CREAT. MATH. INFORM.
Volume **27** (2018), No. 2,
Pages 101 - 110

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2018.02.01

Generalized weighted statistical convergence in intuitionistic fuzzy normed linear spaces

SELMA ALTUNDAĞ and ESRA KAMBER

ABSTRACT. In this paper, we introduce a new statistical convergence type, named weighted λ-statistical convergence to generalize the concept of weighted statistical convergence with respect to the intuitionistic fuzzy norm (μ, υ). Moreover, we establish its relation to weighted statistical convergence and a new summability method, named as $\left(\overline{N}_\lambda, p\right)$-summability with respect to the intuitionistic fuzzy norm (μ, υ).

1. INTRODUCTION

Fuzzy set theory defined by Zadeh [41] has been investigated for the last 40 years. It has been studied many branches of mathematics, engineering and phsiycs such as fuzzy topology [31], metric spaces [9, 18], chaos control [11], nonlinear dynamical systems [17], population dynamics [5] and quantum physics [25].

The theory of intuitionistic fuzzy set introduced by Atanassov [3] was investigated in decision making problems and e^∞-theory [4, 8]. The concept of intuitionistic fuzzy metric spaces was defined by Park [30]. Moreover, the notion of the intuitionistic fuzzy normed linear space was given by Saadati and Park [33].

As an extended definition of usual convergence, the concept of statistical convergence was given by Steinhaus [40] and Fast [10]. Many years later, statistical convergence was discussed by many researchers in the theory of Fourier analysis, ergodic theory and number theory. Furthermore, statistical convergence has been investigated in the view of summability theory benefiting from the works of Schoenberg [38], Salat [34], Connor [7], Fridy [12] and Fridy and Miller [13]. As a variant of statistical convergence, the concept of λ-statistical convergence of real sequences was given by Mursaleen [29]. Applications of statistical convergence and λ-statistical convergence in the study of intuitionistic fuzzy normed linear spaces can be found in [21, 22, 28] and [20, 23, 26, 29, 36 ,37] respectively.

Let $p = (p_k)_{k=0}^{\infty}$ be a sequence of nonnegatif numbers such that $p_0 > 0$ and $P_n = \sum\limits_{k=0}^{n} p_k \to \infty$ as $n \to \infty$. Set

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k, \quad n = 0, 1, 2, ...$$

$x = (x_k)$ is $\left(\overline{N}, p\right)$-summable if $\lim\limits_{n\to\infty} t_n = L$. In this case, we write $\left(\overline{N}, p\right) - \lim x = L$.

Following the introduction of $\left(\overline{N}, p\right)$-summability due to Hardy [15], the concept of statistical convergence was generalized by Karakaya and Chisti [19] and called this new method as weighted statistical convergence. The concept of weighted statistical convergence was altered by Mursaleen et al. [27]. The definition was given as follows:

Received: 18.09.2017. In revised form: 11.12.2017. Accepted: 23.02.2018
2010 *Mathematics Subject Classification.* 40D15, 40G99.
Key words and phrases. *statistical convergence, λ-statistical convergence, weighted statistical convergence, strong summability,intuitionistic fuzzy normed linear space.*
Corresponding author: Esra Kamber; e.burdurlu87@gmail.com

Let define the weighted density of $K \subseteq \mathbb{N}$ by $\delta_{\overline{N}}(K) = \lim_{n\to\infty} \frac{1}{P_n} |K_{P_n}|$ if, the limit exists. A sequence $x = (x_k)$ is weighted statistically convergent (or $S_{\overline{N}}$ - convergent) to L if, for every $\varepsilon > 0$,

$$\delta_{\overline{N}}(\{k \in \mathbb{N} : p_k|x_k - L| \geq \varepsilon\}) = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{P_n} |\{k \leq P_n : p_k |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, it is denoted by $S_{\overline{N}} - \lim x = L$.

In this study, we introduce the concepts of weighted λ-statistical convergence and $(\overline{N}_\lambda, p)$-summability in intuitionistic fuzzy normed linear spaces. We also investigate some inclusion relations for these new concepts in intuitionistic fuzzy normed linear spaces.

2. Basic definitions

In this section, we give the basic definitions satisfied to give new summability methods in the intuitionistic fuzzy normed linear spaces.

Definition 2.1. ([29]) By a sequence $\lambda = (\lambda_n)$, we mean a nondecreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1,\ \lambda_1 = 1$. The sequence $x = (x_k)$ is λ-statistical convergent to L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$. In this case, it is denoted by $S_\lambda - \lim x = L$.

