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Certain Hermite-Hadamard type inequalities associated with conformable fractional integral operators

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ABSTRACT. The aim of this article is to obtain some new Hermite-Hadamard type inequalities for convex functions via conformable fractional integral. The results presented here would provide extensions of those given in earlier works.

1. INTRODUCTION

Historically, the study of convex functions begins in the context of real-valued functions of a real variable. Displayed in this setting where graphic representation guides our intuition, we find variety of theorems having an elegance that is rooted in the very simplicity of their proofs. The results have various applications and a wide variety of generalizations.

Let I be an interval than can be open, half-open, closed, finite or infinite in $\mathbb{R}$. Then A function $f : I \to \mathbb{R}$ called convex if for all $x, y \in I$ and all $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \tag{1.1}$$

holds. One could equivalently take α to be in the open interval $(0,1)$. Then for all $x \neq y$, f is called strictly convex. Geometrically (1.1) means that if K, L and M are any three points on the graph of f with L between K and M, then L is on or bellow chord KM. The functions $f(x) = x^2$ on $(-\infty, \infty)$ and $g(x) = \sin x$ on $[-\pi, 0]$ are simple examples of convex functions. If the inequality in (1.1) is reserved, then f is said to be concave.

Proposition 1.1. [14] *If $f : [a,b] \to \mathbb{R}$ is a convex function, then $f(a+)$ and $f(b-)$ exist in $\mathbb{R}$ and*

$$\tilde{f}(x) = \begin{cases} f(a+) & \text{if } x = a \\ f(x) & \text{if } x \in (a,b) \\ f(b-) & \text{if } x = b \end{cases}$$

is convex too.

As stated in Proposition above, every convex function f on an interval $[a,b]$ can be modified at the end points to become convex and continuous. An immediate consequence of this fact is the (Riemann) integrability of f. The arithmetic mean of f can be estimated as the following:

Theorem 1.1. (see, e.g, [4]) *If $f : I \to \mathbb{R}$ is a convex function, where $I = [a,b]$ and $\mathbb{R}$ are a set of real numbers, then the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

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are valid. This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [3, 6, 7, 8, 12, 15]).

The foundations of fractional integrals were laid by Liouville in a paper from 1832. Various types of fractional integrals as Riemann-Liouville fractional integral, Hadamard fractional integral, k- fractional integral, (k,s)- fractional integral were introduced. Recently, Khalil et al.[11] introduced a new definition of the fractional integral called conformable fractional integral. Then, Abdeljavad gave an extension of conformable fractional integral of any order $\alpha > 0$ as the following:

Definition 1.1. [1] Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of any order $\alpha > 0$ is defined by

$$(I_{\alpha}^{a} f)(t) = \frac{1}{n!}\int_{a}^{t}(t-x)^{n}(x-a)^{\beta-1}f(x)dx$$

Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$({}^{b}I_{\alpha} f)(t) = \frac{1}{n!}\int_{t}^{b}(x-t)^{n}(b-x)^{\beta-1}f(x)dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$ and hence $(I_{\alpha}^{a} f)(t) = (J_{n+1}^{a} f)(t)$. Some recent result and properties concerning the fractional integral operators can be found [1, 2, 5, 9, 10, 11, 13, 16, 17, 18, 19, 20].

The Beta function defined as follows:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_{0}^{1} t^{a-1}(1-t)^{b-1}\,dt, \qquad a, b > 0,$$

where $\Gamma(\alpha)$ is Gamma function.The Incomplete Beta function is defined by

$$B_{x}(a,b) = \int_{0}^{x} t^{a-1}(1-t)^{b-1}dt.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function.

The main goal of this paper, motivated by the above mentions results and results in [21], is to prove some new Hermite-Hadamard type inequalities for conformable fractional integral of convex functions.

2. Main results

Lemma 2.1. *Assume that $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b). If $f' \in (L[a,b])$ then the following equality holds:*

$$\begin{aligned}
\Psi_{\alpha}(a,b) = \frac{-(b-a)\alpha}{16}\Bigg[& \int_{0}^{1} B_{t}(n+1, \alpha-n) f'\left(ta + (1-t)\frac{3a+b}{4}\right)dt \\
& - \int_{0}^{1} B_{1-t}(\alpha-n, n+1) f'\left(t\frac{3a+b}{4} + (1-t)\frac{a+b}{2}\right)dt \\
& + \int_{0}^{1} B_{t}(n+1, \alpha-n) f'\left(t\frac{a+b}{2} + (1-t)\frac{a+3b}{4}\right)dt \\
& - \int_{0}^{1} B_{1-t}(\alpha-n, n+1) f'\left(t\frac{a+3b}{4} + (1-t)b\right)dt\Bigg]
\end{aligned}$$

