CREAT. MATH. INFORM.
Volume **27** (2018), No. 1,
Pages 63 - 70

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2018.01.09

Fixed points of nearly weak uniformly L-Lipschitzian mappings in real Banach spaces

ADESANMI ALAO MOGBADEMU

ABSTRACT. Let K be a nonempty convex subset of a real Banach space X. Let T be a nearly weak uniformly L-Lipschitzian mapping. A modified Mann-type iteration scheme is proved to converge strongly to the unique fixed point of T. Our result is a significant improvement and generalization of several known results in this area of research. We give a specific example to support our result. Furthermore, an interesting equivalence of T-stability result between the convergence of modified Mann-type and modified Mann iterations is included.

1. INTRODUCTION

Let X be an arbitrary real normed space with the dual X^*. We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first recall and define some concepts as follows.

Definition 1.1. Let K be a nonempty subset of a real normed linear space X. Let $T : K \to K$. T is called asymptotically nonexpansive if for each $x, y \in K$

$$\|T^n x - T^n y\| \leq k_n\|x - y\|^2, \forall n \geq 1,$$

where $(k_n) \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. T is called asymptotically pseudocontractive with the sequence$(k_n) \subset [1, \infty)$ if and only if $\lim_{n\to\infty} k_n = 1$, and for all $n \in N$ and all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) >\leq k_n\|x - y\|^2, \forall n \geq 1.$$

Remark 1.1. An asymptotically nonexpansive mapping is asymptotically pseudocontractive. However, the converse may not be true in general (see [2], [3]).

Definition 1.2. Let K be a nonempty subset of a real normed linear space X. Let $T : K \to K$. A mapping T is called uniformly $L-$ Lipschitzian if, for any $x, y \in K$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \forall n \geq 1.$$

Let $\{\sigma_n\}_{n\geq 0}$ be a sequence in $[0, \infty)$ such that $lim_{n\to\infty}\sigma_n = 0$.

A mapping $T : K \to K$ is called nearly Lipschitzian with respect to $\{\sigma_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + \sigma_n), \forall\, x, y \in K. \tag{1.1}$$

Received: 18.09.2017. In revised form: 31.01.2018. Accepted: 07.02.2018
2010 *Mathematics Subject Classification.* 47H09, 47H10, 46A03L..
Key words and phrases. *modified Mann-type iteration process, Banach space, fixed point, nearly weak uniformly L-Lipschitzian.*

A nearly Lipschitzian mapping T with sequence $\{\sigma_n\}$ is said to be nearly uniformly L-Lipschitzian if $k_n = L$, for all $n \in \mathbb{N}$.

Observe that the class of nearly uniformly L-Lipschitzian mapping is more general than the class of uniformly L-Lipschitzian mappings.

The class of nearly uniformly L-Lipschitzian have been studied extensively by many authors: for results in this regard, see e.g. Sahu [16], Kim et al. [8] and Mogbademu [10], [11]; for uniformly L-Lipschitzian mappings, see e.g. Chang [2], Chang et al. [3], Goebel [6], Ofoedu [12] and Rafiq [13]; see also Berinde [1] and the references therein.

Now, we discuss the following new concept.

Definition 1.3. Let K be a subset of a real normed linear space X and $\{a_n\}_{n\geq 1}$ be a sequence in $[0,\infty)$ such that $\lim\limits_{n\to\infty} a_n = 0$. A mapping $T : K \to K$ is called nearly weak uniformly Lipschitzian with respect to the sequence $\{a_n\}$ if for each $n \in N$, there exists a constant $L \geq 1$ such that

$$\|T^n x - T^n y\| \leq L(\|x - y\| + a_n), \forall\, x \in K, \;\; y \in F(T). \tag{1.2}$$

It is easy to see that if T has a bounded range, then it is nearly weak uniformly Lipschitzian. In fact, since $R(T^n) \subset R(T)$, then $\sup_{x\in K} \|T^n x\| \leq \sup_{x\in K} \|T^{n-1}x\| \leq \cdots \leq \sup_{x\in K} \|Tx\| \leq x$, thus $\|T^n x - T^n y\| \leq \|Tx - Ty\| \leq (\|x - y\|) \leq L(\|x - y\| + a_n)$, where $x \in K,\;\; y \in F(T)$. On the contrary, it may not be true in general. Therefore it is of interest to study the class of mappings in fixed point theory and its applications.