Definition 2.2. ([6]) Let $p = (p_k)_{k=0}^{\infty}$ be a sequence of nonnegatif numbers such that $p_0 > 0$ and $P_{\lambda_n} := \sum_{k\in I_n} p_k \to \infty$ as $n \to \infty$ and $\sigma_n := \frac{1}{P_{\lambda_n}} \sum_{k\in I_n} p_k x_k$.

i) A sequence $x = (x_k)$ is $(\overline{N}_\lambda, p)$-summable to L if $\lim_{n\to\infty} \sigma_n = L$. If $\lambda_n = n$, $x = (x_k)$ is $(\overline{N}_\lambda, p)$-summability is reduced to $(\overline{N}, p)$-summability.

ii) A sequence $x = (x_k)$ is strongly $(\overline{N}_\lambda, p)$-summable to L if,

$$\lim_{n\to\infty} \frac{1}{P_{\lambda_n}} \sum_{k\in I_n} p_k |x_k - L| = 0.$$

Definition 2.3. ([6]) Let $K \subseteq \mathbb{N}$. The number

$$\delta_{\overline{N}_\lambda}(K) = \lim_{n\to\infty} \frac{1}{P_{\lambda_n}} |\{k \leq P_{\lambda_n} : k \in K\}|$$

is said to be weighted λ-density of $K \subseteq \mathbb{N}$ if the limit exists. In case $\lambda_n = n$, weighted λ-density reduces to the weighted density defined in [27].

Definition 2.4. ([6]) A sequence $x = (x_k)$ is said to be weighted λ-statistically convergent (or $S_{\overline{N}_\lambda}$- convergent) to L if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{P_{\lambda_n}} |\{k \leq P_{\lambda_n} : p_k |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S_{\overline{N}_\lambda} - \lim x = L$. $S_{\overline{N}_\lambda}$ denotes the set of all weighted λ-statistically convergent sequences.

Definition 2.5. ([39]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuos t-norm if, it satisfies the following conditions:

i) $*$ is associative and commutative,
ii) $*$ is continuous,
iii) $a * 1 = a$ for all $a \in [0,1]$,
iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a,\ b,\ c, d \in [0,1]$.

Definition 2.6. ([39]) A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is said to be a continuos t-conorm if, it satisfies the following conditions:

i) $\circ$ is associative and commutative,
ii) $\circ$ is continuous,
iii) $a \circ 0 = a$ for all $a \in [0,1]$,
iv) $a \circ b \leq c \circ d$ whenever $a \leq c$ and $b \leq d$ for each $a,\ b,\ c, d \in [0,1]$.

Definition 2.7. ([32]) The five-tuple $(X,\ \mu, \upsilon, *, \circ)$ is said to be intuitionistic fuzzy normed linear space (or shortly IFNLS) where X is a linear space over a field F, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, μ, υ are fuzzy sets on $X \times (0,\infty)$, μ denotes the degree of membership and υ denotes the degree of nonmembership of $(x,t) \in X \times (0,\infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

i) $\mu(x,t) + \upsilon(x,t) \leq 1$,
ii) $\mu(x,t) > 0$,
iii) $\mu(x,t) = 1$ if and only if $x = 0$,
iv) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ if $\alpha \neq 0$,
v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
vi) $\mu(x,.) : (0,\infty) \to [0,1]$ is continuous,
vii) $\lim\limits_{t\to\infty} \mu(x,t) = 1$ and $\lim\limits_{t\to 0} \mu(x,t) = 0$,
viii) $\upsilon(x,t) < 1$,
ix) $\upsilon(x,t) = 0$ if and only if $x = 0$,
x) $\upsilon(\alpha x, t) = \upsilon\left(x, \frac{t}{|\alpha|}\right)$ if $\alpha \neq 0$,
xi) $\upsilon(x,t) \circ \upsilon(y,s) \geq \upsilon(x+y, s+t)$,
xii) $\upsilon(x,.) : (0,\infty) \to [0,1]$ is continuous,
xiii) $\lim\limits_{t\to\infty} \upsilon(x,t) = 0$ and $\lim\limits_{t\to 0} \upsilon(x,t) = 1$.

Definition 2.8. ([32]) Let $(X,\ \mu, \upsilon, *, \circ)$ be an IFNLS. A sequence $x = (x_k)$ be a sequence in X is convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, υ) if, for every $\varepsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \varepsilon$ and $\upsilon(x_k - L, t) < \varepsilon$ for all $k \geq k_0$ where $k \in \mathbb{N}$. It is denoted by $(\mu, \upsilon) - \lim x = L$.

Theorem 2.1. ([35]) Let $(X,\ \mu, \upsilon, *, \circ)$ be an IFNLS. Then, a sequence $x = (x_k)$ in X is convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, υ) if and only if $\lim\limits_{k\to\infty} \mu(x_k - L, t) = 1$ and $\lim\limits_{k\to\infty} \upsilon(x_k - L, t) = 0$.

3. Generalized weighted statistical convergence in IFNLS

In this section, we generalize the concept of weighted statistical convergence with respect to the intuitionistic fuzzy norm (μ, υ) following the line of Altundağ and Kamber [2]. Moreover, we investigate some connections between this definition with the concept of weighted statistical convergence and $\left(\overline{N}_\lambda, p\right)$-summability with respect to the intuitionistic fuzzy norm (μ, υ).

