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Fixed point theorems for nonself Bianchini type contractions in Banach spaces endowed with a graph

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ABSTRACT. In this paper we present an extension of fixed point theorem for self mappings on metric spaces endowed with a graph and which satisfies a Bianchini contraction condition. We establish conditions which ensure the existence of fixed point for a non-self Bianchini contractions $T : K \subset X \to X$ that satisfy Rothe's boundary condition $T(\partial K) \subset K$.

1. INTRODUCTION

Starting with well-known Banach contraction principle (see its complete from in [19]), many directions have approached to study the existence of fixed points of a map T. We remember that, for a map $T : X \to X$ the set of fixed point is

$$Fix\,(T) = \{x \in X;\, Tx = x\}\,,$$

where X is a nonempty set. Roughly speaking, the existence conditions of fixed points are a set of rules which reflect the relations between the distances from one element to another of the set $\{x, y, Tx, Ty\}$ and some properties of the map T in the space X. In general, (X, d) is a complete metric space, $T : X \to X$ is a self-mapping which has some specific properties. For example:
a) classical Banach contraction condition

$$d\,(Tx, Ty) \le a \cdot d\,(x, y) \text{ for all } x, y \in X;$$

b) ([45]) Kannan contraction condition

$$d\,(Tx, Ty) \le b\,[d\,(x, Tx)\,, d\,(y, Ty)] \text{ for all } x, y \in X;$$

c) ([24]) Bianchini contraction condition

$$d\,(Tx, Ty) \le a \cdot \max\{d\,(x, Tx)\,, d\,(y, Ty)\} \text{ for all } x, y \in X;$$

d) Rus-Reich contraction condition

$$d\,(Tx, Ty) \le \alpha \cdot d\,(x, Tx) + \beta \cdot d\,(y, Ty) + \gamma d\,(x, y) \text{ for all } x, y \in X;$$

and so on, where $a \in [0, 1)$, $b \in \left[0, \frac{1}{2}\right)$, respectively $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$.

In all these existence results, T is a self-mapping. More details can be found in literature, see [19], [32], [73] and reference therein.

The study of non-self mappings started with the paper of J. Caristi, see [30] for details. The assumption that $T : K \to X$ is non-self, i.e., T maps a subset K of X not into itself and there is at least one $x \in K$ such that $Tx \in X \setminus K$, implies some supplementary conditions which must hold on the boundary of subset K. A list of some type of these conditions can be found in [42]. In this paper we choose the Rothe's boundary condition $T(\partial K) \subset K$. There are a few other results related to the existence of fixed points for

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non-self maps, remind here two fixed point theorems for non self contractions defined on Banach spaces endowed with a graph established by M. Păcurar in [21], while very recently in [15] was extend these results to non-self Kannan type contractions on Banach spaces endowed with a graph. The study of set of fixed points of mappings defined on Banach space endowed with a graph was initiated by J. Jachymski in [44] and continued by work of F. Bojor [25, 26, 27, 28, 29] and others [1], [33] etc.

The present work is organized in two sections. In the first one we remind a few preliminary notions and results, basically taken from [20], regarding the fixed point results for mappings defined on metric spaces endowed with a graph. In the second section there is an existence result of fixed point for an non-self mapping which satisfies a Bianchini contraction condition and is defined on metric spaces endowed with a graph.

2. METRIC SPACES ENDOWED WITH A GRAPH

Let (X, d) be a metric space and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider now a directed simple graph $G = (V(G), E(G))$ such that the set of its vertices, $V(G)$, coincides with X and $E(G)$, the set of its edges, contains all loops, i.e., $\Delta \subset E(G)$.

By G^{-1} we denote the *converse graph* of G, i.e., the graph obtained by G by reversing its edges, i.e.,

$$E(G^{-1}) = \{(y, x) \in X \times X : (x, y) \in E(G)\}.$$

If $x, y \in V(G)$ are vertices in the graph G, then a *path* from x to y of length $N \in \mathbb{N}$ is a sequence $\{x_i\}_{i=1}^{N}$ of $N+1$ vertices of G such that

$$x_0 = x,\ x_N = y \text{ and } (x_{i-1}, x_i) \in E(G),\ i = 1, 2, \ldots, N.$$

A graph G is said to be *connected* if there is at least a path between any two vertices. If $\tilde{G} = (X, E(\tilde{G}))$ is the symmetric graph obtained by putting together the vertices of both G and G^{-1}, i.e.,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}),$$

then G is called *weakly connected* if $\tilde{G}$ is connected. If $G = (V(G),\ E(G))$ is a graph and $H \subset V(G)$, then the graph $(H, E(H))$ with $E(H) = E(G) \cap (H \times H)$ is called the *subgraph of G determined by* H. Denote it by G_H.

