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A note on the morphism theorems for $(n, m)-$semirings

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ABSTRACT. In this paper, some properties of subtractive ideal of $(n, m)-$semirings are investigated. In addition, we study the morphisms of $(n, m)-$semirings starting from the definitions given in the case of universal algebras. We will present several theorems of correspondence for sub-(n, m)-semirings, ideals, subtractive ideals that represent the generalization of the morphism theorems of the binary case.

1. INTRODUCTION

Algebraic polyadic structures are applied in many disciplines such as theoretical physics, computer sciences, coding theory, automata theory and other. The concept of $n-$ary group was introduced by Dörnte [2] and developed by E. Post [14] , J. Timms [17] for commutative case. The $m-$semigroups are studied by F. M. Siosson [16], M. S. Pop [12], A. Pop [8]. I. Purdea [13] and G. Crombez [1] extended the usual ring concept to the case where the underlying group and semigroup is an commutative $n-$ary group and an $m-$ary semigroup, respectively. In some recently appeared papers, various authors continue the study of ordinary semigroups introduced by H. S. Vandiver [18] to the case where the underlying commutative additive semigroup and multiplicative semigroup are not binary but an $n-$ary and one $m-$ary respectively. The new obtained structures are called $(n, m)-$semirings [7], [9], [19].

We begin with some preliminaries about the $m-$semigroups, $n-$groups, $(n, m)-$ semirings and $(n, m)-$rings.

Traditionally in the theory of $n-$groups we use the following abbreviated notation: the sequence $x_i, \dots, x_j$ is denoted by x_i^j (for $j < i$ this symbol is empty). If $x_{i+1} = x_{i+2} = \dots = x_{i+k} = x$, then instead of x_{i+1}^{i+k} we write $\overset{(k)}{x}$. The algebra $(S, (\,)_+)$ is called an $n-$semigroup if for any $i \in \{2, 3, \dots, n\}$ and all $x_1, \dots, x_{2n-1} \in S$, the following associativity laws are true i.e

$$((x_1^n)_+, x_{n+1}^{2n-1})_+ = (x_1^{i-1}, (x_i^{i+n-1})_+, x_{i+n}^{2n-1})_+.$$

An $n-$semigroup $(S, (\,)_+)$ is called $n-$group if for any $i \in \{1, 2, \dots, n\}$ and all $a_1, \dots, a_n \in S$, the equation $(a_1^{i-1}, x, a_{i+1}^n)_+ = a_i$ has a unique solution in S. In some n-groups there is an element $e \in S$ (called identity or neutral element) such that $(\overset{(i-1)}{e}\, x\, \overset{(n-i)}{e})_+ = x$ holds for all $x \in S$ and for all $i \in \{1, \dots, n\}$. It is interesting that there are n-groups with two or more neutral elements or which do not contain such elements [2],[14]. From the definition of the n-group $(S, (\,)_+)$ we can see that for every $x \in S$ there is only one $y \in S$, satisfying the equation $(\overset{(n-1)}{x}\, y)_+ = x$. This element, denoted by $\overline{x}$, so called *querelement* of x, defines the power $x^{[-1]}$. W. Dörnte [2] proved that in any n-group for all $a, x \in A$; $2 \le i, j \le n$, we have $(\overset{(i-2)}{x}\, \overline{x}\, \overset{(n-i)}{x}\, a)_+ = a$ and $(a\, \overset{(n-j)}{x}\, \overline{x}\, \overset{(j-2)}{x})_+ = a$.

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An $n-$semigroup A will be called:
-*semicommutative* [2], if for any $a_1, \ldots, a_n \in A$ we have $(a_1, a_2^{n-1}, a_n)_+ = (a_n, a_2^{n-1}, a_1)_+$;
-*commutative* [2], if $(x_1^n)_+ = (x_{\sigma(1)}^{\sigma(n)})_+$,$(\forall)x_1^n \in A$ and for each permutation σ of $\{1, 2, ..., n\}$.
-*entropic* (*medial*)[2], if for all n^2 elements of A, $a_{ij} \in A,\ i, j \in \{1, 2, \ldots, n\}$ we have

$$((a_{11}^{1n})_\circ, (a_{21}^{2n})_\circ, \ldots, (a_{n1}^{nn})_\circ)_\circ = ((a_{11}^{n1})_\circ, (a_{12}^{n2})_\circ, \ldots, (a_{1n}^{nn})_\circ)_\circ.$$

2. Ideals, Subtractive Ideals

Definition 2.1. ([7]) The algebra $(S, (\,)_+, (\,)_\circ)$ where $(\,)_+ : S^n \to S$; $(\,)_\circ : S^m \to S$; $m, n \in \mathbb{N}$; $m, n \geq 2$ is called an $(n, m)-$semiring if:
1) $(S, (\,)_+)$ is a commutative $n-$semigroup;
2) $(S, (\,)_\circ)$ is an $m-$semigroup ;
3) the "$m-$ary multiplication" is distributive with respect to "$n-$ary addition", i.e.

$$(y_1^{i-1}, (x_1^n)_+, y_{i+1}^m)_\circ = ((y_1^{i-1}\, x_1\, y_{i+1}^m)_\circ, \ldots, (y_1^{i-1}\, x_n\, y_{i+1}^m)_\circ)_+,$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_m \in S$ and all $i \in \{1, 2, \ldots, m\}$.

