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Coefficient estimates for a new subclass of bi-univalent functions defined by convolution

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ABSTRACT. In this paper we introduce general subclasses of bi-univalent functions by using convolution. Bounds for the first two coefficients $|a_2|$ and $|a_3|$ for bi-univalent functions in these classes are obtained. The obtained results generalize the results which are given in [Murugusundaramoorthy, G., Magesh, M., Prameela, V., *Coefficient bounds for certain subclasses of bi-univalent function*, Abstr. Appl. Anal., (2013), Art. ID 573017, 3 pp.] and [Brannan, D. A. and Taha, T. S., *On some classes of bi-univalent functions*, Studia Univ. Babeş Bolyai Math., **31** (1986), No. 2, 70–77].

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and normalized by

$$f(0) = 0 \ \text{ and } f'(0) = 1. \tag{1.2}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions $f(z)$ of the form (1.1).

For the function $f(z)$ defined by (1.1) and the function $h(z)$ defined by

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \quad (h_n \geq 0)\,, \tag{1.3}$$

the Hadamard product (or convolution) of $f(z)$ and $h(z)$ is given by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n h_n z^n = (h * f)(z).$$

According to Koebe-One-Quarter Theorem [10], every function f in $\mathcal{S}$ has an inverse function f^{-1} such that

$$f^{-1}(f(z)) = z, \ \ (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w, \ \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right).$$

Then the inverse function $f^{-1}(w)$ has the following Taylor expansion

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) + \dots. \tag{1.4}$$

Let Σ denote denote the class of univalent functions in $\mathbb{D}$. First, Lewin [15] studied the class of bi-univalent functions finding $|a_2| \leq 1.52$.

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Netenyahu [19] showed that $\max|a_2| = \frac{4}{3}$ for $f \in \Sigma$. After that, Brannan and Taha [5] defined the class of bi-starlike functions of order β and bi-convex functions of order β, denoted by $S^*_\Sigma(\beta)$ and $K_\Sigma(\beta)$, respectively. They found upper bounds on initial coefficients of functions in these classes.

Recently, many interesting results have been obtained in many articles [1], [2], [8], [9], [11], [14], [16],[17], [21], [22], [23], [25]. In the literature, there are certain results investigating the general bounds on $|a_n|$ for analytic bi-univalent functions under some special conditions, [3],[4], [6], [12],[13]. Hence, it is still open problem to find sharp bound on $|a_n|$ for $n \geq 4$.

On the other hand, Murugusundaramoorthy et al. [18] introduced the following two subclasses of the class $\sum$ of bi-univalent functions and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

Definition 1.1 ([12]). A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{G}_\Sigma(\alpha,\lambda)$ if the following conditions are satisfied:

$$f \in \sum, \left|\arg\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right)\right| \leq \frac{\alpha\pi}{2},\ 0<\alpha\leq 1,\ 0\leq\lambda<1,\ z\in\mathbb{D}$$

and

$$\left|\arg\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}\right)\right| \leq \frac{\alpha\pi}{2},\ 0<\alpha\leq 1,\ 0\leq\lambda<1,\ w\in\mathbb{D},$$

where g is the inverse function of f.

Also, they give the following results for functions in $\mathcal{G}_\Sigma(\alpha,\lambda)$.

Theorem 1.1 ([12]). *Let $f(z)$ given by (1.1) in the class $\mathcal{G}_\Sigma(\alpha,\lambda)$, $0<\alpha\leq 1$ and $0\leq\lambda<1$. Then*

$$|a_2| \leq \frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}}, \quad |a_3| \leq \frac{4\alpha^2}{(1-\lambda)^2} + \frac{\alpha}{1-\lambda}.$$

Definition 1.2 ([12]). A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{M}_\Sigma(\alpha,\lambda)$ if the following conditions are satisfied:

$$f \in \sum, \text{ and } Re\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right) > \beta, 0<\alpha\leq 1,\ 0\leq\lambda<1,\ z\in\mathbb{D}$$

and

$$Re\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}\right) > \beta, 0<\alpha\leq 1,\ 0\leq\lambda<1,\ w\in\mathbb{D},$$

where the function g is the inverse function given by (1.4).