Example 1.1. Let $X = R,\;\; K = [0, 1]$. Define $T : K \to K$ by

$$Tx = \begin{cases} \frac{x}{2}, & x \in [0,1], \\ 0, & x > 1. \end{cases}$$

Then $T\rho = \rho$ if and only if $\rho = 0$. In fact, for a real sequence $\{\sigma_n\}_{n\geq 1}$ such that $\sigma_n \to 0$ as $n \to \infty$, we easily compute to have

$$||T^n x - T^n \rho|| \leq \frac{1}{2}(||x - \rho|| + \frac{1}{2^n}), \quad \forall\, x \in K, \quad \rho = 0.$$

Hence, T satisfies the nearly Lipschitzian condition.

Remark 1.2. It is obvious that every nearly Lipschitz map with a fixed point satisfies inequality (1.2). Clearly, the class of nearly weak uniformly L-Lipschitzian mappings is a generalization of the class of nearly uniformly L-Lipschitzian mappings which in turn is a generalization of the class of uniformly L-Lipschitzian mappings (see [5].

It is the interest of this paper to discuss the following modified Mann-type iteration scheme associated with nearly weak uniformly L-Lipschitzian mappings to have a strong convergence in the real Banach spaces setting.

Let $x_1 \in K$ be a nonempty convex subset of a real normed linear space X and $T : K \to K$ be a map. For a sequence $\{v_n = f(x_n)\}$ in K where $f : K \to K$ is a mapping, define $\{x_n\}_{n=1}^{\infty}$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n v_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$. We observe that the iteration process (1.3) is well defined and is a generalization of the modified Mann and modified Ishikawa iterations used by several authors (see [2]-[16]). This is true in the sense that, when $v_n = f(x_n) = x_n$, then (1.3) reduces to the modified Mann iteration define by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \tag{1.4}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$. If in (1.3), $v_n = f(x_n) = (1-\beta_n)x_n + \beta_n T^n x_n$ where β_n is a sequence in $(0,1)$, then it will reduce to the modified Ishikawa iteration method (see [14]).

2. Preliminaries

In the sequel, we shall need the following lemmas.

Lemma 2.1. [4]. *Let X be real Banach Space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$*

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x+y).$$

Lemma 2.2. [4, 10] *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=1}^{\infty}$ be a positive real sequence satisfying*

$$\sum_{n=1}^{\infty} b_n = +\infty \quad \textit{and} \quad \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n \Phi(a_{n+1}), \quad \forall n \geq \mathbb{N}_0$$

where $\lim\limits_{n\to\infty} \frac{o(b_n)}{b_n} = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let K be a nonempty convex subset of a real Banach space X. Let $T : K \to K$ be a nearly weak uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ as defined in equation (1.2). Let $\{\epsilon_n\} \in (0,1)$ and $\{k_n\} \subset [1,\infty)$ be sequences with $lim_{n\to\infty}\epsilon_n = 0$ and $lim_{n\to\infty}k_n = 1$. For a sequence $\{v_n\}$ in K, define a sequence $\{x_n\}$ in K satisfying $\lim_{n\to\infty} ||v_n - x_n|| = 0$ by $x_1 \in K$,*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n v_n, \; n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that (i)$\sum_{n\geq 1} \alpha_n = \infty$ (ii) $\lim_{n\to\infty} \alpha_n = 0$. There exists $\tau_0 > 0$ such that $\alpha_n \leq \tau_0$ $\forall n \geq n_0$, for some $n_0 \in N$. Suppose there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x-\rho) > \leq k_n \|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

for all $x \in K, \rho \in F(T)$. Then $\{x_n\}_{n\geq 1}$ converges strongly to the unique fixed point of T.