for $\alpha \in (n, n+1]$, $n = 0, 1, 2, \ldots$ *where* $B_t(.,.)$ *is incompleted beta function and*

$$\begin{aligned}
&\Psi_\alpha(a,b) \qquad (2.2)\\
= \;& \frac{\alpha}{4}\Bigg[B(n+1,\alpha-n)\left(f(a)+f\left(\frac{a+b}{2}\right)\right)\\
&+B(\alpha-n,n+1)\left(f\left(\frac{a+b}{2}\right)+f(b)\right)\Bigg]-\frac{\alpha 4^{\alpha-1}n!}{(b-a)^\alpha}\\
&\times\Bigg[\left(I_\alpha^a f\left(\frac{3a+b}{4}\right)+I_\alpha^{\frac{3a+b}{4}} f\left(\frac{a+b}{2}\right)+I_\alpha^{\frac{a+b}{2}} f\left(\frac{a+3b}{4}\right)+I_\alpha^{\frac{a+3b}{4}} f(b)\right)\Bigg].
\end{aligned}$$

Proof. Letting $u = ta + (1-t)\frac{3a+b}{4}$ changing variable and integrating by parts yield

$$\begin{aligned}
\Phi_1 &= \frac{-(b-a)\alpha}{16}\int_0^1 B_t(n+1,\alpha-n)f'\left(ta+(1-t)\frac{3a+b}{4}\right)dt\\
&= \frac{-(b-a)\alpha}{16}\Bigg[B_t(n+1,\alpha-n)\frac{f\left(ta+(1-t)\frac{3a+b}{4}\right)}{\left(-\frac{b-a}{4}\right)}\Bigg|_0^1\\
&\quad-\left(\frac{-4}{b-a}\right)\int_0^1 t^n(1-t)^{\alpha-n-1}f\left(ta+(1-t)\frac{3a+b}{4}\right)dt\Bigg]\\
&= \frac{\alpha}{4}\Bigg[B(n+1,\alpha-n)f(a)-\int_0^1 t^n(1-t)^{\alpha-n-1}f\left(ta+(1-t)\frac{3a+b}{4}\right)dt\Bigg]\\
&= \frac{\alpha}{4}\Bigg[B(n+1,\alpha-n)f(a)\\
&\quad-\int_a^{\frac{3a+b}{4}}\left(\frac{4}{b-a}\right)^n\left(\frac{3a+b}{4}-u\right)^n\left(\frac{4}{b-a}\right)^{\alpha-n-1}(u-a)^{\alpha-n-1}\frac{4}{b-a}f(u)du\Bigg]\\
&= \frac{\alpha}{4}\Bigg[B(n+1,\alpha-n)f(a)\\
&\quad-\frac{4^\alpha n!}{(b-a)^\alpha}\frac{1}{n!}\int_a^{\frac{3a+b}{4}}\left(\frac{3a+b}{4}-u\right)^n(u-a)^{\alpha-n-1}f(u)du\Bigg]\\
&= \frac{\alpha}{4}B(n+1,\alpha-n)f(a)-\frac{\alpha 4^{\alpha-1}n!}{(b-a)^\alpha}\left(I_\alpha^a f\right)\left(\frac{3a+b}{4}\right).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\Phi_2 &= \frac{(b-a)\alpha}{16}\int_0^1 B_{1-t}(\alpha-n,n+1)f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)dt\\
&= \frac{(b-a)\alpha}{16}\Bigg[B_{1-t}(\alpha-n,n+1)\frac{f\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)}{\frac{a-b}{4}}\Bigg|_0^1\\
&\quad-\frac{4}{b-a}\int_0^1 t^n(1-t)^{\alpha-n-1}f\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)dt\Bigg]\\
&= \frac{(b-a)\alpha}{16}\Bigg[B(\alpha-n,n+1)f\left(\frac{a+b}{2}\right)\frac{4}{b-a}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4^2}{(b-a)^2}\int_{\frac{a+b}{2}}^{\frac{3a+b}{4}}\left(\frac{4}{a-b}\right)^n\left(u-\frac{a+b}{2}\right)^n \\
&\times\left(\frac{4}{b-a}\right)^{\alpha-n-1}\left(u-\frac{3a+b}{4}\right)^{\alpha-n-1}f(u)du\Bigg] \\
=\;&\frac{\alpha}{4}B(\alpha-n,n+1)f\left(\frac{a+b}{2}\right)-\frac{\alpha 4^{\alpha-1}n!}{(b-a)^\alpha}\left(I_\alpha^{\frac{3a+b}{4}}f\right)\left(\frac{a+b}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3 \;=\;&\frac{-(b-a)\alpha}{16}\int_0^1 B_t(n+1,\alpha-n)f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)dt \\
=\;&\frac{-(b-a)\alpha}{16}\Bigg[B_t(n+1,\alpha-n)\frac{f\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)}{\frac{a-b}{4}}\Bigg|_0^1 \\
&+\frac{4}{b-a}\int_0^1 t^n(1-t)^{\alpha-n-1}f\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)dt\Bigg] \\
=\;&\frac{-(b-a)\alpha}{16}\Bigg[B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)\frac{4}{a-b} \\
&+\frac{4}{b-a}\int_{\frac{a+3b}{4}}^{\frac{a+b}{2}}\left(u-\frac{a+3b}{4}\right)^n\left(\frac{4}{a-b}\right)^n\left(\frac{4}{a-b}\right)^{\alpha-n-1} \\
&\times\left(\frac{a+b}{2}-u\right)^{\alpha-n-1}\frac{4}{a-b}f(u)du\Bigg] \\
=\;&\frac{\alpha}{4}B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)-\frac{\alpha 4^{\alpha-1}n!}{(b-a)^\alpha}\left(I_\alpha^{\frac{a+b}{2}}f\right)\left(\frac{a+3b}{4}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Phi_4 \;=\;&\frac{(b-a)\alpha}{16}\int_0^1 B_{1-t}(\alpha-n,n+1)f'\left(t\frac{a+3b}{4}+(1-t)b\right)dt \\
=\;&\frac{(b-a)\alpha}{16}\Bigg[B_{1-t}(\alpha-n,n+1)\frac{f\left(t\frac{a+3b}{4}+(1-t)b\right)dt}{\frac{a-b}{4}}\Bigg|_0^1 \\
&-\frac{4}{b-a}\int_0^1 t^n(1-t)^{\alpha-n-1}f\left(t\frac{a+3b}{4}+(1-t)b\right)dt\Bigg] \\
=\;&\frac{(b-a)\alpha}{16}\Bigg[B(\alpha-n,n+1)f(b)\frac{4}{b-a}+\frac{4^2}{(b-a)^2}\int_b^{\frac{a+3b}{4}}\left(\frac{4}{b-a}\right)^n(b-u)^n \\
&\times\left(\frac{4}{b-a}\right)^{\alpha-n-1}\left(u-\frac{a+3b}{4}\right)^{\alpha-n-1}f(u)du\Bigg] \\
=\;&\frac{\alpha}{4}B(\alpha-n,n+1)-f(b)\frac{\alpha 4^{\alpha-1}n!}{(b-a)^\alpha}\left(I_\alpha^{\frac{a+3b}{4}}f\right)(b).
\end{aligned}
$$

