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Constant ratio timelike curves in pseudo-Galilean 3-space $\mathbb{G}_3^1$

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ABSTRACT. In this paper, we consider unit speed timelike curves in pseudo-Galilean 3-space $\mathbb{G}_3^1$ as curves whose position vectors can be written as linear combination of their Serret-Frenet vectors. We obtain some results of constant ratio curves and give an example of these curves. Further, we show that there is no T-constant curve and we obtain some results of N-constant type of curves in pseudo-Galilean 3-space $\mathbb{G}_3^1$.

1. INTRODUCTION

The basic concepts of Euclidean plane geometry are points and straight lines. But in nature, every surface is not a plane and every line is not a straight line. Ömer Hayyam and Tusi were the first scholars studying Euclid's postulate. However, in the 19th century by C. F. Gauss, N. I. Lobachevsky and J. Bolyai, non-Euclidean geometries were set forth with the discovery of hyperbolic geometry, which accepts a new postulate (infinite number of parallels can be drawn to a line from a point outside the given line) instead of parallel postulate. G. F. B. Riemann laid the foundations of a new geometry called the elliptic geometry afterwards. F. Klein generalized those geometries, and showed the existence of the nine geometries including the Euclidean, hyperbolic and elliptic ones [20]. Galilean geometry is a non-Euclidean geometry and associated with Galilei principle of relativity. This principle can be explained briefly as " in all inertial frames, all law of physics are the same." (Except for the Euclidean geometry in some cases), Galilean geometry is the easiest of all Klein geometries, and it is revelant to the theory of relativity of Galileo and Einstein. For a comprehensive study of Galilean geometry, one can have a look at the studies of Yaglom [21] and Röschel [19].

In [17], the author explained the projective signature (0,0,+,-) of the pseudo-Galilean geometry which is one of the real Cayley-Klein geometries. Pseudo-Galilean space $\mathbb{G}_3^1$ has been explained in details [9, 10]. Furthermore many works related to pseudo-Galilean space have been done by [1, 12, 16] etc...

For a regular curve $\alpha(x)$, the position vector α can be decomposed into its tangential and normal components at each point:

$$\alpha = \alpha^T + \alpha^N. \tag{1.1}$$

A curve α in $\mathbb{E}^n$ or in $\mathbb{E}_t^n$ is said to be of *constant ratio* if the ratio $\left\|\alpha^T\right\| : \left\|\alpha^N\right\|$ is constant on $\alpha(I)$ where $\left\|\alpha^T\right\|$ and $\left\|\alpha^N\right\|$ denote the length of α^T and α^N, respectively [5, 6].

Moreover, a curve in $\mathbb{E}^n$ or in $\mathbb{E}_t^n$ is called *T-constant* (*N-constant*) if the tangential component α^T (the normal component α^N) of its position vector α is of constant length [7, 8].

Recently, in [13, 18], the authors give the necessary and sufficient conditions for curves in Euclidean spaces to become T-constant and N-constant. In [2, 3, 15], authors study the

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same curves according to Bishop frame in $\mathbb{E}^3$, $\mathbb{E}^3_1$, and $\mathbb{E}^4$. Further, in [4] authors consider these types curves in Galilean 3-space $\mathbb{G}_3$.

In the present study, we consider unit speed timelike curves whose position vectors satisfy the parametric equation

$$\alpha(x) = m_0(x)t(x) + m_1(x)n(x) + m_2(x)b(x), \tag{1.2}$$

for some differentiable functions, $m_i(x)$, $0 \leq i \leq 2$ in pseudo-Galilean space $\mathbb{G}^1_3$. We characterize the curves in terms of their curvature functions κ, τ, and give the necessary and sufficient conditions for these curves to become constant ratio, T-constant and N-constant.

2. Basic notations

The geometry of the pseudo-Galilean space is similar (but not the same) to the Galilean space which is presented in [19].

In [17], the author explained the projective signature (0,0,+,-) of the pseudo-Galilean geometry which is one of the real Cayley-Klein geometries. The absolute of the pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$ where w is the ideal (absolute) plane, f is line in w and I is the fixed hyperbolic involution of the points of f.