Proof. Let $\rho \in F(T)$, it is easy to see that ρ is unique. For, if ρ' is another fixed point of T, then $T^n\rho = \rho$ and $T^n\rho' = \rho'$, $\forall n \geq 1$. That is, $\|\rho-\rho'\|^2 \leq k_n\|\rho-\rho'\|^2 - \Phi(\|\rho-\rho'\|) + \epsilon_n, \; \forall n \geq 1$. Taking limits of bothsides as $n \to \infty$, we get

$$\|\rho-\rho'\|^2 \leq \|\rho-\rho'\|^2 - \Phi(\|\rho-\rho'\|) < \|\rho-\rho'\|^2,$$

a contradiction. Hence, ρ is unique.
Since T is nearly weak uniformly Lipschitzian and $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ is a strictly increasing continuous function such that

$$\langle T^n x - T^n \rho, j(x-\rho)\rangle \leq k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n, \tag{3.5}$$

for $x \in K, \;\; \rho \in F(T)$, implying that

$$\Phi(\|x-\rho\|) \leq k_n\|x-\rho\|^2 + L(\|x-\rho\| + a_n)\|x-\rho\| + \epsilon_n.$$

Taking limit of both sides as $n \to \infty$, we get

$$\Phi(\|x-\rho\|) \leq (1+L)\|x-\rho\|^2.$$

If $\|x-\rho\|=0\ \forall n\in N$, then we are done. So, we assume $x_1\neq Tx_1$ for some $x_1\in K$ such that

$$\epsilon_n+(k_n+L)\|x_1-\rho\|^2+L\|x_1-\rho\|^2\in R(\Phi)$$

and denote that $a_0=\epsilon_n+(k_n+L)\|x_1-\rho\|^2+L\|x_1-\rho\|^2$, $R(\Phi)$ is the range of Φ. Indeed, if $\Phi(a)\to+\infty$ as $a\to\infty$, then $a_0\in R(\Phi)$; if $\sup\{\Phi(a):a\in[0,\infty]\}=a_1<+\infty$ with $a_1<a_0$, then for $\rho\in K$, there exists a sequence $\{u_n\}$ in K such that $u_n\to\rho$ as $n\to\infty$ with $u_n\neq\rho$. Clearly, $Tu_n\to T\rho$ as $n\to\infty$ thus $\{u_n-Tu_n\}$ is a bounded sequence. Therefore, there exists a natural number n_0 such that

$$\epsilon_n+(k_n+L)\|u_n-\rho\|^2+L\|u_n-\rho\|^2<\frac{a_1}{2}$$

for $n\geq n_0$, then we redefine $x_1=u_{n_0}$ and

$$\epsilon_n+(k_n+L)\|x_1-\rho\|^2+L\|x_1-\rho\|^2\in R(\Phi).$$

Step 1. We first prove that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Set $R=\Phi^{-1}(a_0)$, then from above (3.5), we obtain that $\|x_1-\rho\|\leq R$.

Denote

$$B_1=\{x\in K:\|x-\rho\|\leq R\},\quad B_2=\{x\in K:\|x-\rho\|\leq 2R\}. \tag{3.6}$$

Now, we want to prove that $x_n\in B_1$. If $n=1$, then $x_1\in B_1$. Now, assume that it holds for some n, that is, $x_n\in B_1$. Suppose that, it is not the case, then $\|x_{n+1}-\rho\|>R>\frac{R}{2}$. Since $\{a_n\}\in[0,\infty]$ with $a_n\to 0$ as $n\to\infty$, set $M=\sup\{a_n:n\in N\}$. Define $\tau_0\in R^+$ by