They also found upper bounds for initial coefficients of functions in the class $\mathcal{M}_\Sigma(\alpha,\lambda)$.

Theorem 1.2 ([12]). *Let $f(z)$ given by (1.1) in the class $\mathcal{M}_\Sigma(\alpha,\lambda)$, $0\leq\beta<1$ and $0\leq\lambda<1$. Then*

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{1-\lambda}, \quad |a_3| \leq \frac{4(1-\beta)^2}{(1-\lambda)^2} + \frac{1-\beta}{1-\lambda}.$$

In the paper [7], we can find coefficient bounds more general than the results in Theorem 1.1 and Theorem 1.2. In the present paper, we define new subclasses of bi-univalent functions and also generalize the results in [18] and [5].

Definition 1.3. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{M}_{\Sigma}(h,\alpha,\lambda)$, if the following conditions are satisfied:

$$f \in \sum, \ \left|\arg\left(\frac{z(f*h)'(z)}{(1-\lambda)(f*h)(z)+\lambda z(f*h)'(z)}\right)\right| \leq \frac{\alpha\pi}{2}, \tag{1.5}$$

$$\left|\arg\left(\frac{w((f*h)^{-1})'(w)}{(1-\lambda)((f*h)^{-1})(w)+\lambda w((f*h)^{-1})'(w)}\right)\right| \leq \frac{\alpha\pi}{2}, 0<\alpha\leq 1,\ 0\leq\lambda<1,\ z,w\in\mathbb{D},$$

where the function $h(z)$ is defined by (1.3) and $(f*h)^{-1}(w)$ is defined by

$$(f*h)^{-1}(w) = w - a_2h_2w^2 + \left(2a_2^2h_2^2 - a_3h_3\right)w^3 - \left(5a_2^3h_2^3 - 5a_2h_2a_3h_3 + a_4h_4\right)w^4 + \ldots \tag{1.6}$$

and

$$\left((f*h)^{-1}\right)'(w) = 1 - 2a_2h_2w + 3\left(2a_2^2h_2^2 - a_3h_3\right)w^2 - \ldots. \tag{1.7}$$

Remark 1.1. 1) Note that $\mathcal{M}_{\Sigma}(\frac{z}{1-z},\alpha,\lambda) = \mathcal{G}_{\Sigma}(\alpha,\lambda)$, which was studied in [18].
2) $\mathcal{M}_{\Sigma}(\frac{z}{1-z},\alpha,0) = S_{\Sigma}(\alpha)$ is the the class of all strong bi-starlike functions of order α introduced by Brannan and Taha [5].

Definition 1.4. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{F}_{\Sigma}(h,\beta,\lambda)$, if the following conditions are satisfied:

$$f \in \sum, \ Re\left(\frac{z(f*h)'(z)}{(1-\lambda)(f*h)(z)+\lambda z(f*h)'(z)}\right) > \beta, \tag{1.8}$$

$$Re\left(\frac{w((f*h)^{-1})'(w)}{(1-\lambda)((f*h)^{-1})(w)+\lambda w((f*h)^{-1})'(w)}\right) > \beta, 0<\alpha\leq 1,\ 0\leq\lambda<1,\ z,w\in\mathbb{D},$$

where $h(z)$ and $(f*h)^{-1}(w)$ are defined in (1.6) and (1.7), respectively.

Remark 1.2. $\mathcal{F}_{\Sigma}(\frac{z}{1-z},\beta,\lambda) = \mathcal{M}_{\Sigma}(\alpha,\lambda)$, which was studied by Murugusundaramoorthy et al [18]. $\mathcal{F}_{\Sigma}(\frac{z}{1-z},\beta,0)$ is the class of bi-starlike functions of order β which was first defined in [24].

In order to obtain our main results, we need the following lemma.

