* (X, d_{G_p}).
ii. *A* G_p*-metric space* (X, G_p) *is* G_p*-complete if and only if the metric space* (X, d_{G_p}) *is complete. Moreover,* $\lim\limits_{n\to\infty} d_{G_p}(x, x_n) = 0$ *if and only if*

$$\begin{aligned} \lim_{n\to\infty} G_p(x, x_n, x_n) &= \lim_{n\to\infty} G_p(x_n, x, x) = \lim_{n,m\to\infty} G_p(x_n, x_n, x_m) \\ &= \lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(x,x,x). \end{aligned}$$

Lemma 2.2. [2] *Let* (X, G_p) *be a* G_p*-metric space. Then*

i. *If* $G_p(x,y,z) = 0$*, then* $x = y = z$;
ii. *If* $x \neq y$*, then* $G_p(x,y,y) > 0$.

Lemma 2.3. [14] *Assume that* $\{x_n\} \to x$ *as* $n \to \infty$ *in a* G_p*-metric space* (X, G_p) *such that* $G_p(x, x, x) = 0$*. Then, for every* $y \in X$*,*

i. $\lim_{n\to\infty} G_p(x_n, y, y) = G_p(x, y, y)$,
ii. $\lim_{n\to\infty} G_p(x_n, x_n, y) = G_p(x, x, y)$.

The following proposition shows that the concepts of continuity and sequentially continuity are equal in a G_p-metric space.

Proposition 2.4. [20] *Let* (X_1, G_1) *and* (X_2, G_2) *be* G_p*-metric spaces. Then a function* $f : X_1 \to X_2$ *is* G_p*-continuous at a point* $x \in X_1$ *if and only if it is* G_p*-sequentially continuous at* x*; that is, whenever* $\{x_n\}$ *is* G_p*-convergent to* x *one has* $\{f(x_n)\}$ *is* G_p*-convergent to* $f(x)$*.*

Kaya et al. given an important remark, which shows the relationship between G_p-continuity and d_{G_p}-continuity, as follows.

Remark 2.2. [10] It is worth noting that the notions of G_p-continuity and d_{G_p}-continuity of any function in the contex of G_p-metric space are incomparable, in general. Indeed, if $X = [0, +\infty)$, $G_p(x, y, z) = \max\{x, y, z\}$, $d_{G_p}(x, y) = |x - y|$, $f0 = 1$ and $fx = x^2$ for all $x > 0$, $gx = |\sin x|$, then f is a G_p-continuous and d_{G_p}-discontinuous at point $x = 0$; while g is a G_p-discontinuous and d_{G_p}-continuous at point $x = \pi$. Therefore, in this paper, we take that $T : X \to X$ continuous if both $T : (X, G_p) \to (X, G_p)$ and $T : (X, d_{G_p}) \to (X, d_{G_p})$ are continuous.

Definition 2.3. [1] Let f and g be two self mappings of a nonempty set X. If $fx = gx = y$ for some $x \in X$, then x is called the coincidence point of f and g and y is called the point of coincidence of f and g.

Definition 2.4. [9] Two self mappings f and g are said to be weakly compatible if they commute at their coincidence points, that is $fx = gx$ implies that $fgx = gfx$.

Now, we define the concept of expansive mapping in a G_p-metric space in analogy to expansive mapping in a metric space, as follows.

Definition 2.5. Let (X, G_p) be a G_p-metric space and T be a self mapping on X. Then T is called an expansive mapping if

$$G_p(Tx, Ty, Tz) \geq kG_p(x, y, z),$$

where $k > 1$ is a constant for every $x, y, z \in X$.

3. Main results

In this section, we first obtain some fixed point theorems for a single mapping and then prove several common fixed point theorems for expansive mappings of different types on G_p-metric spaces.

Now, we start by stating the following fixed point result.