An $(n, m)-$semiring in which $(S, (\,)_+)$ is a commutative $m-$group is called an $(n, m)-$ring. An $(n, m)-$semiring ($(n, m)-$ring) in which the $m-$ary operation is semicommutative (commutative) is called a semicommutative (commutative) $(n, m)-$semiring ($(n, m)-$ring). For $n = m = 2$, the $(2, 2)-$semiring ($(2, 2)-$ring) is ordinary semiring (ring). For $n = 2$ and $m = 3$, the $(2, 3)-$semiring is the ternary semiring introduced by Dutta and Kar [3].

Definition 2.2. An $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$ is called:
a) Additively idempotent, if $x^{[1]} = x$, for all $x \in S$;
b) Multiplicatively idempotent, if $x^{<1>} = x$, for all $x \in S$.
c) Idempotent, if it is additively idempotent and multiplicatively idempotent $(n, m)-$semiring.

Further, we put $x^{[0]} = x$; $x^{[1]} = (\overset{(n)}{x})_+$ and $x^{[k]} = (x^{[k-1]}, \overset{(n-1)}{x})_+$ for all $x \in S$ and $k \in \mathbb{N}^*$, $x^{[k]}$ having $(n-1)k + 1$ terms.

Similarly, for $m-$ary operation we put $x^{<0>} = x$; $x^{<1>} = (\overset{(m)}{x})_\circ$ and $x^{<k>} = (x^{<k-1>}, \overset{(m-1)}{x})_\circ$ for all $x \in S$, $x^{<k>}$ having $(m-1)k + 1$ terms.

We denote the set of additively idempotents and the set of multiplicatively idempotents of $(n, m)-$semiring, with Ida(S) and Idm(S), respectively. We observe that an $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$ is additively idempotent (multiplicatively idempotent), if and only if Ida$(S) = S$ (Idm$(S) = S$). An $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$ is idempotent if and only if Ida$(S) = S =$Idm(S).

Definition 2.3. The subset $H \subseteq S$ of an $(n, m)-$semiring is called a sub-(n, m)-semiring if $(x_1^n)_+, (x_1^m)_\circ \in H$ for all $x_1, \ldots, x_p \in H$, $p = \max(m, n)$. An element $e \in S$ is called aditive neutral element or identity if $(\overset{(n-1)}{e}\ x)_+ = x$, for every $x \in S$. An element z is said to be zero element (multiplicative absorbing) if $(x_1^{i-1}\, z\, x_{i+1}^m)_\circ = z$ for all $x_1, \ldots, x_m \in S$ and $i \in \{1, \ldots, m\}$.

Definition 2.4. ([11]) An $(n, m)-$semiring$(S, (\,)_+, (\,)_\circ)$ is called:
a) additively cancellative, if the $n-$ary semigroup $(S, (\,)_+)$ is cancellative, i.e.,

$$(x_1^{i-1}\, a\, x_{i+1}^n)_+ = (x_1^{i-1}\, b\, x_{i+1}^n)_+ \Rightarrow a = b,$$

for all $x_1, x_2, .., x_n \in S \setminus \{0\}$ (if zero element, 0 exists) and for every $a, b \in S$

b) multiplicatively cancellative, if the $m-$semigroup $(S, ( )_\circ)$ is cancellative, i.e.,

$$(x_1^{i-1}\, a\, x_{i+1}^m)_\circ = (x_1^{i-1}\, b\, x_{i+1}^m)_\circ \Rightarrow a = b,$$

for all $x_1, x_2, .., x_m \in S \setminus \{0\}$ (if zero element, 0 exists) and for every $a, b \in S$

An element $u \in S$ is called multiplicative neutral element or unity if $(\overset{(i-1)}{u}\; x\; \overset{(m-i)}{u})_\circ = x$ for all $x \in S$ and $i \in \{1, \ldots, m\}$.

Note that, unlike the case of usual semirings there are (n, m)-semirings which have more identities (only one of which is zero) and / or more units.

Definition 2.5. A semidomain is an additively and multiplicatively cancellative $(n, m)-$semiring with additive neutral element which is also zero element and with multiplicative neutral element.

Definition 2.6. An integral semidomain is a semidomain that has no divisors of zero.

Example 2.1. ([11]) Let n, m be the positive integers, $n, m \geq 2$. The $(n, m)-$semiring $(\mathbb{N}, ( )_+, ( )_\circ)$ $(k_1^n)_+ = k_1 + \ldots + k_n + 1$ and $(k_1^m)_\circ = \frac{\prod_{j=1}^{m}((n-1)k_j+1)-1}{n-1}$, $k_1, \ldots, k_p \in \mathbb{N}$, $p = \max(m, n)$, has no zero element, but it has multiplicative neutral element, namely 0.

Example 2.2. ([10]) The set of all integers $\mathbb{Z}$ endowed with the above defined $n-$ary operation with $n = 2$, $k_1 * k_2 = k_1 + k_2 + 1$ and $2m+1-$ary operation, $m \geq 2$, $m \in \mathbb{N}$ defined by $(k_1^{2m-1})_\circ = \prod_{i=1}^{2m+1}(k_i + 1) - 1$, $k_1, \ldots, k_p \in \mathbb{N}$, $p = \max(m, n)$, is a commutative and multiplicatively cancellative $(2, 2m+1)-$ring. It has a neutral aditive element, -1 which is also the zero element, and two neutral multiplicative elements, namely 0 and -2.