$$\tau_0=min\left\{1,\frac{R}{L(2R+M)},\frac{R}{(L(2R+M)+R)},\frac{\Phi(\frac{R}{2})}{32R^2},\frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]},\frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]},\frac{\Phi(\frac{R}{2})}{8}\right\}. \tag{3.7}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}k_n=1$. Without loss of generality, let $0\leq\alpha_n,k_n-1$, $\epsilon_n\leq\tau_0$ for any $n\geq 1$. We get

$$\begin{aligned}
\|x_{n+1}-\rho\| &\leq (1-\alpha_n)\|x_n-\rho\|+\alpha_n\|T^nv_n-\rho\| \\
&\leq R+\tau_0L(2R+M) \\
&\leq 2R, \\
\|x_{n+1}-x_n\| &\leq \alpha_n\|T^nv_n-x_n\| \\
&\leq \alpha_n(\|T^nv_n-\rho\|+\|x_n-\rho\|) \\
&\leq \tau_0(L(2R+M)+R), \\
\|v_n-x_{n+1}\| &\leq \|v_n-x_n\|+\alpha_n\|T^nv_n-x_n\| \\
&\leq \|v_n-x_n\|+\alpha_n(\|T^nv_n-\rho\|+\|x_n-\rho\|) \\
&\leq R+\tau_0(L(2R+M)+R).
\end{aligned} \tag{3.8}$$

Using Lemma 2.1 and the above estimates, we compute

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 \le\ & (1-\alpha_n)^2\|x_n-\rho\|^2 + 2\alpha_n < T^n v_n - x_n, j(x_{n+1}-\rho) > \\
=\ & (1-\alpha_n)^2\|x_n-\rho\|^2 + 2\alpha_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1}-\rho) > \\
& + 2\alpha_n < x_{n+1} - x_n, j(x_{n+1}-\rho) > \\
& + 2\alpha_n < T^n v_n - T^n x_{n+1}, j(x_{n+1}-\rho) > \\
\le\ & \|x_n-\rho\|^2 + 2\alpha_n(k_n\|x_{n+1}-\rho\|^2 - \Phi(\|x_{n+1}-\rho\|) + \epsilon_n) \\
& - 2\alpha_n\|x_{n+1}-\rho\|^2 + 2\alpha_n L(\|v_n - x_{n+1}\| + a_n)\|x_{n+1}-\rho\| \\
& + 2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
\le\ & \|x_n-\rho\|^2 - 2\alpha_n\Phi(\frac{R}{2}) + 2\alpha_n\frac{\Phi(\frac{R}{2})}{32R^2}4R^2 + 2\alpha_n\frac{\Phi(\frac{R}{2})}{8} \\
& + 2\alpha_n L\frac{\Phi(\frac{R}{2})}{16R[2(L(2R+M)+R)+M]}2R[2(L(2R+M)+R)+M] \\
& + 2\alpha_n\frac{\Phi(\frac{R}{2})}{16R[L(2R+M)+R]}2R[L(2R+M)+R] \\
\le\ & \|x_n-\rho\|^2 - \alpha_n\Phi(\frac{R}{2}) \\
\le\ & R^2.
\end{aligned} \tag{3.9}$$

which is a contradiction. Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.

Step 2. We want to prove that $\|x_n-\rho\| \to 0$ as $n \to \infty$. By step 1, we obtain that $\{\|x_n-\rho\|\}$ is a bounded sequence. Let $M_1 = \sup\{\|x_n-\rho\|\} + sup\{\|v_n-\rho\|\}$. Observe that

$$\begin{aligned}
\|x_{n+1}-\rho\| \le\ & (1-\alpha_n)\|x_n-\rho\| + \alpha_n\|T^n v_n - \rho\| \\
\le\ & R + \tau_0 L(2R+M) \\
\le\ & 2R, \\
\|x_{n+1}-x_n\| \le\ & \alpha_n\|T^n v_n - x_n\| \\
\le\ & \alpha_n(\|T^n v_n - \rho\| + \|x_n-\rho\|) \\
\le\ & \alpha_n(L(\|v_n-\rho\| + a_n) + \|x_n-\rho\|) \\
\le\ & \alpha_n(L(M_1+a_n) + M_1), \\
\|v_n - x_{n+1}\| \le\ & \|v_n - x_n\| + \alpha_n\|T^n v_n - x_n\| \\
\le\ & \|v_n - x_n\| + \alpha_n(\|T^n v_n - \rho\| + \|x_n-\rho\|) \\
\le\ & \|v_n - x_n\| + \alpha_n(L(M_1+a_n) + M_1).
\end{aligned} \tag{3.10}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}k_n = 1$ and $\{x_n\}_{n=1}^{\infty}$ is bounded. From (3.9), we observed that