Theorem 3.1. *Let* (X, G_p) *be a* G_p*-complete* G_p*-metric space and* $T : X \to X$ *be a surjective mapping satisfying the following condition*

$$G_p(Tx, Ty, Tz) \geq aG_p(x, y, z) + bG_p(x, x, Tx) + cG_p(y, y, Ty) + dG_p(z, z, Tz) \tag{3.1}$$

for all $x, y, z \in X$ *where* $a, b, c, d \geq 0$ *with* $a + b + c + d > 1$*. Then* T *has a fixed point in* X*.*

Proof. Suppose x_0 is an arbitrary point in X. Since T is a surjective mapping, there exists $x_1 \in X$ such that $x_0 = Tx_1$. Continuing in this way, we can define a sequence $\{x_n\}$ by $x_{n-1} = Tx_n$, $n \geq 1$. If $x_n = x_{n-1}$ for some $n \geq 1$ then we see that $x_n = Tx_n$. Thus x_n is a fixed point of T. Therefore, we assume that two consecutive terms of sequence $\{x_n\}$ are not equal. Now, using inequality (3.1), we have

$$\begin{aligned} G_p(x_{n-1}, x_n, x_n) &= G_p(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\geq aG_p(x_n, x_{n+1}, x_{n+1}) + bG_p(x_n, x_n, Tx_n) \\ &\quad +cG_p(x_{n+1}, x_{n+1}, Tx_{n+1}) + dG_p(x_{n+1}, x_{n+1}, Tx_{n+1}) \\ &= aG_p(x_n, x_{n+1}, x_{n+1}) + bG_p(x_n, x_n, x_{n-1}) + cG_p(x_{n+1}, x_{n+1}, x_n) \\ &\quad +dG_p(x_{n+1}, x_{n+1}, x_n), \end{aligned}$$

which yields

$$(1-b)G_p(x_{n-1}, x_n, x_n) \geq (a+c+d)G_p(x_n, x_{n+1}, x_{n+1}).$$

If $a + c + d = 0$, then $b > 1$ from hypothesis. The above inequality implies that a negative number is greater than or equal to zero. That is impossible. So, $a+c+d \neq 0$ and $(1-b) > 0$. Therefore, we can write

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq qG_p(x_{n-1}, x_n, x_n), \tag{3.2}$$

where $q = \dfrac{1-b}{a+c+d}$. Since $a + b + c + d > 1$, it implies $q < 1$. Using (3.2) repeatedly, we obtain

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq q^n G_p(x_0, x_1, x_1). \tag{3.3}$$

Then, for all $n, m \in \mathbb{N}$, $m > n$, we have by repeated use of the rectangle inequality and equation (3.3) that

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + \dots \\ &\quad +G_p(x_{m-1}, x_m, x_m) \\ &\leq [q^n + q^{n+1} + \cdots + q^{m-1}]G_p(x_0, x_1, x_1) \\ &\leq \frac{q^n}{1-q}G_p(x_0, x_1, x_1), \end{aligned}$$

which shows that $G_p(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$. Hence, $\{x_n\}$ is a G_p-Cauchy sequence in X. By the completeness of (X, G_p), there exists $z \in X$ such that $\{x_n\}$ is G_p-converges to z, that is

$$\lim_{n\to\infty} G_p(x_n, x_n, z) = \lim_{n\to\infty} G_p(x_n, z, z) = \lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(z, z, z) = 0.$$

Since T is a surjective mapping, there exists $u \in X$ such that $Tu = z$. Now, we denote that $z = u$. Then, from condition (3.1), we obtain

$$\begin{aligned} G_p(x_n, z, z) &= G_p(Tx_{n+1}, Tu, Tu) \\ &\geq aG_p(x_{n+1}, u, u) + bG_p(x_{n+1}, x_{n+1}, Tx_{n+1}) + (c+d)G_p(u, u, Tu) \\ &= aG_p(x_{n+1}, u, u) + bG_p(x_{n+1}, x_{n+1}, x_n) + (c+d)G_p(u, u, z). \end{aligned}$$

Taking the limit as $n \to \infty$ in the previous inequality, we get

$$0 = G_p(z, z, z) \geq aG_p(z, u, u) + (c+d)G_p(u, u, z) = (a+c+d)G_p(u, u, z),$$

which means that $G_p(u, u, z) = 0$, that is $u = z = Tu$. This gives that z is a fixed point of T.

We can deduce that if $a < 1$, then fixed point of T is not unique, since the mapping will provide Condition (3.1). But, if $a > 1$ this fixed point is unique. □

Corollary 3.1. *Let (X, G_p) be a G_p-complete G_p-metric space and $T : X \to X$ be a surjective mapping satisfying the following condition*

$$G_p(Tx, Ty, Tz) \geq \lambda G_p(x, y, z) \tag{3.4}$$

for all $x, y, z \in X$ where $\lambda > 1$. Then T has a unique fixed point in X.