Lemma 1.1. [20] *If $p(z)\in\mathcal{P}$, then $|p_n|\leq 1$ for each n, where $\mathcal{P}$ is the class of functions $p(z)$,*

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \ldots,\ \forall z\in\mathbb{D},$$

analytic in $\mathbb{D}$ for which $Re\,(p(z)) > 0$.

2. Main results

Theorem 2.3. *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{M}_{\Sigma}(h,\alpha,\lambda), 0<\alpha\leq 1$ and $0\leq\lambda<1$. Then*

$$|a_2| \leq \frac{2\alpha}{h_2(1-\lambda)\sqrt{1+\alpha}}, \quad |a_3| \leq \frac{\alpha}{(1-\lambda)h_3}\left\{\frac{4\alpha}{(1-\lambda)}+1\right\}.$$

Proof. It is obvious from definition of the class $\mathcal{M}_{\Sigma}(h,\alpha,\lambda)$,

$$\begin{aligned}\frac{z(f*h)'(z)}{(1-\lambda)(f*h)(z)+\lambda z(f*h)'(z)} &= [p(z)]^{\alpha}\\ \frac{w((f*h)^{-1})'(w)}{(1-\lambda)((f*h)^{-1})(w)+\lambda w((f*h)^{-1})'(w)} &= [q(w)]^{\alpha},\end{aligned} \tag{2.9}$$

where the functions $p(z)$ and $q(w)$ satisfy the following

$$Re(p(z)) > 0,\ z \in \mathbb{D}, \quad Re(q(w)) > 0,\ w \in \mathbb{D}.$$

Also, these functions have the following expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \tag{2.10}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots \tag{2.11}$$

Equating the coefficients in (2.9), we obtain

$$(1-\lambda)a_2 h_2 = \alpha p_1 \tag{2.12}$$

$$2(1-\lambda)a_3 h_3 = \alpha\left[p_2 + \frac{(\alpha-1)}{2}p_1^2\right] + \alpha^2 p_1^2 \frac{(1+\lambda)}{(1-\lambda)} \tag{2.13}$$

$$-(1-\lambda)a_2 h_2 = \alpha q_1 \tag{2.14}$$

$$2(1-\lambda)\left(2a_2^2 h_2^2 - a_3 h_3\right) = \alpha\left[q_2 + \frac{(\alpha-1)}{2}q_1^2\right] + \alpha^2 q_1^2 \frac{(1+\lambda)}{(1-\lambda)} \tag{2.15}$$

From (2.12) and (2.14) we get

$$p_1 = -q_1 \tag{2.16}$$

and

$$2(1-\lambda)^2 a_2^2 h_2^2 = \alpha^2\left(p_1^2 + q_1^2\right). \tag{2.17}$$

Using equations (2.13) and (2.15), we obtain

$$4(1-\lambda)a_2^2 h_2^2 = \alpha\left(p_2 + q_2\right) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 + q_1^2\right) + \alpha^2\frac{(1+\lambda)}{(1-\lambda)}\left(p_1^2 + q_1^2\right). \tag{2.18}$$

From equality (2.17), it is obvious that

$$a_2^2 = \frac{\alpha^2\left(p_2 + q_2\right)}{h_2^2(1-\lambda)^2(1+\alpha)}. \tag{2.19}$$

According to Lemma 1.1, we have

$$|a_2| \le \frac{2\alpha}{h_2(1-\lambda)\sqrt{1+\alpha}}.$$

In order to find the bound of $|a_3|$, we substract (2.13) from (2.15) and obtain

$$4(1-\lambda)\left(a_3 h_3 - a_2^2 h_2^2\right) = \alpha\left(p_2 - q_2\right) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 - q_1^2\right) + \alpha^2\frac{(1+\lambda)}{(1-\lambda)}\left(p_1^2 - q_1^2\right). \tag{2.20}$$

By using equation (2.17) and observing that $p_1^2 = q_1^2$, equation (2.20) can be reduced to the following form

$$4(1-\lambda)a_3 h_3 = \frac{2\alpha^2\left(p_1^2 + q_1^2\right)}{(1-\lambda)} + \alpha(p_2 - q_2). \tag{2.21}$$

Now, applying Lemma 1.1, we get the desired result

$$|a_3| \le \frac{\alpha}{(1-\lambda)h_3}\left\{\frac{4\alpha}{(1-\lambda)} + 1\right\}. \tag{2.22}$$

□

According to Remark 1.1, we obtain the following special results if we specialize function $h(z)$ and parameter λ.