Definition 2.1. Let (X, d, G) be a Banach space endowed with a simple directed and weakly connected graph G. We say that the property (L) holds if

$$\begin{cases} \text{for any sequence } \{x_n\}_{n=1}^{\infty} \subset X \text{ with } x_n \to x \text{ as } n \to \infty \\ \text{and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N}, \\ \text{there exists a subsequence } \{x_{k_n}\}_{n=1}^{\infty} \text{ satisfying } (x_{k_n}, x) \in E(G), \text{ for all } n \in \mathbb{N}. \end{cases} \tag{L}$$

A mapping $T : X \to X$ is said to be (well) defined on a metric space endowed with a graph G if it has the property

$$\forall x, y \in X,\ (x, y) \in E(G) \text{ implies } (Tx, Ty) \in E(G). \tag{2.1}$$

For a non self mapping $T : K \to X$ we shall say that it is (well) defined on the Banach space X endowed with the graph G if it has this property for the subgraph of G induced by K, that is,

$$(x, y) \in E(G) \text{ with } Tx, Ty \in K \text{ implies } (Tx, Ty) \in E(G) \cap (K \times K), \tag{2.2}$$

for all $x, y \in K$.

According to [44], a mapping $T : X \to X$, which is well defined on a metric space endowed with a graph G, is called a *G-contraction* if there exists a constant $\alpha \in (0,1)$ such that for all $x, y \in X$ with $(x,y) \in E(G)$ we have

$$d(Tx, Ty) \leq \alpha \cdot d(x,y). \tag{2.3}$$

Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. If $x \in K$ is such that $Tx \notin K$, then we can always choose an $y \in \partial K$ (the boundary of K) such that $y = (1-\lambda)x + \lambda Tx\,(0 < \lambda < 1)$, which actually expresses the fact that

$$d(x,Tx) = d(x,y) + d(y,Tx),\ y \in \partial K = Fr\,(K)\,, \tag{2.4}$$

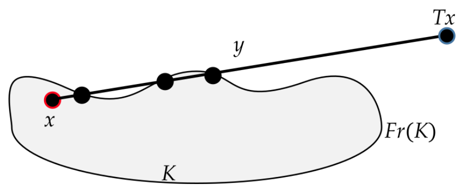


FIGURE 1

where we use the notation

$$d(x,y) = \|x - y\|\,.$$

In general, the set Y of points y satisfying condition (2.4) from above may contain more than one element.
We suppose Y is always nonempty.

In this context we shall need the following important concept first introduced and used in [20].

Definition 2.2. ([20]) Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. We say that T has property (M) if

$$\begin{cases} \text{for any elements } x \in K \text{ with } Tx \notin K \text{ the inequality} \\ \qquad\qquad d(y,Ty) \leq d(x,Tx) \\ \text{holds for at least one corresponding } y \in Y \subset \partial K \text{ given by (2.4).} \end{cases} \tag{M}$$

Examples of non-self mapping T which has property (M) can be found in work of V. Berinde and M. Păcurar (see [20], [21]) or in the next example.

Example 2.1. Let $K = [0,1] \times [0,1]$ be a subset of $X = \mathbb{R}^2$, where X is endowed with the Chebyshev metric, i.e., $d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X. Consider the map $T : K \to X$ given by $Tx = T(x_1, x_2) = (-x_1, x_2)$ for all $x = (x_1, x_2) \in K$. Remark that $Tx = x$ for any $x \in \{(0,b) : b \in [0,1]\} := K_0$, $K_0 \subset \partial K$ and $Tx \notin K$ for all $x \in K \setminus K_0$. Since

$$d_\infty(x, Tx) = 2x_1 \text{ for all } x = (x_1, x_2) \in K$$

and

$$d_\infty(y, Ty) = 0 \text{ for all } y \in K_0,$$

we have

$$d_\infty(y, Ty) < d_\infty(x, Tx)$$

for all $x \in K \setminus K_0$ with $y \in K_0 = Y$. So, T has property (M) and Y is not singleton.