Adding the above quantities, desired result is obtained. So, the proof is completed.

Remark 2.1. If we choose $\alpha = n+1$ in the equality (2.2), then Lemma 2.1 reduces to the Lemma 2.1 in [21].

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I^o such that $f' \in L([a,b])$ with $a, b \in I$, $a < b$ and $\alpha > 0$. If $|f'|^q$ is convex function on $[a,b]$ and $q \geq 1$, then we have the following inequality:*

$$\begin{aligned}
&|\Psi_\alpha(a,b)| \\
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[(B(n+1,\alpha-n+1))^{1-\frac{1}{q}}\left(\Phi_1^{\frac{1}{q}}+\Phi_3^{\frac{1}{q}}\right) \\
&+(B(n+2,\alpha-n))^{1-\frac{1}{q}}\left(\Phi_2^{\frac{1}{q}}+\Phi_4^{\frac{1}{q}}\right)\Bigg],
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
\Phi_1 &= \left(\frac{1}{2}B(n+1,\alpha-n)-\frac{1}{2}B(n+3,\alpha-n)\right)|f'(a)|^q \\
&\quad+\left(\frac{1}{2}B(n+1,\alpha-n+2)\right)\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \\
\Phi_2 &= \left(\frac{1}{2}B(\alpha-n+2,n+1)\right)\left|f'\left(\frac{3a+b}{4}\right)\right|^q \\
&\quad+\left(\frac{1}{2}B(\alpha-n,n+1)-\frac{1}{2}B(\alpha-n,n+3)\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q, \\
\Phi_3 &= \left(\frac{1}{2}B(n+1,\alpha-n)-\frac{1}{2}B(n+3,\alpha-n)\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q \\
&\quad+\left(\frac{1}{2}B(n+1,\alpha-n+2)\right)\left|f'\left(\frac{a+3b}{4}\right)\right|^q, \\
\Phi_4 &= \left(\frac{1}{2}B(\alpha-n+2,n+1)\right)\left|f'\left(\frac{a+3b}{4}\right)\right|^q \\
&\quad+\left(\frac{1}{2}B(\alpha-n,n+1)-\frac{1}{2}B(\alpha-n,n+3)\right)|f'(b)|^q
\end{aligned}$$