Definition 2.7. Let $(S, ( )_+, ( )_\circ)$ be an $(n, m)-$semiring, Then an $i-$ideal A of $(S, ( )_+, ( )_\circ)$, $i \in \{1, 2, ..., n\}$ is defined as a sub-n-semigroup $(A, ( )_+)$ of $(S, ( )_+)$ (i.e. $A^{[1]} \subseteq A$) satisfying $(\overset{(i-1)}{S}\; A\; \overset{(m-i)}{S})_\circ \subseteq A$. If A is an $i-$ideal of S for every i, then it is called an ideal of S.

Remark 2.1. If $(S, ( )_+, ( )_\circ)$ is an $(n, m)-$semiring, then:

1) An ideal I of S is a sub-$(n, m)-$semiring of $(n, m)-$semiring $(S, ( )_+, ( )_\circ$;

2) If S has a zero element, then it belongs to all $i-$ideals and ideal of S, too. In addition, the subset $\{0\} \subseteq S$ is an ideal, called null ideal and noted (0).

Definition 2.8. Let A be an ideal of an $(n, m)-$semiring $(S, ( )_+, ( )_\circ)$. The set

$$cl(A) = \{x \in S \mid \text{ there are } a_1, \ldots, a_{n-1} \in A \text{ such that } (xa_1^{n-1})_+ \in A\}$$

is called the $k-$closure of A.

Proposition 2.1. *Let A and B be $(n, m)-$semiring ideals of an $(n, m)-$semiring $(S, ( )_+, ( )_\circ)$. Then*

1) cl(A) is an $(n, m)-$semiring ideal of S and $A \subseteq$cl(A);

2) If $A \subseteq B$, then cl$A \subseteq$ cl$B$;

3) cl(cl$A$) =cl$A$.

Proof. 1) For all $x_1, \ldots, x_n \in$ cl$A$, there are $a_{i1}, \ldots, a_{i,n-1} \in A$ such that $(x_i a_{i1}^{i,n-1})_+ \in A$, for $i \in \{1, 2, \ldots, n\}$. Since A is an $(n, m)-$semiring ideal, by commutativity and associativity of $n-$ary operation, we have:

$$((x_1^n)_+(a_{11}^{n1})_+ \cdots (a_{1,n-1}^{n,n-1})_+)_+ = ((x_1 a_{11}^{1,n-1})_+, \ldots, (x_n a_{n1}^{n,n-1})_+)_+ \in A^{[1]} \subseteq A$$

Also, for all $s_1, \ldots, s_m \in S,\ x \in \mathrm{cl}A$ and $i \in \{1, ..., n\}$, there are $a_1, \ldots, a_{n-1} \in A$ such that $(xa_1^{n-1})_+ \in A$ and $(s_1^{i-1} a_j s_{i+1}^m)_\circ \in A$ for every $j \in \{1, 2, \ldots, n-1\}$.

But

$$\left((s_1^{i-1} x\, s_{i+1}^m)_\circ, (s_1^{i-1} a_1\, s_{i+1}^m)_\circ, \ldots, (s_1^{i-1} a_{n-1}\, s_{i+1}^m)_\circ\right)_+ =$$

$$= (s_1^{i-1}(x\, a_1^{n-1})_+\, s_{i+1}^m)_\circ \in A.$$

Consequently, $(s_1^{i-1} x\ s_{i+1}^m)_\circ \in \mathrm{cl}A$ and so $\mathrm{cl}A$ is an ideal of S.

From $A^{[1]} \subseteq A$ results $A \subseteq \mathrm{cl}A$.

2) If $A \subseteq B$ and $x \in \mathrm{cl}A$ there are $a_1, \ldots, a_{n-1} \in A \subseteq B$ such that $(xa_1^{n-1})_+ \in A \subseteq B$, hence $x \in \mathrm{cl}B$.

3) By 2) the inclusion $A \subseteq \mathrm{cl}A$ implies $\mathrm{cl}A \subseteq \mathrm{cl}(\mathrm{cl}A)$.

If $x \in \mathrm{cl}(\mathrm{cl}A)$, then there are $x_1, \ldots, x_{n-1} \in \mathrm{cl}A$ such that $\left(xx_1^{n-1}\right)_+ \in \mathrm{cl}A$. Hence, there are $y_1, \ldots y_{n-1} \in A$ such that $\left((xx_1^{n-1})_+ y_1^{n-1}\right)_+ \in A$. Since $(x_1^{n-1} y_1)_+ \in A$ and using the associativity of the n-ary operation "$(\,)_+$" we have $\left(x(x_1^{n-1}y_1)_+ y_2^{n-1}\right)_+ \in A$, whence $x \in \mathrm{cl}A$. From this it follows that $\mathrm{cl}(\mathrm{cl}A) \subseteq \mathrm{cl}A$ and $\mathrm{cl}A = \mathrm{cl}(\mathrm{cl}A)$. □

Definition 2.9. In the special case where $A = \mathrm{cl}A$ holds, the ideal A is called subtractive ideal, $k-$closed or $k-$ideal of $(S, (\,)_+, (\,)_\circ)$. The $k-$closure, $\mathrm{cl}A$, of an $(n, m)-$semiring ideal is always a $k-$ideal.

An equivalent definition of subtractive ideal is the following:

Definition 2.10. ([19]) The ideal A of an $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$ is a subtractive ideal if $a_2, \ldots a_n,\ (a_1^n)_+ \in A$ implies $a_1 \in A$.