In the next example we can fond another non-self mapping which satisfies some contraction condition and the Rothe's boundary condition, which has the (M) properties and iterations sequence converges to fixed point of it.

Example 2.2. Let $X = \mathbb{R}^2$ endowed with the metric

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \text{ for all } x = (x_1, x_2), y = (y_1, y_2) \in X$$

and $K = [-1, 1] \times [-1, 1]$, i.e.,

$$K = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} \leq 1\right\} \subset X.$$

Let

$K_1 = \{(x_1, x_2) \in K : 0 > x_2 \geq x_1\}$

$K_2 = \{(x_1, x_2) \in K : 0 > x_1 > x_2\}$

and

$$K_3 = K \setminus (K_1 \cup K_2).$$

Let $T : K \to X$ given by

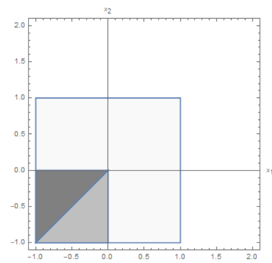
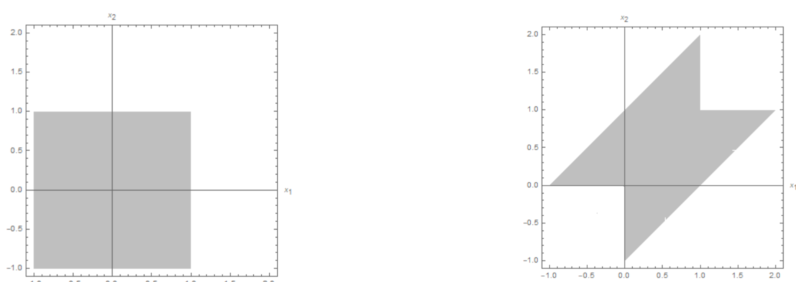


FIGURE 2. The set K and the three subsets K_1(the gray one), K_2 (the dark one) and K_3 with $K = K_1 \cup K_2 \cup K_3$.

$$Tx = T(x_1, x_2) = \begin{cases} (2x_1 + 2, 2x_2 + 1), & (x_1, x_2) \in K_1 \\ (2x_1 + 1, 2x_2 + 2), & (x_1, x_2) \in K_2 \\ (\alpha x_2, \alpha x_1), & (x_1, x_2) \in K_3 \end{cases} \tag{2.5}$$

where $\alpha = \frac{1}{2} \in (0, 1)$.



The set K The set $T(K)$

FIGURE 3. The set K and its image by the map T.

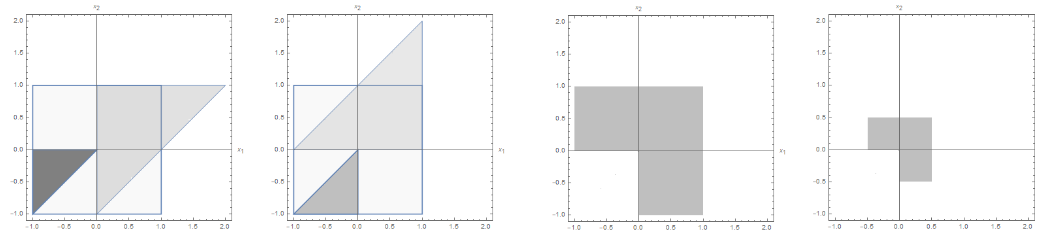
Note here some properties of T defined by (2.5).

I. The map T has a unique fix point, $Fix(T) = \{(0, 0)\}$.

II. The Rothe's boundary condition $T(\partial K) \subset K$ holds, where

$$\partial K = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} = 1\right\}.$$

III. T is a non-self map, i.e., there are $x \in K$ such that $Tx \notin K$. Practically, for every $x = (x_1, x_2) \in K_1$ with $x_1 \in \left(-\frac{1}{2}, 0\right)$ we have $Tx = (2x_1 + 2, 2x_2 + 1)$ with $2x_1 + 2 > 1$, so $Tx \notin K$.



K_1 and $T(K_1)$ $\quad$ K_2 and $T(K_2)$ $\quad$ K_3 $\quad$ $T(K_3)$

FIGURE 4. The subsets K_1, K_2, K_3 of K and their image by the map T.