$$\lim_{n\to\infty} L\|v_n - x_{n+1}\| = 0. \tag{3.11}$$

So from (1.3), we have

$$\begin{aligned}\|x_{n+1}-\rho\|^2 &\le (1-\alpha_n)^2\|x_n-\rho\|^2 + 2\alpha_n < T^n v_n - x_n, j(x_{n+1}-\rho) > \qquad (3.12)\\ &= (1-\alpha_n)^2\|x_n-\rho\|^2 + 2\alpha_n < T^n x_{n+1} - x_{n+1}, j(x_{n+1}-\rho) >\\ &\quad + < x_{n+1}-x_n, j(x_{n+1}-\rho) >\\ &\quad + < T^n v_n - T^n x_{n+1}, j(x_{n+1}-\rho) >\\ &\le \quad \|x_n-\rho\|^2 + 2\alpha_n(k_n\|x_{n+1}-\rho\|^2 - \Phi(\|x_{n+1}-\rho\|) + \epsilon_n)\\ &\quad - 2\alpha_n\|x_{n+1}-\rho\|^2 + 2\alpha_n L(\|v_n - x_{n+1}\| + a_n)\|x_{n+1}-\rho\|\\ &\quad + 2\alpha_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\|\\ &\le \|x_n-\rho\|^2 + 2\alpha_n(k_n-1)M_1^2\\ &\quad - 2\alpha_n\Phi(\|x_{n+1}-\rho\|) + 2\alpha_n\epsilon_n\\ &\quad + 2\alpha_n L(\|v_n-x_n\| + \alpha_n(L(M_1+a_n)+M_1)) + a_n)M_1\\ &\quad + 2\alpha_n(\alpha_n(L(M_1+a_n)+M_1))M_1\\ &= \|x_n-\rho\|^2 - 2\alpha_n\Phi(\|x_{n+1}-\rho\|) + o(\alpha_n),\end{aligned}$$

where

$$\begin{aligned}&2\alpha_n(k_n-1)M_1^2 + 2\alpha_n L(\|v_n-x_n\| + \alpha_n(L(M_1+a_n)+M_1)) + a_n)M_1\\ &+2\alpha_n(\alpha_n(L(M_1+a_n)+M_1))M_1 + 2\alpha_n\epsilon_n\\ &= o(\alpha_n).\end{aligned}$$

Thus, by Lemma 2.2, we obtain that $\lim_{n\to\infty}\|x_n-\rho\| = 0$. This completes the proof. □

From Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let K be a nonempty convex subset of a real Banach space X. Let $T : K \to K$ be a nearly weak uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ as defined in equation* (1.2). *Let $\{\epsilon_n\} \in (0,1)$ and $\{k_n\} \subset [1,\infty)$ be sequences with $lim_{n\to\infty}\epsilon_n = 0$ and $lim_{n\to\infty}k_n = 1$. For some $x_0 \in K$, define the modified Mann iterative sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ n \ge 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that (i)$\sum_{n\ge1}\alpha_n = \infty$ (ii) $\lim_{n\to\infty}\alpha_n = 0$. Then, there exists $\tau_0 > 0$ such that $\alpha_n \le \tau_0$ $\forall n \ge n_0$, for some $n_0 \in N$. Suppose there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n\rho, j(x-\rho) >\le k_n\|x-\rho\|^2 - \Phi(\|x-\rho\|) + \epsilon_n$$

for all $x \in K, \rho \in F(T)$. Then $\{x_n\}_{n\ge0}$ converges strongly to the unique fixed point of T.