Example 2.3. ([10]) For the commutative $(n, m)-$semiring $\left(\mathbb{N}^*, \sum\limits_{i=1}^{n}, \prod\limits_{j=1}^{m}\right)$, where $N^* = \mathbb{N}\backslash\{0\}$, derived from the semiring $(\mathbb{N}^*, +, \cdot)$, by repeating the binary operations, each sub-n-semigroup of $(\mathbb{N}^*, \sum\limits_{i=1}^{n})$, $k\mathbb{N}^*$ is an $(n, m)-$semiring ideal of $\left(\mathbb{N}^*, \sum\limits_{i=1}^{n}, \prod\limits_{j=1}^{m}\right)$, too. Moreover $k\mathbb{N}^*$ is a subtractive ideal.

Also, for every $k, b \in \mathbb{N}^*$ the subset $k\mathbb{N}^*$; $A_b = \{a \in \mathbb{N}^*; a \geq b\}$ and $A_{k,b} = k\mathbb{N}^* \cap A_b$ are examples of $(n, m)-$semiring ideals, but there are various others. We note that unlike $k\mathbb{N}^*$, A_b and $A_{k,b}$ are not subtractive ideals.

Example 2.4. The set $A = \{0, 1, 2, 3\}$ with the operations $(\,)_+ : A^3 \to A$, $\circ : A^2 \to A$

$$(a_1^3)_+ = \begin{cases} a_1 + a_2 + a_3 & \text{if } a_1 + a_2 + a_3 \leq 3 \\ r \equiv a_1 + a_2 + a_3 (\mathrm{mod} 2); 2 \leq r < 4 & \text{if } a_1 + a_2 + a_3 \geq 4 \end{cases}$$

respectively,

$$a_1 \circ a_2 = \begin{cases} a_1 \cdot a_2 & \text{if } a_1 \cdot a_2 \leq 3 \\ r \equiv a_1 \cdot a_2 (\mathrm{mod} 2); 2 \leq r < 4 & \text{if } a_1 \cdot a_2 \geq 4 \end{cases}$$

is a commutative $(3, 2)$-semiring with zero element 0 and one multiplicative identity 1. The set of all additive idempotents $\mathrm{Ida}(A) = \{0, 2, 3\}$ is an ideal of $(3, 2)-$semiring A, but is not subtractive ideal since $\mathrm{cl}\{0, 2, 3\} = S \neq \mathrm{Ida}(S)$.

Remark 2.2. In general, the set of all additive idempotents $\mathrm{Ida}(A)$ is not necessarily subtractive ideal.

Example 2.5. Let $\mathbb{Z}_0^-$ be the set of all negative integers with zero. Then $\mathbb{Z}_0^-$ endowed with the usual binary addition and $(2m+1)-$ary multiplication $(x_1^{2m+1})_\circ = x_1 \cdot x_2 \cdot \ldots \cdot x_{2m+1}$, form a commutative $(2, 2m+1)-$semiring with zero and identity element. For $m = 1$ we obtain the $(2, 3)-$semiring, so called "ternary semiring" defined by Kar S [6]. The subsets $k\mathbb{Z}_0^-$; where $k \in \mathbb{N}$ are subtractive ideals of this $(2, 2m+1)-$semiring.

Proposition 2.2. ([9]) *Let $(S, (\,)_+, (\,)_\circ)$ be an $(n, m)-$semiring. If U is a sub$-(n, m)-$semiring of S and A is an ideal of $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$, then:*

$i)$ $U \cap A$ is either empty set or a $(n, m)-$semiring ideal of the $(n, m)-$semiring $(U, (\,)_+, (\,)_\circ)$.

$ii)$ If A is a subtractive ideal and $U \cap A \neq \emptyset$, then $U \cap A$ is also subtractive.

Proof. i) Assume that $U \cap A \neq \varnothing$. Since $(U, (\,)_+)$ and $(A, (\,)_+)$ are, in particular sub-n-semigroups of S, if $x_1, x_2, \ldots, x_n \in U \cap A$ it follows that $(x_1^n)_+ \in U \cap A$.

Let $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots u_m \in U$ be any elements in U and $x \in U \cap A$. Since $(U, (\,)_\circ)$ is a sub-n-semigroup, we will have $(u_1^{i-1}\, x\, u_{i+1}^m) \in U$.
But A is an ideal of the $(n, m)-$ semiring S and $U \subseteq S$. It follows that $(u_1^{i-1}\, x\, u_{i+1}^m)_\circ \in A$. Consequently, $(u_1^{i-1} x\, u_{i+1}^m)_\circ \in U \cap A$.

ii) Assume that $U \cap A = \varnothing$. If $y_2, ..., y_m \in U \cap A$, and $x \in U$ with $(x\, y_2^m)_+ \in U \cap A$, considering that A is an $k-$ ideal, it results that $x \in A$. Therefore $x \in U \cap A$ which shows that $U \cap A$ is an subtractive ideal of the $(n, m)-$ semiring U. □

Remark 2.3. The intersection of $k-$ideals is again a $k-$ideal, whereas the semiring ideal $A \cup B \bigcup\limits_{i=1}^{n-1} (\overset{(i)}{A}, \overset{(n-i)}{B})_+$ need not be $k-$ideal. Indeed in Example 2.3, if we consider the $k-$ideals $2\mathbb{N}^*$,$3\mathbb{N}^*$, then the subset $2\mathbb{N}^* \cup 3\mathbb{N}^* \cup (\overset{(i)}{2\mathbb{N}^*}, \overset{(n-i)}{3\mathbb{N}^*})_+ = \mathbb{N}^* \setminus \{1\}$ is not a subtractive ideal.