IV. T has property (M). Indeed, if $x = \left(x_1, \frac{1}{2}x_1\right) \in K_1$ with $x_1 \in \left(-\frac{1}{3}, 0\right)$, then for any $y = (1, \varepsilon) \in \partial K$ with $\varepsilon \in (-x_1, 1)$ the equality

$$d(x, y) + d(y, Tx) = d(x, Tx) \tag{2.6}$$

holds. Much more, if $x = \left(x_1, \frac{1}{2}x_1\right)$ with $x_1 \in \left(-\frac{1}{2}, -\frac{1}{3}\right)$, then for any $y = (1, \varepsilon) \in \partial K$ with $\varepsilon \in (-x_1, 2 + 3x_1)$ we have $d(x, y) = 1 - x_1$, $d(x, Tx) = 2 + x_1$ and $d(y, Tx) = 1 + 2x_1$ for $y = (1, \varepsilon) \in \partial K$. So, the equality (2.6) holds. Hence, T has property (M) and for some x the corresponding set Y is not singleton.
A similar result holds for some specific $x \in K_2$.

V. T satisfies a contraction condition.
A) For $x = (x_1, x_2) \in K_1$ and $y = (y_1, y_2) \in K_1$ we have $d(Tx, Ty) = 2 \cdot d(x, y)$, $d(y, Ty) = y_1 + 2$ and

$$d(x, Tx) = \max\{|x_1 + 2|, |x_2 + 1|\} = \frac{|x_1 + 2| + |x_2 + 1| + |x_1 + 2 - x_2 - 1|}{2} = x_1 + 2.$$

B) For $x = (x_1, x_2) \in K_2$ and $y = (y_1, y_2) \in K_2$ we have $d(Tx, Ty) = 2 \cdot d(x, y)$, $d(y, Ty) = y_2 + 2$ and

$$d(x, Tx) = \max\{|x_1 + 1|, |x_2 + 2|\} = \frac{|x_1 + 1| + |x_2 + 2| + |x_1 + 1 - x_1 - 2|}{2} = x_2 + 2.$$

VI. Picard iterations of T.

On K_3 the map T is a classical contraction and, as we can see in first representation from Figure 5, the sequence of Picard iteration converges.

If the initial point of the Picard iterations is situated in K_1, $x_0 \in K_1$, then there are two possible situations: first one has $Tx_0 \in K_3$ and the sequence of Picard iterations converges and second one has $Tx_0 \notin K$, but we can choose $x_1 \in \partial K \cap K_3$, so Picard iterations converges, too. Such situation is depicted on the left image from Figure 5.

A similar situation can be obtain if we choose the initial point from K_2, see the right image from Figure 5.

3. FIXED POINT THEOREM FOR NONSELF BIANCHINI TYPE CONTRACTIONS IN BANACH SPACES ENDOWED WITH A GRAPH

In that follow, we establish some conditions which ensure that a nonself Bianchini type contraction has a fixed point in (X, d, G), a Banach space endowed with a simple directed and weakly connected graph.

Let $K \subset X$ a nonempty closed subset of X. We say that $T : K \to X$ is a *Bianchini contraction* if there exists a constant $a \in [0, 1)$ such that

$$d(Tx, Ty) \leq a \cdot \max\{d(x, Tx), d(y, Ty)\}, \text{for all } (x, y) \in E(G_K), \tag{3.7}$$

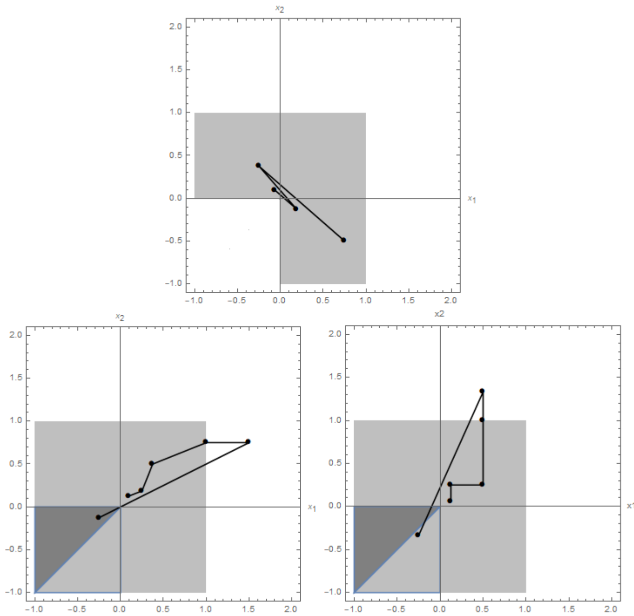


FIGURE 5. Example of Picard iterations.

where G_K is the subgraph of G determined by K.