Proof. From Theorem 3.1, we can conclude that T has a fixed point z in X by taking $b = c = d = 0$ and $a = \lambda$ in Condition (3.1).

Uniqueness of fixed point: Let $z \neq w$ be another fixed point of T, that is $Tw = w$. Then, by (3.4) we get

$$G_p(z, w, w) = G_p(Tz, Tw, Tw) \geq \lambda G_p(z, w, w),$$

which is a contradiction and hence $z = w$. This proves uniqueness. □

Corollary 3.2. *Let (X, G_p) be a G_p-complete G_p-metric space and $T : X \to X$ be a surjective mapping. Suppose that there exists a positive integer n such that*

$$G_p(T^n x, T^n y, T^n z) \geq \lambda G_p(x, y, z)$$

for all $x, y, z \in X$ where $\lambda > 1$. Then T has a unique fixed point in X.

Proof. From Corollary 3.1, T^n has a unique fixed point z. Furthermore, since we have $T^n(Tz) = T(T^n z) = Tz$, Tz is also a fixed point of T^n. This shows that $Tz = z$, that is, z is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique. □

Next theorem is one of the main results of this study.

Theorem 3.2. *Let (X, G_p) be a G_p-complete G_p-metric space and $T : X \to X$ be a continuous surjective mapping satisfying the following condition*

$$G_p(Tx, Ty, Tz) \geq \lambda u \tag{3.5}$$

where $u = \min\{G_p(x, y, z), G_p(x, x, Tx), G_p(y, y, Ty), G_p(z, z, Tz)\}$, for all $x, y, z \in X$ and $\lambda > 1$. Then T has a fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_{n-1} = Tx_n$. If $x_{n-1} = x_n$ for some $n \geq 1$, then T has a fixed point in X, which is x_n. Assuming $x_{n-1} \neq x_n$ for each $n \geq 1$, the we have from (3.5)

$$G_p(x_{n-1}, x_n, x_n) = G_p(Tx_n, Tx_{n+1}, Tx_{n+1}) \geq \lambda u$$

where $u = \min\{G_p(x_n, x_{n+1}, x_{n+1}), G_p(x_{n-1}, x_n, x_n)\}$.

Now we have to consider the following two cases.

If $u = G_p(x_{n-1}, x_n, x_n)$, then

$$G_p(x_{n-1}, x_n, x_n) \geq \lambda G_p(x_{n-1}, x_n, x_n),$$

which is impossible since $\lambda > 1$.

Also if $u = G_p(x_n, x_{n+1}, x_{n+1})$, then

$$G_p(x_{n-1}, x_n, x_n) \geq \lambda G_p(x_n, x_{n+1}, x_{n+1}),$$

i.e.,

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq q G_p(x_{n-1}, x_n, x_n),$$

where $q = \dfrac{1}{\lambda}$ and $q < 1$. Continuing in this way, we obtain

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq q^n G_p(x_0, x_1, x_1). \tag{3.6}$$

We can prove that $\{x_n\}$ is a G_p-Cauchy sequence in X using rectangle inequality and (3.6) as proved in Theorem 3.1. Since (X, G_p) is G_p-complete, the sequence $\{x_n\}$ G_p-converges to a point $z \in X$. So, we can conclude that $\lim_{n\to\infty} d_{G_p}(z, x_n) = 0$ if and only if

$$\lim_{n\to\infty} G_p(x_n, x_n, z) = \lim_{n\to\infty} G_p(x_n, z, z) = \lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(z, z, z) = 0.$$

Since T is a continuous mapping, thanks to Remark 2.2 we have Tx_{n+1} converges to Tz in (X, d_{G_p}). On the other hand, $Tx_{n+1} = x_n$ converges to z in (X, d_{G_p}) because of $\lim_{n\to\infty} d_{G_p}(z, x_n) = 0$. By uniqueness of the limit in metric space (X, d_{G_p}), we deduce that $Tz = z$. □

Let's state and prove a fixed point theorem for expansive condition given by a rational expression.