3. Morphism of $(n, m)-$ semirings

Definition 3.11. Let $(S, (\,)_+, (\,)_\circ)$ and $(S', (\,)_*, (\,)_\bullet)$ be $(n, m)-$semirings. The function $f : S \to S'$ is called the morphism of $(n, m)-$semirings, if for any $x_i \in S$ with $i \in \{1, 2, \ldots, \max(n, m)\}$ the following statements are true:

$$f((x_1^n)_+) = (f(x_1), \ldots, f(x_n))_*,$$

$$f((x_1^m)_\circ) = (f(x_1), \ldots, f(x_m))_\bullet.$$

A morphism from the semiring S into the semiring S is called endomorphism. A semiring isomorphism is both injective and surjective . The semirings S and S' will be called isomorphic semirings if there exists an isomorphism from S onto S'. In this case we will write $(S, (\,)_+, (\,)_\circ) \cong (S', (\,)_*, (\,)_\bullet)$.

Next, for simplification, we will write the operations of the two $(n, m)-$semirings in the same way.

Theorem 3.1. *Let $(S, (\,)_+, (\,)_\circ)$ and $(S', (\,)_+, (\,)_\circ)$, be $(n, m)-$semirings and $f : S \to S'$ a morphism of $(n, m)-$semirings.*

i) If f is a surjective morphism and the $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$ has a zero element 0, then $f(0) = 0'$ is a zero element in the $(n, m)-$semiring $(S', (\,)_+, (\,)_\circ)$.

ii) If $a \in S$ is an additive (multiplicative) idempontent of $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$, then $f(a)$ is an additive (multiplicative) idempontent in the $(n, m)-$semiring $(S', (\,)_+, (\,)_\circ)$.

iii) If the $(n, m)-$ semiring $(S, (\,)_+, (\,)_\circ)$ has an additive (multiplicative) neutral element, and f is surjective, then $f(e)$ is an additive (multiplicative) neutral element.

iv) If an element $x \in S$ admits an additive querelement $\overline{x} \in S$ (multiplicative querelement $\underline{x} \in S$), then there exists a querelement of $f(x)$ and the following applies:

$$f(\overline{x}) = \overline{f(x)} \ (f(\underline{x}) = \underline{f(x)}).$$

Proof. i) Since f is a surjective morphism, then for every $y_1, y_2, ..., y_m \in S'$ there exists $x_1, x_2, ..., x_m \in S$ such that $f(x_i) = y_i$, $i \in \{1, 2, ..., m\}$. If we denote by $f(0) = 0'$, we will have:

$$\begin{aligned}\left(y_1^{i-1}\ 0'\ y_{i+1}^m\right)_\circ &= (f(x_1), ..., f(x_{i-1}), f(0), f(x_i), ..., f(x_m))_\circ \\ &= f\left((x_1^{i-1}\ 0\ x_{i+1}^m)_\circ\right) = f(0) = 0'.\end{aligned}$$

ii) If $a \in S$ is an additive (multiplicative) idempotent in $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$, that is, $a^{[1]} = a$ (respectively $a^{<1>} = a$) then

$$f(a) = f(a^{[1]}) = (\overset{(n)}{f(a)})_+ = f(a)^{[1]} \ \left(f(a) = f(a^{<1>}) = f(a)^{<1>}\right).$$

iii) If $e \in S$ is a neutral additive (multiplicative) element, then for any $x \in S$ and any $i \in \{1, 2, ..., p\}$, $p = max(n, m)$ we have:

$$(\overset{(i-1)}{e}\ x\ \overset{(n-i)}{e})_+ = x \ \ \left((\overset{(i-1)}{e}\ x\ \overset{(m-i)}{e})_\circ = x\right)$$

Since f is a morphism, then

$$f(x) = f((\overset{(i-1)}{e}\ x\ \overset{(n-i)}{e})_+) = (\overset{(i-1)}{f\ (e)}, f(x), \overset{(n-i)}{f\ (e)})_+.$$

$$\left(f(x) = f((\overset{(i-1)}{e}\ x\ \overset{(m-i)}{e})_\circ) = (\overset{(i-1)}{f\ (e)}, f(x), \overset{(m-i)}{f\ (e)})_\circ\right).$$

From the definition of the additive (multiplicative) neutral element, it follows that $f(e)$ is an additive (multiplicative) neutral element.

iv) If $x \in S$ has an additive querelement $\overline{x} \in S$, then x is unique and $(\overset{(n-1)}{x}\ \overline{x})_+ = x$. Considering the properties of morphism f and the uniqueness of the querelement element, it follows that $f(x) = f((\overset{(n-1)}{x}\ \overline{x})_+) = (\overset{(n-1)}{f\ (x)}\ f(\overline{x}))_+$. Consequently, we obtain $f(\overline{x}) = \overline{f(x)}$.

□

It is easy to prove the following:

Theorem 3.2. *Let $(S, (\,)_+, (\,)_\circ)$, $(T, (\,)_+, (\,)_\circ)$ and $(R, (\,)_+, (\,)_\circ)$ be $(n, m)-$semirings.*

i) If $f : S \to T$ and $\varphi : T \to R$ are morphisms of $(n, m)-$semirings, then $\varphi \circ f : S \to R$ is a morphism of $(n, m)-$semirings.

ii) The identity function on S, $1_S : S \to S$, $1_S(x) = x$ for any $x \in S$ is an isomorphism of the $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$.

iii) If $f : S \to T$ is an isomorphism of $(n, m)-$semirings, then the inverse function $f^{-1} : T \to S$ is an isomorphism of $(n, m)-$semirings, too.