A vector $v = (x, y, z)$ in $\mathbb{G}^1_3$ is said to be non-isotropic if $x \neq 0$. All unit non-isotropic vectors have the form $(1, y, z)$. For isotropic vectors $x = 0$ holds. There are four types of isotropic vectors: spacelike $(y^2 - z^2 > 0)$, timelike $(y^2 - z^2 < 0)$ and two types of lightlike vectors $(y = \pm z)$. A non-lightlike isotropic vector is unit vector if $y^2 - z^2 = \pm 1$.

A trihedron (T_0, e_1, e_2, e_3), with a proper origin $T_0(x_0, y_0, z_0) \sim (1 : x_0 : y_0 : z_0)$ is orthonormal in pseudo-Galilean sense iff the vectors e_1, e_2, e_3 have the following form: $e_1 = (1, y_1, z_1)$, $e_2 = (0, y_2, z_2)$, $e_3 = (0, \varepsilon z_2, \varepsilon y_2)$ with $y^2 - z^2 = \delta$, where each of ε, δ is of $+1$ or -1. An above trihedron (T_0, e_1, e_2, e_3) is called positively oriented if for its vectors $\det(e_1, e_2, e_3) = 1$, i.e. $y_2^2 - z_2^2 = \varepsilon$ stand.

The scalar product between two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ in $\mathbb{G}^1_3$ is defined as

$$\langle v_1, v_2 \rangle = \begin{cases} x_1 x_2, & \text{if } x_1 \neq 0 \vee x_2 \neq 0 \\ y_1 y_2 - z_1 z_2 & \text{if } x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.3}$$

[9].

The length of the vector $v = (x, y, z)$ is defined as

$$\|v\| = \begin{cases} x & \text{if } x \neq 0 \\ \sqrt{|y^2 - z^2|} & \text{if } x = 0 \end{cases}$$

[11].

Definition 2.1. [9] Let $\alpha(x) = (\alpha_1(x), \alpha_2(x), \alpha_3(x))$ be a spatial curve with the three times continuously differentiable functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$ and x run through a real interval. α is called admissible if $\alpha_1'(x) \neq 0$. Then the curve α can be given by $\alpha(x) = (x, y(x), z(x))$ and we assume in addition that ${y''}^2 - {z''}^2 \neq 0$.

The curvature $\kappa(x)$ and the torsion $\tau(x)$ of an admissible curve are given by the following formulas

$$\begin{aligned} \kappa(x) &= \sqrt{\left|y''^2(x) - z''^2(x)\right|}, \\ \tau(x) &= \det \frac{(\alpha'(x), \alpha''(x), \alpha'''(x))}{\kappa^2(x)}. \end{aligned}$$

Furthermore, the associated moving trihedron is given by

$$\begin{aligned} t(x) &= \alpha'(x) = (1, y'(x), z'(x)), \\ n(x) &= \frac{\alpha''(x)}{\kappa(x)} = \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \\ b(x) &= \frac{1}{\kappa(x)}(0, \varepsilon z''(x), \varepsilon y''(x)), \end{aligned} \tag{2.4}$$

where t, n and b are called the vectors of tangent, principal normal and binormal line of the curve α, respectively. Then, the curve α is timelike (spacelike) if $n(x)$ is a spacelike (timelike) vector. The principal normal vector is spacelike if $\varepsilon = 1$ and timelike if $\varepsilon = -1$. Consequently, the following Frenet's formulas are true

$$t' = \kappa n, \qquad n' = \tau b, \qquad b' = \tau n, \tag{2.5}$$

where t is spacelike, n is spacelike and b is a timelike vector [9, 10, 14].

3. Characterization of curves in $\mathbb{G}_3^1$

In the present section, we characterize the unit speed timelike curves given with the invariant parameter x in $\mathbb{G}_3^1$ in terms of their curvatures. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed timelike curve with curvatures $\kappa(x) \geq 0$ and $\tau(x)$. The position vector of the curve (also defined by α) satisfies the vectorial equation (1.2) for some differentiable functions $m_i(x)$, $0 \leq i \leq 2$. Differentiating (1.2) with respect to the arclength parameter x, and using the Serret-Frenet equations (2.5),

$$\begin{aligned} \alpha'(x) = m_0'(x)t(x) &+ (m_1'(x) + \kappa(x)m_0(x) + \tau(x)m_2(x))n(x) \\ &+ (m_2'(x) + \tau(x)m_1(x))b(x). \end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned} m_0'(x) &= 1, \\ m_1'(x) + \kappa(x)m_0(x) + \tau(x)m_2(x) &= 0, \\ m_2'(x) + \tau(x)m_1(x) &= 0 \end{aligned} \tag{3.7}$$

[16].