If T maps K into K, i.e., T is self-mapping, then there are hypotheses which implie that $Fix\,(T) \neq \emptyset$. Such rezults are Bianchini fixed point theorems [24] with $X = K$, some theorems establised by F. Bojor [26] and other authors.

The next theorem establishes a fixed point theorem for non-self Bianchini contractions defined on a Banach space endowed with a graph.

Theorem 3.1. *Let (X, d, G) be a Banach space endowed with a simple directed and weakly connected graph G such that the property (L) holds. Let K be a nonempty closed subset of X and let $T : K \to X$ be a Bianchini contraction. If $K_T := \{x \in \partial K : (x, Tx) \in E(G)\} \neq \emptyset$, T has property (M) and T satisfies the Rothe's boundary condition $T(\partial K) \subset K$, then*

(i) $Fix\,(T) = \{x^*\}$;

(ii) *Picard iteration $\{x_n = T^n x_0\}_{n=1}^{\infty}$ converges to x^*, for all $x_0 \in K_T$, and the following estimate holds*

$$d\,(x^*, x_n) \leq \frac{a^{n-1}}{a-1} \cdot \max\left\{d\,(x_0, x_1)\,, d\,(x_1, x_2)\right\}, \quad n = 0, 1, 2, \ldots. \tag{3.8}$$

Proof. If $T\,(K) \subset K$, then T is self-mapping and the prove is given by Bianchini fixed point theorem, see [24]. Therefore, we consider only the case $T(K) \cap (X \setminus K) \neq \emptyset$, i.e., there is at least one $x \in K$ such that $Tx \in X \setminus K$. The proof has two parts: first we construct the Picard iteration and establish some properties of this sequence and after that we prove that Picard iteration is a Cauchy sequence.

Let $x_0 \in K_T$. Since $(x_0, Tx_0) \in E(G)$ and T is well defined on a metric space endowed with the graph G, then we have $(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$.

In the following, we construct the Picard iteration $\Xi := \{x_n\}_{n \geq 1}$. First, we denote $x_1 = y_1 = Tx_0$ and for $n \geq 2$ we proceed in the following way: if $Tx_{n-1} \in K$ then $x_n = y_n = Tx_{n-1}$, else $Tx_{n-1} \notin K$ and we can choose a $\lambda_n \in (0, 1)$ such that

$$x_n = (1 - \lambda_n)\,x_{n-1} + \lambda_n T x_{n-1} \in \partial K.$$

Now, we can consider two disjoint subsets of the Picard iteration Ξ. One set is

$$P = \{x_k \in \Xi;\ x_k = y_k = Tx_{k-1}, k \in N_P \subset \mathbb{N}\} \subset K$$

and other one is

$$Q = \{x_k \in \Xi;\ x_k \neq Tx_{k-1}, k \in N_Q \subset \mathbb{N}\} \subset \partial K.$$

By virtue of Rothe's boundary condition, in Q there is no two consecutive terms of Ξ, but in P we can have consecutive terms of Ξ. So, for a given $n \in \mathbb{N}$ we can have the following three hypothetical cases: (1) $x_n, x_{n+1} \in P$; (2) $x_n \in P$, $x_{n+1} \in Q$ and (3) $x_n \in Q$, $x_{n+1} \in P$.

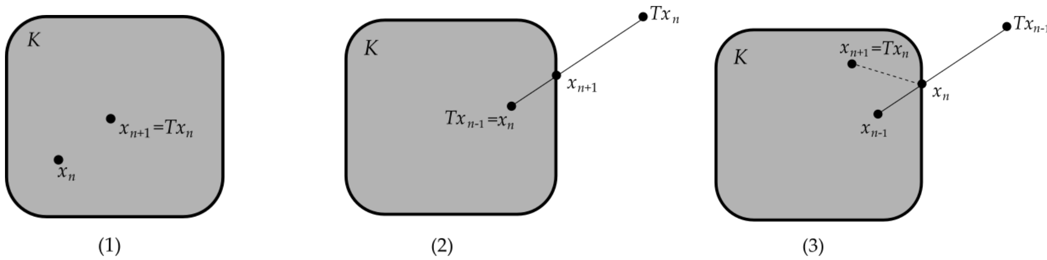


FIGURE 6. The three hypothetical cases:
(1) $x_n, x_{n+1} \in P$; (2) $x_n \in P$, $x_{n+1} \in Q$; (3) $x_n \in Q$, $x_{n+1} \in P$.