Theorem 3.3. *Let (X, G_p) be a G_p-complete G_p-metric space and $T : X \to X$ be a surjective mapping satisfying the following condition*

$$\begin{aligned} G_p(Tx, Tx, Ty) \quad \geq \quad & \frac{aG_p(x, x, Tx)G_p(y, y, Ty)}{G_p(x, x, y)} + b[G_p(x, x, Tx) + G_p(y, y, Ty)] \\ & + cG_p(x, x, y) \end{aligned} \tag{3.7}$$

for all $x, y \in X$ with $G_p(x, x, y) > 0$, where $a, b \geq 0$, $c > 1$ with $a + 2b + c > 1$. Then T has a unique fixed point in X.

Proof. As in the proof of the previous theorems, we define a sequence $\{x_n\}$ by $x_{n-1} = Tx_n$ for each $n \geq 1$. Without loss of generality, we suppose that two successive terms of sequence $\{x_n\}$ are different. Then, from condition (3.7), we get

$$\begin{aligned} G_p(x_{n-1}, x_{n-1}, x_n) &= G_p(Tx_n, Tx_n, Tx_{n+1}) \\ &\geq \frac{aG_p(x_n, x_n, Tx_n)G_p(x_{n+1}, x_{n+1}, Tx_{n+1})}{G_p(x_n, x_n, x_{n+1})} \\ &\quad + b[G_p(x_n, x_n, Tx_n) + G_p(x_{n+1}, x_{n+1}, Tx_{n+1})] \\ &\quad + cG_p(x_n, x_n, x_{n+1}) \\ &= \frac{aG_p(x_n, x_n, x_{n-1})G_p(x_{n+1}, x_{n+1}, x_n)}{G_p(x_n, x_n, x_{n+1})} \\ &\quad + b[G_p(x_n, x_n, x_{n-1}) + G_p(x_{n+1}, x_{n+1}, x_n)] \\ &\quad + cG_p(x_n, x_n, x_{n+1}). \end{aligned}$$

Then, from the above inequality, we can deduce that

$$(1 - a - b)G_p(x_{n-1}, x_{n-1}, x_n) \geq (b + c)G_p(x_n, x_n, x_{n+1}).$$

Since $b + c > 0$, we have $a + b < 1$. Therefore, we have

$$\begin{aligned} G_p(x_n, x_n, x_{n+1}) &\leq \frac{1 - a - b}{b + c} G_p(x_{n-1}, x_{n-1}, x_n) \\ &= qG_p(x_{n-1}, x_{n-1}, x_n), \end{aligned}$$

where $q = \dfrac{1 - a - b}{b + c} < 1$. In an analogous way, we can calculate

$$G_p(x_{n-1}, x_{n-1}, x_n) \leq qG_p(x_{n-2}, x_{n-2}, x_{n-1}).$$

By induction, we get

$$\begin{aligned} G_p(x_n, x_n, x_{n+1}) &\leq qG_p(x_{n-1}, x_{n-1}, x_n) \\ &\leq q^2 G_p(x_{n-2}, x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq q^n G_p(x_0, x_0, x_1). \end{aligned}$$

By using the same arguments as in Theorem 3.1, we can say that the sequence $\{x_n\}$ is a G_p-Cauchy sequence in X, which is G_p-complete, and so G_p-converges to $z \in X$. Hence, we obtain

$$\lim_{n\to\infty} G_p(z, x_n, x_n) = \lim_{n\to\infty} G_p(x_n, z, z) = \lim_{n,m\to\infty} G_p(x_n, x_n, x_m) = G_p(z, z, z) = 0.$$

Thereby, we can say that $Tu = z$ for $u \in X$, since T is a surjective mapping. We will show that $z = u$. Let's assert the contrary, that is $z \neq u$. Then, from condition (3.7), we get

$$\begin{aligned} G_p(x_n, x_n, z) &= G_p(Tx_{n+1}, Tx_{n+1}, Tu) \\ &\geq \frac{aG_p(x_{n+1}, x_{n+1}, Tx_{n+1})G_p(u, u, Tu)}{G_p(x_{n+1}, x_{n+1}, u)} \\ &\quad +b[G_p(x_{n+1}, x_{n+1}, Tx_{n+1}) + G_p(u, u, Tu)] + cG_p(x_{n+1}, x_{n+1}, u) \\ &= \frac{aG_p(x_{n+1}, x_{n+1}, x_n)G_p(u, u, z)}{G_p(x_{n+1}, x_{n+1}, u)} \\ &\quad +b[G_p(x_{n+1}, x_{n+1}, x_n) + G_p(u, u, z)] + cG_p(x_{n+1}, x_{n+1}, u). \end{aligned}$$