3.1. Curves of constant-ratio in $\mathbb{G}_3^1$.

In [5, 6], B. Y. Chen introduced the curves of constant ratio. Similarly, we give the following definition:

Definition 3.2. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed timelike curve given with the invariant parameter x in pseudo-Galilean space $\mathbb{G}_3^1$. Then the position vector α can be decomposed into its tangential and normal components at each point as in (1.1). If the ratio $\left\|\alpha^T\right\| : \left\|\alpha^N\right\|$ is constant on $\alpha(I)$, then α is said to be of *constant ratio.*

Clearly, for a constant ratio curve in pseudo-Galilean space $\mathbb{G}_3^1$, we have

$$\frac{m_0^2}{m_1^2 - m_2^2} = c_1 \tag{3.8}$$

for some constant c_1.

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed timelike constant ratio curve given with the invariant parameter x in $\mathbb{G}_3^1$. Then, α is of constant ratio if and only if*

$$\left(\frac{\kappa' + c_1\kappa^3(x + c)}{c_1\kappa^2\tau}\right)' = -\frac{\tau}{c_1\kappa}$$

holds.

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed timelike constant ratio curve given with the invariant parameter x in $\mathbb{G}_3^1$. Then from the equality (3.8), the curvature functions satisfy

$$m_1 m_1' - m_2 m_2' = \frac{x+c}{c_1}.$$

Substituting the last equality into the second and the third equalities of (3.7), we get $m_1 = -\frac{1}{c_1\kappa}, m_2 = -\frac{\kappa' + c_1\kappa^3(x+c)}{c_1\kappa^2\tau}$ and $m_2' = \frac{\tau}{c_1\kappa}$. □

Example 3.1. Let us consider the timelike curve (general helix) $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$,

$$\alpha(x) = \left(x, \frac{x^4+3}{12x}, \frac{x^4-3}{12x}\right).$$

The Frenet vectors of the curve α are as follows:

$$\begin{aligned} t(x) &= \left(1, \frac{x^4-1}{4x^2}, \frac{x^4+1}{4x^2}\right), \\ n(x) &= \left(0, \frac{x^4+1}{2x^2}, \frac{x^4-1}{2x^2}\right), \\ b(x) &= \left(0, \frac{x^4-1}{2x^2}, \frac{x^4+1}{2x^2}\right). \end{aligned}$$

By a direct computation, we obtain the following curvature functions

$$m_0 = x,\ m_1 = \frac{x}{3},\ m_2 = -\frac{2x}{3},$$

which means

$$\frac{m_0^2}{m_1^2 - m_2^2} = -3.$$

Therefore, α is a timelike curve of constant-ratio.

3.2. **T-constant Curves in $\mathbb{G}_3^1$.**

As in [7, 8], we define T-constant curves in pseudo-Galilean space $\mathbb{G}_3^1$.

Definition 3.3. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed curve in $\mathbb{G}_3^1$. If $\|\alpha^T\|$ is constant, then α is called a *T-constant curve* in $\mathbb{G}_3^1$. Further, a T-constant curve α is called first kind if $\|\alpha^T\| = 0$, otherwise second kind.

As a consequence of (1.2) with (3.7), we get the following result.

Proposition 3.1. *There is no unit speed timelike T-constant curve in pseudo-Galilean space $\mathbb{G}_3^1$.*

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed timelike T-constant curve in $\mathbb{G}_3^1$. Then from definition, $\|\alpha^T\| = m_0$ is zero or a nonzero constant. However, we know that $m_0 = x + c$ from the equalities (3.7), a contradiction. Thus, we say that there is no unit speed timelike T-constant curve in pseudo-Galilean space $\mathbb{G}_3^1$. □

3.3. **N-constant curves in $\mathbb{G}_3^1$.**

As in [7, 8], we define N-constant curves in pseudo-Galilean space $\mathbb{G}_3^1$.

Definition 3.4. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed curve in $\mathbb{G}_3^1$. If $\|\alpha^N\|$ is constant, then α is called a N-constant curve. For a N-constant curve α, either $\|\alpha^N\| = 0$ or $\|\alpha^N\| = \mu$ for some non-zero smooth function μ. Further, a N-constant curve α is called first kind if $\|\alpha^N\| = 0$, otherwise second kind.