Using the properties of universal algebras we obtain:

Proposition 3.3. *Let $(S, (\,)_+, (\,)_\circ)$ and $(S', (\,)_+, (\,)_\circ)$ be two universal algebras of the same type, with the $n-$ary operation "$()_+$", the $m-$ary operation "$()_\circ$", and $f : S \to S'$ a morphism.*

i) If $(S, (\,)_+, (\,)_\circ)$ is an $(n, m)-$semiring, then the subalgebra $(f(S), (\,)_+, (\,)_\circ)$ of S' is an $(n, m)-$semiring. If $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$ is semicomutative (commutative), respectively with multiplicative neutral element, then $(n, m)-$semiring $(f(S), (\,)_+, (\,)_\circ)$ is semicomutative (commutative) with a multiplicative neutral element. Generally, if A is a sub-(n, m)-semiring of $(S, (\,)_+, (\,)_\circ)$, then the homomorphic image $f(A)$ is a sub-(n, m)-semiring of $(S', (\,)_+, (\,)_\circ)$.

ii) If $(S,(\,)_+,(\,)_\circ)$ and $(S',(\,)_+,(\,)_\circ)$ are $(n,m)-$semirings and A' is a sub-(n,m)-semiring of $(S',(\,)_+,(\,)_\circ)$, then the inverse image of A' by f, $\overset{-1}{f}(A')$, is either the empty set or a sub-(n,m)-semiring of $(S,(\,)_+,(\,)_\circ)$.

Proof. (i) Immediately verified

(ii) Let us observe that $\overset{-1}{f}(A')=\overset{-1}{f}(A'\cap f(S))$. Notice that $\overset{-1}{f}(A')=\emptyset$ if and only if $A'\cap f(S)=\emptyset$. Otherwise, since A' and $f(S)$ are sub-(n,m)-semirings , it follows that $A'\cap f(S)$ is a sub-(n,m)-semiring of $(n,m)-$semiring $(S',(\,)_+,(\,)_\circ)$. If $a_1,a_2,\dots,a_p\in\overset{-1}{f}(A')$, $p=max(n,m)$, then $f(a_1),f(a_2),...,f(a_p)\in A'\cap f(S)$.
It follows that $(f(a_1),\dots,f(a_n))_+\in A'\cap f(S)$, respectively $(f(a_1),\dots,f(a_m))_\circ\in A'\cap f(S)$.
Therefore $f((a_1^n)_+)\in A'\cap f(S)$ and $f((a_1^m)_\circ)\in A'\cap f(S)$ which yields to $(a_1^n)_+\in\overset{-1}{f}(A')$, respectively $(a_1^m)_\circ\in\overset{-1}{f}(A')$. □

If $(n,m)-$semirings are $(n,m)-$rings, then we find the Theorem 2 of I. Purdea ([13]).

Corollary 3.1. *The algebraic structure $(Imf,()_+()_\circ)$ is a sub-(n,m)-semiring of $(n,m)-$semiring $(S',(\,)_+,(\,)_\circ)$.*

Corollary 3.2. *Let $(S,(\,)_+,(\,)_\circ)$ be an $(n,m)-$semiring with zero element 0, $(S',(\,)_+,(\,)_\circ)$ an universal algebra.*

(i) If $(n,m)-$semiring $(S,(\,)_+,(\,)_\circ)$ has no divisors of zero and the morphism $f:S\to S'$ is injective, then $f(S)$ has no divisors of zero.

(ii) If $(n,m)-$semiring $(S,(\,)_+,(\,)_\circ)$ is an integral semidomain and the morphism $f:S\to S'$ is injective, then $(f(S),(\,)_+(\,)_\circ)$ is an integral semidomain.

(iii) If the morphism $f:S\to S'$ is surjective, then the universal algebra $(S',(\,)_+,(\,)_\circ)$ is an $(n,m)-$semiring.

Proof. i) Indeed, for any $y_1,y_2,\dots,y_m\in f(S)$ there are $x_1,x_2,\dots,x_m\in S$ with the property that $f(x_i)=y_i$, $i\in\{1,2,..,m\}$. If $(y_1^m)_\circ=f(0)$, then we obtain $(f(x_1),\dots,f(x_m))_\circ=f(0)$, respectively $f((x_1^m)_\circ)=f(0)$. Since the morphism f is injective, it results $(x_1^m)_\circ=0$. But $(n,m)-$semiring $(S,(\,)_+,(\,)_\circ)$ has no divisors of zero, so there exists an $i\in\{1,2,\dots,m\}$ such that $x_i=0$. It follows that $f(x_i)=f(0)$, $f(0)$ being zero element in $(S',(\,)_+,(\,)_\circ)$ in accordance with Theorem 3.1. (i). Therefore there exists $i\in\{1,2,\dots,m\}$ such that $y_i=f(0)$, which shows us that $(n,m)-$semiring $(f(S),(\,)_+,(\,)_\circ)$ has no divisors of zero. □

It is easy to prove the following:

Theorem 3.3. *Let $(S,(\,)_+)$ be the $n-$semigrup comutative of $(n,m)-$semiring $(S,(\,)_+,(\,)_\circ)$. Then the set of endomorphisms of this $n-$semigrup, denoted End$(S,(\,)_+)$, forms an $(n,m)-$semiring endowed with operations*

$$[f_1,f_2,\dots,f_n]_+(x)=(f_1(x),\dots,f_n(x))_+,$$

respectively

$$(g_1,g_2,\dots,g_n)_\star(x)=(g_1(g_2(\dots(g_m(x))\dots)))$$

where $f_i,g_j\in$ End$(S,(\,)_+)$; $i\in\{1,2\dots,n\}$, $j\in\{1,2,\dots,m\}$. The identity function $1_S:S\to S$, $1_S(x)=x$ is a multiplicative neutral element in $(n,m)-$semiring (End$(S,(\,)_+),[\,]_+,(\,)_\star$).