Proof. By letting $v_n = x_n$ in Theorem 3.1, we get the convergence of modified Mann iteration (1.4). □

Remark 3.3. Unlike as in several existing results in the literature (see [2, 3, 6], [10]-[16]), Theorem 3.1 and Corollary 3.1 are applicable for any nearly weak uniformly L-Lipschitzian mapping. It is also well known that whenever a theorem is proved using Mann-type iteration (without error term), the method of proof follows easily to the case of Mann-type iteration process with error term.
Now, we give an example to support practical application of our main result.

Example 3.2. Let $X = R$ be the set of real numbers with the usual norm, $K = [0,\infty)$ and $T : K \to K$ be a mapping define by

$$Tx = \frac{x^2}{1+x^3}, \forall x \in K.$$

It is easy to show that T is nearly weak uniformly L-Lipschitzian with sequence $\{a_n\}$ having fixed point $\rho = 0$ and strictly monotonic increasing. Observe that for any $x \in K$, $T^n x \leq T^{n-1}x \leq ... \leq Tx$. Let define a function $\Phi : [0, \infty) \to [0, \infty)$ by $\Phi(t) = \frac{t^2}{1+t^2}$. Then Φ is a strictly increasing continuous function with $\Phi(0) = 0$. For all $x \in K$ and $\rho \in F(T)$, set $k_n = 1 + \frac{1}{n}$, $\epsilon_n = \frac{1}{1+n}$, $a_n = \frac{1}{2^n}$ and $L \geq 1$, then we get

$$\begin{aligned} < T^n x - T^n 0, j(x-0) > &\leq < Tx - 0, j(x-0) > \\ &\leq |x-0|^2 - \Phi(|x-0|) + 0 \\ &\leq k_n|x-0|^2 - \Phi(|x-0|) + \epsilon_n, \forall n \geq 1 \end{aligned}$$

and

$$\begin{aligned} |T^n x - T^n 0| &\leq |Tx - 0| \\ &\leq L(|x-0| + 0) \\ &\leq L(|x-0| + a_n), \forall n \geq 1. \end{aligned}$$

Clearly, T is nearly weak uniformly L-Lipschitzian and applicable to Theorem 3.1 and Corollary 3.1.

Prototype. An example of our control sequence α_n in our Theorem 3.1 and Corollary 3.1 is $\alpha_n = \frac{1}{1+n}$.

4. The equivalence of stability between modified Mann-type and modified Mann iterations

The following definition is well known (see [7]).

Definition 4.4. Let X be a real Banach space. Suppose that $F(T)$, the fixed point set of T, is nonempty and that the sequence $\{x_n\}$ converges to a point $\rho \in F(T)$.
(i). Let $\{y_n\} \subset X$, and define $\epsilon_n = \|y_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n f(y_n)\|$. If $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = \rho$, then the modified Mann-type iteration scheme (1.3) is said to be T-stable or stable with respect to T.
(ii). Let $\{y_n\} \subset X$, and define $\epsilon_n = \|y_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n y_n\|$. If $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = \rho$, then the modified Mann iteration scheme (1.4) is said to be T-stable or stable with respect to T.
According to Definition 4.4, we shall prove that the modified Mann-type and modified Mann iterations are equivalent for a nearly weak uniformly L-Lipschitzian map:

$$\|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n v_n\| = 0 \implies \lim_{n\to\infty} u_n = \rho. \tag{4.13}$$

$$\|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| = 0 \implies \lim_{n\to\infty} x_n = \rho. \tag{4.14}$$

Theorem 4.2. *Let X be a real Banach space . Let T be a nearly weak uniformly L-Lipschitzian self mapping with sequence $\{a_n\}$ as defined in equation* (1.2)*, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, then the following are equivalent:*
(i). The modified Mann-type iteration is T stable.
(ii). The modified Mann iteration is T stable.