Corollary 2.1. *If we set* $h(z) = \frac{z}{1-z}$, *we obtain the results in Theorem 1.1.*

Remark 2.3. If we set $h(z) = \dfrac{z}{1-z}$ and $\lambda = 0$, we get the result for bi-strongly starlike functions given by Brannan and Taha [5].

Theorem 2.4. *Let* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{F}_\Sigma(h, \alpha, \lambda), 0 < \alpha \leq 1$ *and* $0 \leq \lambda < 1$. *Then*

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{(1-\lambda)\, h_2}, |a_3| \leq \frac{(1-\beta)}{(1-\lambda)\, h_3} \left\{1 + \frac{4(1-\beta)}{(1-\lambda)}\right\}.$$

Proof. We have the following relations from definition of the class $\mathcal{F}_\Sigma(h, \alpha, \lambda)$,

$$\frac{z(f*h)'(z)}{(1-\lambda)(f*h)(z) + \lambda z(f*h)'(z)} = \beta + (1-\beta)p(z) \tag{2.23}$$

$$\frac{w((f*h)^{-1})'(w)}{(1-\lambda)((f*h)^{-1})(w) + \lambda w((f*h)^{-1})'(w)} = \beta + (1-\beta)q(w), \tag{2.24}$$

where the functions $p(z)$ and $q(w)$ are given by (2.10) and (2.11), respectively.

From (2.24) we have

$$(1-\lambda)a_2 h_2 = (1-\beta)p_1 \tag{2.25}$$

$$2(1-\lambda)a_3 h_3 = (1-\beta)\left\{p_2 + (1-\beta)\frac{(1+\lambda)}{(1-\lambda)}p_1^2\right\} \tag{2.26}$$

$$-(1-\lambda)a_2 h_2 = (1-\beta)q_1 \tag{2.27}$$

$$2(1-\lambda)\left(2a_2^2 h_2^2 - a_3 h_3\right) = (1-\beta)\left\{q_2 + (1-\beta)\frac{(1+\lambda)}{(1-\lambda)}q_1^2\right\}. \tag{2.28}$$

From equations (2.25) and (2.27), it follows that

$$p_1 = -q_1 \tag{2.29}$$

and

$$2(1-\lambda)^2 a_2^2 h_2^2 = (1-\beta)^2\left(p_1^2 + q_1^2\right). \tag{2.30}$$

Now, from (2.26), (2.28) and (2.30), we have

$$4(1-\lambda)a_2^2 h_2^2 = (1-\beta)(p_2 + q_2) + (1-\beta)^2\frac{(1+\lambda)}{(1-\lambda)}\left(p_1^2 + q_1^2\right).$$

Now, according to Lemma 1.1, we obtain the following result

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{(1-\lambda)h_2}.$$

In order to find the upper bound for $|a_3|$ we extract (2.26) from (2.28) to get

$$4(1-\lambda)a_3 h_3 = \frac{2(1-\beta)^2\left(p_1^2 + q_1^2\right)}{(1-\lambda)} + (1-\beta)(p_2 - q_2).$$

By applying Lemma 1.1, we obtain

$$|a_3| \leq \frac{(1-\beta)}{(1-\lambda)h_3}\left\{\frac{4(1-\beta)}{(1-\lambda)} + 1\right\}.$$

□

In view of Remark 1.2, we obtain the following special results if we specialize the function $h(z)$ and our parameter λ.

Corollary 2.2. *If we set $h(z) = \frac{z}{1-z}$, we obtain the results given in Theorem 1.2.*

Remark 2.4. If we set $h(z) = \frac{z}{1-z}$ and $\lambda = 0$, we get the result for bi-starlike functions given by Taha [24].

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