Next, we study the influence of Bianchini condition (3.7) upon the distance between the terms of Ξ in all three cases from above.

Case 1. Assume that $x_n, x_{n+1} \in P$ and

$$\max\{d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} = d(x_n, Tx_n).$$

Since $d(x_n, Tx_n) = d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$, by (3.7) we have

$$d(x_n, x_{n+1}) \leq a \cdot d(x_n, x_{n+1})$$

which implies the inequality

$$(1-a) \cdot d(x_n, x_{n+1}) \leq 0$$

and this cannot be hold. So, this situation can not occurs.

Assume that $x_n, x_{n+1} \in P$ and

$$\max\{d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} = d(x_{n-1}, Tx_{n-1}) = d(x_{n-1}, x_n).$$

In this case, $d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1})$ and (3.7) implies

$$d(x_n, x_{n+1}) \leq a \cdot d(x_{n-1}, x_n). \tag{3.9}$$

Case 2. Assume that $x_n \in P$ and $x_{n+1} \in Q$. Hence, there is $\lambda_{n+1} \in (0,1)$ such that

$$x_{n+1} = (1-\lambda_{n+1}) x_n + \lambda_{n+1} Tx_n \in \partial K,$$

which actually express the fact the

$$d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n).$$

Hence, we have

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) - d(x_{n+1}, Tx_n) \leq d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n).$$

Now, by (3.7) we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq a \cdot \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &\leq a \cdot \max\{d(x_{n-1}, x_n), d(x_n, Tx_n)\}. \end{aligned} \tag{3.10}$$

If we consider that $\max\{d(x_{n-1},x_n), d(x_n,Tx_n)\} = d(x_{n-1},x_n)$, then (3.10) is equivalent to (3.9). On the other hand, if $\max\{d(x_{n-1},x_n), d(x_n,Tx_n)\} = d(x_n,Tx_n)$ then (3.10) implies

$$\begin{aligned} d(x_n,Tx_n) = d(x_n,x_{n+1}) &\le a\cdot d(x_n,Tx_n) = a\cdot d(Tx_{n-1},Tx_n) \\ &\le a^2\cdot\max\{d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n)\} \\ &= a^2\cdot d(x_n,Tx_n). \end{aligned}$$

Since $a\in(0,1)$, the last inequality implies $a\cdot d(x_n,Tx_n) < a^2\cdot d(x_n,Tx_n)$ which is equivalent to

$$a(1-a)\cdot d(x_n,Tx_n) < 0$$

and this cannot be hold in our hypotheses.

Case 3. Assume that $x_n\in Q$ and $x_{n+1}\in P$, i.e., $x_n\neq Tx_{n-1}=y_n$, $x_n\in\partial K$ and $x_{n+1}=Tx_n$. In this case, $0<d(x_{n+1},x_n)=d(x_n,Tx_n)$ and the property (M) implies

$$d(x_n,Tx_n)\le d(x_{n-1},Tx_{n-1}) = d(Tx_{n-2},Tx_{n-1}). \tag{3.11}$$

Hence, by (3.7) we obtain

$$d(x_{n+1},x_n)\le a\cdot\max\{d(x_{n-2},Tx_{n-2}), d(x_{n-1},Tx_{n-1})\}. \tag{3.12}$$

Now, if we assume that

$$\max\{d(x_{n-2},Tx_{n-2}), d(x_{n-1},Tx_{n-1})\} = d(x_{n-2},Tx_{n-2}) = d(x_{n-2},x_{n-1}),$$

then $d(x_n,x_{n+1})\le a\cdot d(x_{n-2},x_{n-1})$.