The following theorem is a generalization of Theorem 6 ([13])

Theorem 3.4. *If $(S,(\,)_+,(\,)_\circ)$ is an $(n,m)-$semiring, $c_1,c_2,\ldots,c_{m-2}\in S$ are fixed elements, not necessarily distinct, then:*

i) the function $\varphi:(S,(\,)_+)\to End(S,(\,)_+)$, $\varphi(a)=t_a$ with $t_a:S\to S,\ t_a(x)=(a\,c_1^{m-2}\,x)_\circ$ is a morphism of $n-$semigroups.

ii) If $(n,m)-$semiring S is commutative and $c_1,c_2,\ldots,c_{m-2}\in Idm(S)$, then φ is a morphism of $(n,m)-$semirings. If S is a multiplicatively cancellative $(n,m)-$semiring , then the φ is an injective morphism.

Proof. i) Let the elements $a_1,a_2,\ldots,a_n\in S$ be. Then

$$\varphi((a_1^n)_+)=t_{(a_1^n)_+}\ \text{ and }\ t_{(a_1^n)_+}(x)=((a_1^n)_+,c_1^{m-2},x)_\circ,\ \text{for any } x\in S.$$

Applying the distributivity law, we will have

$$t_{(a_1^n)_+}(x)=((a_1\,c_1^{m-2}\,x)_\circ,\ldots,(a_n\,c_1^{m-2}\,x)_\circ)_+=(t_{a_1}(x),\ldots,t_{a_n}(x))_+$$

for any $x\in S$. Therefore $\varphi((a_1^n)_+)=(\varphi(a_1),\ldots,\varphi(a_n))_+$ for all $a_1,a_2,\ldots,a_n\in S$, so φ is a morphism of $n-$semigroups.

ii) If we consider the elements $a_1,a_2,\ldots,a_m\in S$, then we have

$$\varphi((a_1^m)_\circ)=t_{(a_1^m)_\circ}\ \text{ and }\ t_{(a_1^m)_\circ}(x)=((a_1^m)_\circ\,c_1^{m-2}\,x)_\circ,\ \text{ for any } x\in S.$$

Using the commutativity and the associativity of the $m-$ary operation and also the fact that $c_i^{<1>}=c_i$ for any $i\in\{1,2,\ldots,m-2\}$, it follows

$$t_{(a_1^m)_\circ}(x)=((a_1^m)_\circ,c_1^{<1>},\ldots,c_{m-2}^{<1>},x)_\circ=(a_1,c_1^{m-2},(a_2c_1^{m-2},(\ldots(a_m\,c_1^{m-2}\,x)_\circ\ldots)))_\circ$$
$$=t_{a_1}(t_{a_2}(\ldots(t_{a_m}(x))\ldots))=(t_{a_1}\circ t_{a_2}\circ\ldots\circ t_{a_m})(x),$$

for any $x\in S$. Therefore $\varphi((a_1^m)_\circ)=(\varphi(a_1),\varphi(a_2),\ldots\varphi(a_m))_\star$.

Next we want to show that the morphism φ is injective. If we assume that $\varphi(a)=\varphi(b)$ it follows that $t_a=t_b$ and so $t_a(x)=t_b(x)$ for any $x\in S$. Since S is a multiplicatively cancellative $(n,m)-$semiring, it follows that $(a\,c_1^{m-2}\,x)_\circ=(b\,c_1^{m-2}\,x)_\circ\Rightarrow a=b$.

□

In the particular case of $(n,2)-$semirings, a generalization of the Theorem 7 ([13]) is obtained.

Corollary 3.3. *If $(S,(\,)_+,\cdot)$ is an $(n,2)-$semiring, then function $\varphi:S\to End\,(S,(\,)_+),\varphi(a)=t_a$, $t_a:S\to S$, with $t_a(x)=a\cdot x$ is a morphism of $(n,2)-$semirings. If $(n,2)-$semiring S has a multiplicative neutral element, then function φ is an injective morphism.*

Remark 3.4. Theorem 3.3 and Theorem 3.4 are true even when we consider a generalization of $(n,m)-$semiring , namely the $n-$ary operation $(\,)_+$ is not commutative, but it is entropic (medial) (see [9]).

Proposition 3.4. *Let $(S,(\,)_+,(\,)_\circ)$ and $(S',(\,)_+,(\,)_\circ)$ be $(n,m)-$semirings and $f:S\to S'$ a surjective morphism of $(n,m)-$semirings. If A and A' are ideals in $(n,m)-$semirings S and S', then:*

i) The set $f(A)=\{f(a)|a\in A\}$ is an ideal in the $(n,m)-$semiring $(S',(\,)_+,(\,)_\circ)$.

ii) the inverse image of A' by f, $\overset{-1}{f}(A')=\{a\in A|\,f(a)\in A'\}$ is either the empty set or an ideal in $(n,m)-$semiring $(S,(\,)_+,(\,)_\circ)$. In addition, if A' is a subtractive ideal, then its inverse image $\overset{-1}{f}\ (A')$ is a subtractive ideal (if it is not the empty set).