If we take the limit as $n \to \infty$ in the last inequality, we have

$$0 = G_p(z, z, z) \geq (b + c)G_p(u, u, z),$$

which implies that $G_p(u, u, z) = 0$, that is $z = u$. This is a contradiction. So, our assumption that $z \neq u$ is not true. Hence, we can deduce that $z = u$.

Now, we denote that z is a unique fixed point of T. Assume the contrary. Let w be another fixed point of T, in other words $z \neq w$ and $Tw = w$. Then, we gain

$$\begin{aligned} G_p(z, z, w) &= G_p(Tz, Tz, Tw) \\ &\geq \frac{aG_p(z, z, Tz)G_p(w, w, Tw)}{G_p(z, z, w)} + b[G_p(z, z, Tz) + G_p(w, w, Tw)] \\ &\quad +cG_p(z, z, w) \\ &= \frac{aG_p(z, z, z)G_p(w, w, w)}{G_p(z, z, w)} + b[G_p(z, z, z) + G_p(w, w, w)] + cG_p(z, z, w). \end{aligned}$$

This means that $(c - 1)G_p(z, z, w) \leq 0$, which is possible with $G_p(z, z, w) = 0$ since $c > 1$. This results in a contradiction and so we get $z = w$. □

We now present a concrete example in support of previous result.

Example 3.4. $X = [0, \infty)$ and $G_p(x, y, z) = \max\{x, y, z\}$. Then (X, G_p) is a G_p-complete G_p-metric space. Define the surjective self mapping $T : X \to X$ by $Tx = 2x$. Without loss of generality assume that $x > y$. Then for all $x, y \in [0, \infty)$ with $x > y$ we have

$$\begin{aligned} G_p(Tx, Tx, Ty) = 2x &> \frac{7x}{4} \\ &> a\frac{G_p(x, x, Tx)G_p(y, y, Ty)}{G_p(x, x, y)} \\ &\quad +b[G_p(x, x, Tx) + G_p(y, y, Ty)] + cG_p(x, x, y) \end{aligned}$$

where $a = b = \frac{1}{32}$ and $c = \frac{3}{2}$. Thus T satisfies all the conditions of Theorem 3.3 and hence T has a unique fixed point. Clearly, $x = 0$ is a unique fixed point of T.

Now, we introduce a common fixed point theorem for expansive mappings by using the notion of weakly compatibility in G_p-metric spaces.

Theorem 3.4. *Let (X, G_p) be a G_p-metric space and S and T be weakly compatible self mappings of X satisfying*

$$\begin{aligned} G_p(Sx, Sy, Sz) \quad &\geq \quad aG_p(Tx, Ty, Tz) + bG_p(Tx, Tx, Sx) + cG_p(Ty, Ty, Sy) \\ &\quad +dG_p(Tz, Tz, Sz) \end{aligned} \tag{3.8}$$

for all $x, y, z \in X$, where $b, c, d \geq 0$, $a > 1$ with $a + b + c + d > 1$. If $T(X) \subseteq S(X)$ and one of the subspaces $T(X)$ or $S(X)$ is G_p-complete, then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Due to $T(X) \subseteq S(X)$, we can pick a point $x_1 \in X$ such that $Tx_0 = Sx_1 = y_1$ and for this point x_1, there exists a point $x_2 \in X$ such that $Tx_1 = Sx_2 = y_2$. So, there exists a sequence of points $\{x_n\}$ such that $Tx_n = Sx_{n+1} = y_{n+1}$. Note that, if $Tx_n = Tx_{n-1}$ for some $n \geq 1$, then $Tx_n = Sx_n$ and x_n is a coincidence point of T and S. Hence, let us suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Now, using equation (3.8), we get