Note that, for a N-constant curve α in $\mathbb{G}_3^1$, we can write;

$$\left\|\alpha^N(x)\right\|^2 = m_1^2(x) - m_2^2(x) = c_1, \tag{3.9}$$

where c_1 is a real constant.

As a consequence of (1.2), (3.7) and (3.9), we get the following result.

Lemma 3.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed curve in $\mathbb{G}_3^1$. Then α is a N-constant curve if and only if*

$$\begin{aligned} m_0'(x) &= 1, \\ m_1'(x) + \kappa(x)m_0(x) + \tau(x)m_2(x) &= 0, \\ m_2'(x) + \tau(x)m_1(x) &= 0, \\ m_1(x)m_1'(x) - m_2(x)m_2'(x) &= 0 \end{aligned} \tag{3.10}$$

hold, where $m_i(x)$, $0 \leq i \leq 2$ are differentiable functions.

Proposition 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed curve in $\mathbb{G}_3^1$. Then α is a $N-$constant curve of first kind if and only if α is a straight line.*

Proof. Suppose that α is N-constant curve of first kind in $\mathbb{G}_3^1$. Then $m_1^2 - m_2^2 = 0$. If $m_1 = m_2 = 0$, then $\kappa = 0$, which means α is a straight line. If we use $m_1^2 - m_2^2 = 0$, then from the equalities (3.7), we get

$$m_1 m_2' - m_1' m_2 - \kappa m_0 m_2 = 0,$$

which means

$$\left(\frac{m_1(x)}{m_2(x)}\right)' m_2(x) = -\kappa(x + c). \tag{3.11}$$

Since $m_1(x) = \pm m_2(x)$, we get $\kappa = 0$ from the equality (3.11). Thus, α is a straight line. $\square$

Proposition 3.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3^1$ be a unit speed curve in $\mathbb{G}_3^1$. If α is a $N-$constant curve of second kind, then the position vector of α has one of the following parametrizations:*

i)

$$\alpha(x) = (x + c)\, t(x) + c_0 b(x), \tag{3.12}$$

ii)

$$\begin{aligned} \alpha(x) = (x + c)\, t(x) &- \frac{1}{2c_2}\left(c_2^2 e^{\int \tau(x)dx} + c_1 e^{-\int \tau(x)dx}\right) n(x) \\ &+ \frac{1}{2c_2}\left(-c_1 e^{-\int \tau(x)dx} + c_2^2 e^{\int \tau(x)dx}\right) b(x), \end{aligned}$$

iii)

$$\begin{aligned} \alpha(x) = (x + c)\, t(x) &+ \frac{1}{2c_2}\left(e^{-\int \tau(x)dx} + c_1 c_2^2 e^{\int \tau(x)dx}\right) n(x) \\ &+ \frac{1}{2c_2}\left(-c_1 c_2^2 e^{\int \tau(x)dx} + e^{-\int \tau(x)dx}\right) b(x). \end{aligned}$$

Proof. Let α be N-constant curve of second kind in $\mathbb{G}_3^1$. Then $m_1^2 - m_2^2 = c_1$, where c_1 is a real constant. By multiplying the second and the third equalities of (3.10) with m_1 and $-m_2$, respectively, and combining them, we get $\kappa m_0 m_1 = 0$. Therefore there are two possibilities; $\kappa = 0$ or $m_1 = 0$. Furthermore, we differ three cases. Case 1: If $\kappa = 0$ and $m_1 = 0$, then the curve is congruent to a N-constant curve of first kind. Case 2: If $\kappa \neq 0$ and $m_1 = 0$, then from the equalities (3.10), we get $m_2 = c_0$ which is constant. Case 3: If $\kappa = 0$ and $m_1 \neq 0$, then using the equalities (3.9) and (3.10), we get the differential equation

$$m_2'^2 - \tau^2 m_2^2 - \tau^2 c_1 = 0,$$

which has two following solutions:

$$m_2 = \frac{1}{2c_2}\left(-c_1 e^{-\int \tau(x)dx} + c_2^2 e^{\int \tau(x)dx}\right) \quad \text{and} \quad m_2 = \frac{1}{2c_2}\left(-c_1 c_2^2 e^{\int \tau(x)dx} + e^{-\int \tau(x)dx}\right).$$

Then writing these solutions in the third equation of (3.10), we get m_1 in two different cases given in the parametrizations ii) or iii). □

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