and $\alpha\in(n,n+1]$, $n=0,1,2,\ldots$, $B(a,b)$ *is Euler beta function.*

Proof. Using Lemma 2.1, well known power mean inequality and the convexity of $|f'|^q$ on $[a,b]$, we get

$$\begin{aligned}
&|\Psi_\alpha(a,b)| \\
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|dt \\
&+\int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \\
&+\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|dt \\
&+\int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|dt\Bigg]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[\left(\int_0^1 B_t(n+1,\alpha-n)dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 B_t(n+1,\alpha-n)\left(t\left|f'(a)\right|^q+(1-t)\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 B_{1-t}(\alpha-n,n+1)dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 B_{1-t}(\alpha-n,n+1)\left(t\left|f'\left(\frac{3a+b}{4}\right)\right|^q+(1-t)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 B_t(n+1,\alpha-n)dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 B_t(n+1,\alpha-n)\left(t\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 B_{1-t}(\alpha-n,n+1)dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 B_{1-t}(\alpha-n,n+1)\left(t\left|f'\left(\frac{a+3b}{4}\right)\right|^q+(1-t)\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

In fact notice that using the properties of beta functions and integrating by parts formula we can write;

$$
\int_0^1 B_t(n+1,\alpha-n)dt = B(n+1,\alpha-n+1), \tag{2.4}
$$

$$
\int_0^1 B_{1-t}(\alpha-n,n+1)dt = B(n+2,\alpha-n). \tag{2.5}
$$

Substituting

$$
\begin{aligned}
\Phi_1 &= \int_0^1 B_t(n+1,\alpha-n)t|f'(a)|^q dt \\
&\quad+\int_0^1 B_t(n+1,\alpha-n)(1-t)\left|f'\left(\frac{3a+b}{4}\right)\right|^q dt \\
&= \left(B_t(n+1,\alpha-n)\frac{t^2}{2}\Big|_0^1-\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{t^2}{2}dt\right)|f'(a)|^q \\
&\quad+\Bigg(B_t(n+1,\alpha-n)\frac{-(1-t)^2}{2}\Big|_0^1 \\
&\quad-\int_0^1 t^n(1-t)^{\alpha-n-1}\frac{-(1-t)^2}{2}dt\Bigg)\left|f'\left(\frac{3a+b}{4}\right)\right|^q \\
&= \left(\frac{1}{2}B(n+1,\alpha-n)-\frac{1}{2}B(n+3,\alpha-n)\right)|f'(a)|^q \\
&\quad+\left(\frac{1}{2}B(n+1,\alpha-n+2)\right)\left|f'\left(\frac{3a+b}{4}\right)\right|^q,
\end{aligned}
$$