$$\begin{aligned} G_p(y_{n-1}, y_n, y_n) \quad &= \quad G_p(Sx_{n-1}, Sx_n, Sx_n) \\ &\geq \quad aG_p(Tx_{n-1}, Tx_n, Tx_n) + bG_p(Tx_{n-1}, Tx_{n-1}, Sx_{n-1}) \\ &\quad\quad +cG_p(Tx_n, Tx_n, Sx_n) + dG_p(Tx_n, Tx_n, Sx_n) \\ &= \quad aG_p(y_n, y_{n+1}, y_{n+1}) + bG_p(y_n, y_n, y_{n-1}) + cG_p(y_{n+1}, y_{n+1}, y_n) \\ &\quad\quad +dG_p(y_{n+1}, y_{n+1}, y_n). \end{aligned}$$

Thus, we can conclude that

$$G_p(y_n, y_{n+1}, y_{n+1}) \leq qG_p(y_{n-1}, y_n, y_n),$$

where $q = \dfrac{1-b}{a+c+d} \in (0, 1)$.

By induction, we get

$$\begin{aligned} G_p(y_n, y_{n+1}, y_{n+1}) \quad &\leq \quad qG_p(y_{n-1}, y_n, y_n) \\ &\leq \quad q^2G_p(y_{n-2}, y_{n-1}, y_{n-1}) \\ &\ \vdots \\ &\leq \quad q^nG_p(y_0, y_1, y_1). \end{aligned}$$

Therefore, for all $n, m \in \mathbb{N}$, $n < m$, we obtain

$$\begin{aligned} G_p(y_n, y_m, y_m) \quad &\leq \quad G_p(y_n, y_{n+1}, y_{n+1}) + G_p(y_{n+1}, y_{n+2}, y_{n+2}) + \ldots \\ &\quad\quad +G_p(y_{m-1}, y_m, y_m) \\ &\leq \quad [q^n + q^{n+1} + \cdots + q^{m-1}]G_p(y_0, y_1, y_1) \\ &\leq \quad \frac{q^n}{1-q}G_p(y_0, y_1, y_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, $\{y_n\}$ is a G_p-Cauchy sequence in X. Since $T(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a complete subspace of X, there exists $z \in S(X)$ such that

$$\lim_{n\to\infty} G_p(y_n, y_n, z) = \lim_{n\to\infty} G_p(y_n, z, z) = \lim_{n,m\to\infty} G_p(y_n, y_m, y_m) = G_p(z, z, z) = 0.$$

As a consequence, we can find $u \in X$ such that $Su = z$. Now, we show that $Tu = z$. We have by (3.8),

$$\begin{aligned} G_p(Tx_{n-1}, Su, Su) &= G_p(Sx_n, Su, Su) \\ &\geq aG_p(Tx_n, Tu, Tu) + bG_p(Tx_n, Tx_n, Sx_n) + cG_p(Tu, Tu, Su) \\ &\quad + dG_p(Tu, Tu, Su) \\ &= aG_p(y_{n+1}, Tu, Tu) + bG_p(y_{n+1}, y_{n+1}, y_n) + cG_p(Tu, Tu, z) \\ &\quad + dG_p(Tu, Tu, z). \end{aligned}$$

Taking the limit as $n \to \infty$, we gain

$$0 = G_p(z, z, z) \geq (a + c + d)G_p(z, Tu, Tu),$$

which means that $G_p(z, Tu, Tu) = 0$ since $a + c + d \neq 0$, that is $z = Tu$. Hence, we have $z = Su = Tu$. Because S and T are weakly compatible, we can say that $STu = TSu$, in other word $Sz = Tz$.

Now, let us prove that z is a common fixed point of S and T. From (3.8),

$$\begin{aligned} G_p(Sz, Sx_n, Sx_n) &\geq aG_p(Tz, Tx_n, Tx_n) + bG_p(Tz, Tz, Sz) + cG_p(Tx_n, Tx_n, Sx_n) \\ &\quad + dG_p(Tx_n, Tx_n, Sx_n) \\ &= aG_p(Tz, y_{n+1}, y_{n+1}) + bG_p(Tz, Tz, Sz) + cG_p(y_{n+1}, y_{n+1}, y_n) \\ &\quad + dG_p(y_{n+1}, y_{n+1}, y_n). \end{aligned}$$

If we take the limit as $n \to \infty$ in the previous inequality, we obtain

$$\begin{aligned} G_p(Sz, z, z) &\geq aG_p(Tz, z, z) + bG_p(Tz, Tz, Sz) \\ &\geq aG_p(Tz, z, z) = aG_p(Sz, z, z). \end{aligned}$$

This leads to $G_p(Sz, z, z) = 0$ since $a > 1$, which means that $Sz = z$. Hence, we get $Sz = Tz = z$.