Proof. i) Since $f(A)$ is in particular a sub-(n,m)-semiring of $(S,()_+,()_\circ)$, the pair $(f(A),()_+)$ is a sub-n-semigroup of the $n-$semigroup $(S,(\,)_+)$. We will further show that $f(A)$ is an ideal in $(n,m)-$semiring $(S',(\,)_+,(\,)_\circ)$.

Let $a' \in f(A)$ and $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_m \in S'$.. Considering that f is a surjective morphism, it follows that there exists $a \in A$ and $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m \in S$ with the property that $f(a) = a'$ and $f(s_j) = s'_j$, for any $j \in \{1, 2, \ldots, i-1, i+1, \ldots, m\}$. Since A is an ideal in $(n, m)-$semiring S, it follows that $(s_1^{i-1} a\, s_{i+1}^m)_\circ \in A$ and we will have $(s_1'^{i-1}\, a'\, s_{i+1}'^m)_\circ = (f(s_1), \ldots, f(s_{i-1}), f(a), f(s_{i+1}), \ldots, f(s_m))_\circ = f((s_1^{i-1}\, a\, s_{i+1}^m)_\circ) \in f(A)$.

ii) Let $\overset{-1}{f}(A') \neq \emptyset$ and $x_1, x_2, \ldots, x_n \in \overset{-1}{f}(A')$. It follows that $f(x_1), \ldots, f(x_n) \in A'$. Since, in particular, A' is a sub-n-semigrup of the semigroup $(S', (\,)_+)$ we have $(f(x_1), \ldots, f(x_n))_+ \in A'$. The function f is a morphism of $(n, m)-$semirings, therefore $f((x_1^n)_+) \in A'$. In conclusion $(x_1^n)_+ \in \overset{-1}{f}(A')$.

If $a \in \overset{-1}{f}(A')$ and $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m \in S$, considering that A' is an ideal in S', then

$$f((s_1^{i-1}\, a\, s_{i+1}^m)_\circ) = (f(s_1), \ldots, f(s_{i-1}), f(a), f(s_{i+1}), \ldots, f(s_m))_\circ \in A'.$$

It follows that $(s_1^{i-1}\, a\, s_{i+1}^m)_\circ \in \overset{-1}{f}(A')$. Therefore, $\overset{-1}{f}(A')$ is an ideal of $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$.

Further we will show that $\overset{-1}{f}(A')$ is a subtractive ideal. Indeed, if $x \in S$ and $a_2, \ldots, a_m \in \overset{-1}{f}(A')$ with the property that $(x\, a_2^m)_+ \in \overset{-1}{f}(A')$, then $f((x\, a_2^m)_+) \in A'$. Considering the fact that f is a morphism of $(n, m)-$semirings, it follows $(f(x), f(a_2), \ldots, f(a_m))_+ \in A'$. But A' is a subtractive ideal in $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$. Hence $f(x) \in A'$ and therefore $x \in \overset{-1}{f}(A')$. In conclusion, $\overset{-1}{f}(A')$ is a subtractive ideal in $(n, m)-$semiring S. □

Remark 3.5. If A is a subtractive ideal of $(n, m)-$semiring $(S, (\,)_+, (\,)_\circ)$, it does not generally result that $f(A)$ is a subtractive ideal in $(n, m)-$semiring $(S', (\,)_+, (\,)_\circ)$.

Example 3.6. Let the set $S = \{0, a, b, c\}$ endowed with a ternary operation $(\,)_+ : S^3 \to S$ defined as follows

$$(x, 0, 0)_+ = x\,; (x, c, c)_+ = c \text{ for all } x \in S,$$

$$(x, a, a)_+ = \begin{cases} a, & \text{if } x \in \{0, a\} \\ c, & \text{if } x \in \{b, c\} \end{cases}$$

$$(x, b, b)_+ = \begin{cases} b, & \text{if } x \in \{0, b\} \\ c, & \text{if } x \in \{a, c\} \end{cases}$$

$$(0, a, b)_+ = (c, a, b)_+ = (0, c, a)_+ = (0, c, b)_+ = c,$$

and the m-ary operation $(\,)_\circ : S^m \to S$ defined by $(x_1^m)_\circ = 0$, for any $x_1, x_2, ..., x_m \in S$. The algebraic structure $(S, (\,)_+, (\,)_\circ)$ is a commutative $(3, m)-$semiring with additive neutral element 0 which is also the zero element of the semiring and the set of additive idempotents $\mathrm{Ida}(S) = S$.

Let $T = \{0', a', c'\}$ be a set endowed with the ternary operation "$(\,)_\star$" defined as follows $(x', 0', 0')_\star = x'$; $(x', c', c')_\star = c'$, for all $x' \in T$, $(0', a', c')_* = c'$,

$$(x', a', a')_\star = \begin{cases} a', & \text{if } x' \in \{0', a'\} \\ c', & \text{if } x' = c' \end{cases}$$

and the $m-$ary operation "$(\,)_\circ$" defined by $(x_1'^m)_\circ = 0'$ for all $x'_1, x'_2, \ldots, x'_m \in T$.

Then $(T, (\,)_\star, (\,)_\circ)$ is also a commutative $(3, m)-$semiring with an additive neutral element $0'$, which is also the zero element of the semiring T and $\mathrm{Ida}(T) = T$.

The function $f : S \to T$; $f(0) = 0'$, $f(a) = f(c) = c'$, and $f(b) = a'$ is a surjective morphism.

The set $A = \{0, a\}$ is a subtractive ideal of S, but $f(A) = \{0', c'\}$ is not a subtractive ideal, because $(a'\,0'\,c')_\star = c' \in f(A)$, but $a' \notin f(A)$.

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