Proof. Firstly, we prove that (4.13) $\implies$ (4.14).
Suppose $\lim_{n\to\infty} \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| = 0$, then

$$\begin{aligned} \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n v_n\| &\leq \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| + \|\alpha_n T^n u_n - \alpha_n T^n v_n\| \\ &\leq \|u_{n+1} - (1-\alpha_n)u_n - \alpha_n T^n u_n\| + \alpha_n L(\|u_n - v_n\| + a_n) \\ &\to 0. \end{aligned}$$

So according to (4.13), we have $\lim_{n\to\infty} u_n = \rho$.
Conversely, we prove that (4.14) $\implies$ (4.13).
Suppose $\lim_{n\to\infty} \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n v_n\| = 0$, then

$$\begin{aligned}\|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n x_n\| &\le \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n v_n\| + \|\alpha_n T^n v_n - \alpha_n T^n x_n\| \\ &\le \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T^n v_n\| + \alpha_n L(\|v_n - x_n\| + a_n) \\ &\to 0.\end{aligned}$$

□

By (4.14), we obtain $\lim\limits_{n\to\infty} x_n = \rho$.

Acknowlegments. The author acknowledge the referees and Professor Z. Xue for his useful suggestions and comments to improve this paper.

REFERENCES

[1] Berinde, V., *Iterative approximation of fixed points*, Lecture Notes in Math., 1912 (2007), Springer Berlin
[2] Chang, S. S., *Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **129** (2000), 845–853
[3] Chang, S. S., Cho, Y. J. and Kim, J. K., *Some results for uniformly L-Lipschitzian mappings in Banach spaces* , Appl. Math. Lett., **22** (2009), 121–125
[4] Chidume, C. E., *Geometric properties of Banach spaces and nonlinear iterations*, Lecture Notes in Math., 1965 (2009), Springer-Verlag series
[5] Chidume, C. E. and Chidume, C. O., *Convergence theorems for fixed points of uniformly continuous generalized Φ hemicontractive mappings*, J. Math. Anal. Appl., **303** (2005), 545–554
[6] Goebel, K. and Kirk, W. A., *A fixed point theorem for asymptotically nonexpansive mappings* , Proc. Amer. Math. Soc. 35(1972), 171-174
[7] Harder, A. M. and Hicks, T. L., *Stability results for fixed point iteration procedures*, Math. Japon., **33** (1988), 693–706
[8] Kim, J. K., Sahu, D. R. and Nam, Y. M., *Convergence theorem for fixed points of nearly uniformly $L-$ Lipschitzian asymptotically generalized Φ-hemicontractive mappings*, Nonl. Anal., **71** (2009), e2833– e2838
[9] Mann, W. R., *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–610
[10] Mogbademu, A. A., *A convergence theorem for multistep iterative scheme for nonlinear maps*, Publ. Inst. Math. (Beograd) (N. S.), 98 (112) (2015), 281–285
[11] Mogbademu, A. A., *Strong convergence results for nonlinear mappings in Banach spaces*, Creat. Math. Inform., **25** (2016), No. 1, 85–92
[12] Ofoedu, E. U., *Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space*, J. Math. Anal. Appl., **321** (2006), 722–728
[13] Rafiq, A.,*Fixed point iterations of three asymptotically pseudocontractive mappings*, Demonstratio Math., **46** (2013), No. 3, 563–573
[14] Rhoades, B. E. and Soltuz, S. M., *The equivalence between Mann-Ishikawa iterations and multistep iteration*, Nonlinear Anal., **58** (2004), No. 1-2, 218–228
[15] Sahu, D. R., *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces*, Comment. Math. Univ. Carolin., **46** (2005), No. 4, 653–666
[16] Schu, J., *Iterative construction of fixed points of asymptotically nonexpansive mappings* , J. Math. Anal. Appl., **158** (1999), 407–413

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LAGOS
LAGOS, NIGERIA
Email address: amogbademu@unilag.edu.ng