To prove uniqueness, suppose that $z \neq w$ is another common fixed point of S and T, then we get $Sw = Tw = w$. By using (3.8),

$$\begin{aligned} G_p(z, w, w) &= G_p(Sz, Sw, Sw) \\ &\geq aG_p(Tz, Tw, Tw) + bG_p(Tz, Tz, Sz) + cG_p(Tw, Tw, Sw) \\ &\quad + dG_p(Tw, Tw, Sw) \\ &= aG_p(z, w, w) + (c + d)G_p(w, w, w) \\ &\geq aG_p(z, w, w), \end{aligned}$$

which need that $z = w$. This completes the proof. □

We give an example to illustrate Theorem 3.4.

Example 3.5. $X = [0, 1]$ and $G_p(x, y, z) = \max\{x, y, z\}$. Let $S(x) = \dfrac{x}{2}$ and $T(x) = \dfrac{x}{6}$ for all $x \in X$. It is clear that $T(X) \subseteq S(X)$ and $S(X)$ is G_p-complete. Further, for all $x, y, z \in [0, 1]$ with $x \geq y \geq z$, we obtain

$$\begin{aligned} G_p(Sx, Sy, Sz) &= \max\left\{\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right\} = \frac{x}{2} \\ &\geq \frac{4x}{9} \\ &\geq aG_p(Tx, Ty, Tz) + bG_p(Tx, Tx, Sx) + cG_p(Ty, Ty, Sy) \\ &\quad + dG_p(Tz, Tz, Sz) \end{aligned}$$

for $a = 2$, $b = c = d = \dfrac{2}{27}$. Moreover, mappings S and T are weakly compatible at $x = 0$. Consequently, all assumptions of Theorem 3.4 are satisfied and hence 0 is the unique common fixed point.

Taking $a = \lambda$ and $b = c = d = 0$ in Theorem 3.4, we get the following corollary generalizing the results of Daffer and Kaneko [6] and Rhoades [17] to weakly compatible mappings in a G_p-metric space.

Corollary 3.3. *Let (X, G_p) be a G_p-metric space and S and T be weakly compatible self mappings of X satisfying*

$$G_p(Sx, Sy, Sz) \geq \lambda G_p(Tx, Ty, Tz)$$

for all $x, y, z \in X$, where $\lambda > 1$ is a constant. If $T(X) \subseteq S(X)$ and one of the subspaces $T(X)$ or $S(X)$ is G_p-complete, then S and T have a unique common fixed point in X.

Now, we shall construct an example and show that the necessary condition of weakly compatible can not be removed.

Example 3.6. Let $X = [0, 1]$ with the G_p-metric $G_p(x, y, z) = \max\{x, y, z\}$. Define the mappings

$$S(x) = 1 - x \text{ and } T(x) = \frac{1 - x}{2}$$

for all $x \in X$. Then $T(X) \subseteq S(X)$ and $S(X)$ is G_p-complete. Furthermore, for all $x, y, z \in [0, 1]$ we get

$$\begin{aligned} G_p(Sx, Sy, Sz) &= \max\{1 - x, 1 - y, 1 - z\} \\ &\geq a \max\left\{\frac{1 - x}{2}, \frac{1 - y}{2}, \frac{1 - z}{2}\right\} \\ &= aG_p(Tx, Ty, Tz) \end{aligned}$$

where $1 < a \leq 2$ and $b = c = d = 0$. Also, $S1 = T1 = 0$ but $ST1 = 1$ and $TS1 = \dfrac{1}{2}$. Hence, S and T are not weakly compatible mappings. It follows that except for the weakly compatibility of S and T, all other hypothesis of Theorem 3.4 and Corollary 3.3 are satisfied. But, the mappings S and T do not have a common fixed point. This shows that the weakly compatibility of S and T in Theorem 3.4 and Corollary 3.3 is an essential condition.