$$
\begin{aligned}
\Phi_2 &= \int_0^1 B_{1-t}(\alpha-n,n+1)t\left|f'\left(\frac{3a+b}{4}\right)\right|^q dt \\
&\quad + \int_0^1 B_{1-t}(\alpha-n,n+1)(1-t)\left|f'\left(\frac{a+b}{2}\right)\right|^q dt \\
&= \left(B_{1-t}(\alpha-n,n+1)\frac{t^2}{2}\Big|_0^1 + \int_0^1 (1-t)^n t^{\alpha-n-1}\frac{t^2}{2}dt\right)\left|f'\left(\frac{3a+b}{4}\right)\right|^q \\
&\quad + \Bigg(B_{1-t}(\alpha-n,n+1)\frac{-(1-t)^2}{2}\Big|_0^1 \\
&\quad + \int_0^1 (1-t)^n t^{\alpha-n-1}\frac{-(1-t)^2}{2}dt\Bigg)\left|f'\left(\frac{a+b}{2}\right)\right|^q \\
&= \left(\frac{1}{2}B(\alpha-n+2,n+1)\right)\left|f'\left(\frac{3a+b}{4}\right)\right|^q \\
&\quad + \left(\frac{1}{2}B(\alpha-n,n+1)-\frac{1}{2}B(\alpha-n,n+3)\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q, \\
\Phi_3 &= \int_0^1 B_t(n+1,\alpha-n)t\left|f'\left(\frac{a+b}{2}\right)\right|^q dt \\
&\quad + \int_0^1 B_t(n+1,\alpha-n)(1-t)\left|f'\left(\frac{a+3b}{4}\right)\right|^q dt \\
&= \left(B_t(n+1,\alpha-n)\frac{t^2}{2}\Big|_0^1 - \int_0^1 t^n(1-t)^{\alpha-n-1}\frac{t^2}{2}dt\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q \\
&\quad + \Bigg(B_t(n+1,\alpha-n)\frac{-(1-t)^2}{2}\Big|_0^1 \\
&\quad - \int_0^1 t^n(1-t)^{\alpha-n-1}\frac{-(1-t)^2}{2}dt\Bigg)\left|f'\left(\frac{a+3b}{4}\right)\right|^q \\
&= \left(\frac{1}{2}B(n+1,\alpha-n)-\frac{1}{2}B(n+3,\alpha-n)\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q \\
&\quad + \left(\frac{1}{2}B(n+1,\alpha-n+2)\right)\left|f'\left(\frac{a+3b}{4}\right)\right|^q, \\
\Phi_4 &= \int_0^1 B_{1-t}(\alpha-n,n+1)t\left|f'\left(\frac{a+3b}{4}\right)\right|^q dt \\
&\quad + \int_0^1 B_{1-t}(\alpha-n,n+1)(1-t)\left|f'(b)\right|^q dt \\
&= \left(B_{1-t}(\alpha-n,n+1)\frac{t^2}{2}\Big|_0^1 + \int_0^1 (1-t)^n t^{\alpha-n-1}\frac{t^2}{2}dt\right)\left|f'\left(\frac{a+3b}{4}\right)\right|^q \\
&\quad + \Bigg(B_{1-t}(\alpha-n,n+1)\frac{-(1-t)^2}{2}\Big|_0^1 \\
&\quad + \int_0^1 (1-t)^n t^{\alpha-n-1}\frac{-(1-t)^2}{2}dt\Bigg)\left|f'(b)\right|^q \\
&= \left(\frac{1}{2}B(\alpha-n+2,n+1)\right)\left|f'\left(\frac{a+3b}{4}\right)\right|^q
\end{aligned}
$$

$$+\left(\frac{1}{2}B(\alpha-n,n+1)-\frac{1}{2}B(\alpha-n,n+3)\right)|f'(b)|^q$$

into the above inequality and simplifying result via the equality (2.4) and (2.5) in the required inequality. So, the proof is completed.

Remark 2.2. If we choose $\alpha = n+1$ in the inequality (2.3), then Theorem 2.1 reduces to the special case of Corollary 3.1 for $\alpha_1 = m = 1$ in [21].

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I^o such that $f' \in L([a,b])$ with $a, b \in I$, $a < b$ and $\alpha > 0$. If $|f'|^q$ is convex function on $[a,b]$, $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then we have the following inequality:*

$$\begin{aligned}
&|\Psi_\alpha(a,b)| \\
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}} \\
&\times\left(\frac{|f'(a)|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}} \\
&\times\left(\frac{\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+3b}{4}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

for $\alpha \in (n, n+1]$, $n = 0, 1, 2, \dots$ where $B_t(.,.)$ is incompleted beta function.

Proof. Using Lemma 2.1, well known Hölder inequality and the convexity of $|f'|^q$ on $[a,b]$, we get

$$\begin{aligned}
&|\Psi_\alpha(a,b)| \\
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|dt \\
&+\int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \\
&+\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|dt \\
&+\int_0^1 B_{1-t}(\alpha-n,n+1)\left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|dt\Bigg] \\
\leq\ & \frac{(b-a)\alpha}{16}\Bigg[\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg].$$

Substituting

$$\begin{aligned}
&\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(ta+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t|f'(a)|^q+(1-t)\left|f'\left(\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\\
=\ &\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1 t\left|f'\left(\frac{3a+b}{4}\right)\right|^q+(1-t)\left|f'\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
=\ &\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1 t\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f'\left(\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\\
=\ &\left(\int_0^1 |B_t(n+1,\alpha-n)|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(t\frac{a+3b}{4}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(\frac{a+3b}{4}\right)\right|^q+(1-t)\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\\
=\ &\left(\int_0^1 |B_{1-t}(\alpha-n,n+1)|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{a+3b}{4}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

into the above inequality and simplifying lead to the required inequality. So, the proof is completed.

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