Else, if we consider that

$$\max\{d(x_{n-2},Tx_{n-2}), d(x_{n-1},Tx_{n-1})\} = d(x_{n-1},Tx_{n-1}) = d(Tx_{n-2},Tx_{n-1}),$$

then

$$\begin{aligned} d(x_n,x_{n+1}) &\le a\cdot d(Tx_{n-2},Tx_{n-1}) \\ &\le a^2\cdot\max\{d(x_{n-2},Tx_{n-2}), d(x_{n-1},Tx_{n-1})\} \\ &= a^2\cdot d(x_{n-1},Tx_{n-1}) = a^2\cdot d(Tx_{n-2},Tx_{n-1}). \end{aligned}$$

This implies $d(x_n,x_{n+1})\le a^k\cdot d(Tx_{n-2},Tx_{n-1})$ for any $k\in\mathbb{N}$ and this cannot occurs.

At the end of the first part of the proof, we can say that for the elements from the iterative sequence $\Xi=\{x_n\}_{n\ge 0}$ the following inequality holds

$$d(x_n,x_{n+1})\le a\cdot\max\{d(x_{n-1},x_n), d(x_{n-2},x_{n-1})\}, \tag{3.13}$$

for all $n\ge 2$. Now, using consecutively these inequalities (3.13), we obtain

$$d(x_n,x_{n+1})\le a^{n-1}\cdot\max\{d(x_0,x_1), d(x_1,x_2)\},\ n\ge 2. \tag{3.14}$$

In the second part of this proof, we show that Ξ is Cauchy sequence and T has at least one fixed point. For any $n,p\in\mathbb{N}$ we have

$$d(x_n,x_{n+p})\le\sum_{k=0}^{p-1} d(x_{n+k},x_{n+k+1}).$$

Now, by (3.14) we obtain

$$d(x_n, x_{n+p}) \leq \sum_{k=0}^{p-1} a^{n+k-1} \cdot \max\{d(x_0, x_1), d(x_1, x_2)\} \tag{3.15}$$

$$= \frac{a^{n-1}(1-a^p)}{1-a} \cdot \max\{d(x_0, x_1), d(x_1, x_2)\}$$

$$< \frac{a^{n-1}}{a-1} \cdot \max\{d(x_0, x_1), d(x_1, x_2)\},$$

for any $p \in \mathbb{N}$ and this shows that $\{x_n\}_{n\geq 1}$ is Cauchy sequence in closed set K. So, the sequence $\{x_n\}_{n\geq 1}$ converges to some point x^* in K. By triangle inequality we have

$$d(x^*, Tx^*) \leq d(x^*, y) + d(y, Tx^*), \; y \in K. \tag{3.16}$$

Property (L) implies there is a subsequence $\{x_{k_n}\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ satisfying

$$(x_{k_n}, x^*) \in E(G) \text{ for all } n \in \mathbb{N}.$$

Hence, if we choose $y = x_{k_n+1} = Tx_{k_n}$, then (3.16) implies

$$d(x^*, Tx^*) \leq d(x^*, x_{k_n+1}) + d(x_{k_n+1}, Tx^*). \tag{3.17}$$

By Bianchini's type contraction condition (3.7) we have

$$d(x_{k_n+1}, Tx^*) \leq a \cdot \max\{d(x_{k_n}, Tx_{k_n}), d(x^*, Tx^*)\}. \tag{3.18}$$

So, the inequalities (3.17) and (3.18) imply

$$d(x^*, Tx^*) \leq d(x^*, x_{k_n+1}) + a \cdot \max\{d(x_{k_n}, Tx_{k_n}), d(x^*, Tx^*)\}.$$

Therefore, we can estimate the distance between x^* and Tx^* by

$$d(x^*, Tx^*) \leq \frac{1}{1-a} \cdot d(x^*, x_{k_n+1}) + \frac{a}{1-a} \cdot d(x_{k_n}, Tx_{k_n}) \text{ for all } n \geq 1.$$

and by (3.14) we obtain

$$d(x^*, Tx^*) \leq \frac{1}{1-a} \cdot d(x^*, x_{k_n+1}) + \frac{a^n}{1-a} \cdot \max\{d(x_{k_0}, x_{k_1}), d(x_{k_1}, x_{k_2})\}. \tag{3.19}$$

Letting now $n \to \infty$ in (3.19), results $d(x^*, Tx^*) = 0$, which shows that x^* is a fixed point of T.

Letting now $p \to \infty$ in (3.15), results the error estimate given by (3.8). □

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