Acknowledgements. The authors would like to thank the reviewers for pointing out some mistakes and misprints in the earlier version of this paper. So, They would like to express their pleasure to the reviewers for their careful reading and making some useful comments which improved the presentation of the paper. M. Kaya has been supported by the Scientific and Technological Research Council of Turkey (TUBITAK Programme, 2211-A).

References

[1] Abbas, M. and Jungck, G., *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341** (2008), 416–420

[2] Aydi, H., Karapınar, E. and Salimi, P., *Some fixed point results in G_p-metric spaces*, J. Appl. Math., **2012** (2012), Article ID 891713, 15 pages, doi:10.1155/2012/891713

[3] Barakat, M. A. and Zidan, A. M., *A common fixed point theorem for weak contractive maps in G_p-metric spaces*, J. Egyptian Math. Soc., **23** (2015), 309–314

[4] Bilgili, N., Karapınar, E. and Salimi, P., *Fixed point theorems for generalized contractions on G_p-metric spaces*, J. Inequal Appl., **2013**:39 (2013), 1–13

[5] Ciric, L., Alsulami, S. M., Parvaneh, V. and Roshan, R., *Some fixed point results in ordered G_p-metric spaces*, Fixed Point Theory Appl., *2013*:317 (2013), 1–25

[6] Daffer, P. Z. and Kaneko, H., *On expansive mappings*, Math. Japon, **37** (1992), 733–735
[7] Huang, X., Zhu, C. and Wen, X., *Fixed point theorems for expanding mappings in partial metric spaces*, An. Şt. Univ. Ovidius Constanta, **20** (2012), No. 1, 213–224
[8] Imdad, M., Ali, J. and Popa, V., *Impact of Occasionally Weakly Compatible Property on Common Fixed Point Theorems for Expansive Mappings*, Filomat, **25** (2011), No. 2, 79–89
[9] Jungck, G., *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **9** (1986), 771–779
[10] Kaya, M., Öztürk, M. and Furkan, H., *Some Common Fixed Point Theorems for (F, f)-Contraction Mappings in 0-G_p-Complete G_p-Metric Spaces*, British Journal of Mathematics & Computer Science, **16** (2016), No. 2, 1–23
[11] Kumar, S., *Common fixed point theorems for expansion mappings in various spaces*, Acta. Math. Hunger., **118** (2008), 9–28
[12] Matthews, S. G., *Partial metric topology*, in: Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., **728** (1994), 183–197
[13] Mustafa, Z. and Sims, B., *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), No. 2, 289–297
[14] Parvaneh, V., Roshan, J. R. and Kadelburg, Z., *On generalized weakly GP-contractive mappings in ordered GP-metric spaces*, Gulf J. Math., **1** (2013), 78–97
[15] Parvaneh, V., Salimi, P., Vetro, P., Nezhad, A. D. and Radenović, S., *Fixed point results for $GP_{(\Lambda\Theta)}$-contractive mappings*, J. Nonlinear Sci. Appl., **7** (2014), 150–159
[16] Popa, V. and Patriciu, A. M., *Two general fixed point theorems for a sequence of mappings satisfying implicit relations in Gp-metric spaces*, Appl. Gen. Topol., **16** (2015), 225-231
[17] Rhoades, B. E., *An expansion mapping theorem*, Jnanabha, **23** (1993), 151–152
[18] Salimi, P. and Vetro, P., *A result of Suzuki type in partial G-metric spaces*, Acta Math. Sci. Ser. B Engl. Ed., **34** (2014), No. 2, 274–284
[19] Wang, S. Z., Li, B. Y., Gao, Z. M. and Iseki, K., *Some fixed point theorems for expansion mappings*, Math. Japon, **29** (1984), 631–636
[20] Zand, M. R. A. and Nezhad, A. D., *A generalization of partial metric spaces*, J. Contemp. Appl. Math., **24** (2011), 86–93

DEPARTMENT OF MATHEMATICS
KAHRAMANMARAŞ SÜTÇÜ İMAM UNIVERSITY
46100, KAHRAMANMARAŞ, TURKEY
Email address: meltemkaya55@hotmail.com
Email address: hasanfurkan@gmail.com