Definition 3.9. Let $(X, \mu, \upsilon, *, \circ)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be $(\overline{N}_\lambda, p)$-summable to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, υ) (or $(\overline{N}_\lambda, p)^{(\mu,\upsilon)}$-summable to $L \in X$) if, for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{P_{\lambda_n}} \sum_{k \in I_n} \mu\left(p_k\left(x_k - L\right), t\right) > 1 - \varepsilon$$

and

$$\frac{1}{P_{\lambda_n}} \sum_{k \in I_n} \upsilon\left(p_k\left(x_k - L\right), t\right) < \varepsilon$$

for all $n \geq n_0$. In this case, we write $(\overline{N}_\lambda, p)^{(\mu,\upsilon)} - \lim x = L$.

Definition 3.10. Let $(X, \mu, \upsilon, *, \circ)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be weighted λ-statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, υ) (or $S_{\overline{N}_\lambda}^{(\mu,\upsilon)}$- convergent to $L \in X$) if, for every $\varepsilon > 0$ and $t > 0$,

$$\delta_{\overline{N}_\lambda}\left(\left\{k \in \mathbb{N} : \mu\left(p_k\left(x_k - L\right), t\right) \leq 1 - \varepsilon \text{ or } \upsilon\left(p_k\left(x_k - L\right), t\right) \geq \varepsilon\right\}\right) = 0, \tag{3.1}$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{P_{\lambda_n}} \left|\left\{k \leq P_{\lambda_n} : \mu\left(p_k\left(x_k - L\right), t\right) \leq 1 - \varepsilon \text{ or } \upsilon\left(p_k\left(x_k - L\right), t\right) \geq \varepsilon\right\}\right| = 0.$$

In this case, we write $S_{\overline{N}_\lambda}^{(\mu,\upsilon)} - \lim x = L$.

Remark 3.1.
- (i) If $\lambda_n = n$ for all $n \in \mathbb{N}$ in Definition 3.9, then $(\overline{N}_\lambda, p)^{(\mu,\upsilon)}$- summability is reduced to $(\overline{N}, p)^{(\mu,\upsilon)}$- summability introduced in [2].
- (ii) If $p_k = 1$ for all $k \in \mathbb{N}$ in Definition 3.9, then $(\overline{N}_\lambda, p)^{(\mu,\upsilon)}$- summability is reduced to $(V, \lambda)^{(\mu,\upsilon)}$- summability introduced in [16].
- (iii) If $p_k = 1$ for all $k \in \mathbb{N}$ in Definition 3.10, then the concept of $S_{\overline{N}_\lambda}^{(\mu,\upsilon)}$- convergence is reduced to the concept of $S_\lambda^{(\mu,\upsilon)}$- convergence introduced in [26].
- (iv) If $\lambda_n = n$ for all $n \in \mathbb{N}$ in Definition 3.10, then the concept of $S_{\overline{N}_\lambda}^{(\mu,\upsilon)}$- convergence is reduced to the concept of $S_{\overline{N}}^{(\mu,\upsilon)}$- convergence introduced in [2].

It is easily seen that if $S_{\overline{N}_\lambda}^{(\mu,\upsilon)} - \lim x = L$, then $S_{\overline{N}}^{(\mu,\upsilon)} - \lim x = L$, since $\frac{\lambda_n}{n}$ is bounded by 1 and so $\frac{P_{\lambda_n}}{P_n}$ is bounded by 1. We prove the following theorem for the reverse situation.

Theorem 3.2. *Let $(X, \mu, \upsilon, *, \circ)$ be an IFNLS and $x = (x_k)$ be a sequence in X. If $\liminf_{n\to\infty} \frac{P_{\lambda_n}}{P_n} > 0$ and $S_{\overline{N}}^{(\mu,\upsilon)} - \lim x = L$, then $S_{\overline{N}_\lambda}^{(\mu,\upsilon)} - \lim x = L$.*

Proof. Suppose that $\liminf_{n\to\infty} \frac{P_{\lambda_n}}{P_n} > 0$. Then, there exists $\delta > 0$ such that $\delta \leq \frac{P_{\lambda_n}}{P_n} \leq 1$. Then for every $\varepsilon > 0$ and $t > 0$, we have

$$\frac{1}{P_n} \left|\left\{k \leq P_n : \mu\left(p_k\left(x_k - L\right), t\right) \leq 1 - \varepsilon \text{ or } \upsilon\left(p_k\left(x_k - L\right), t\right) \geq \varepsilon\right\}\right|$$

$$\geq \frac{\delta}{P_{\lambda_n}} \left|\{k \leq P_{\lambda_n} : \mu\left(p_k(x_k - L), t\right) \leq 1-\varepsilon \text{ or } \upsilon\left(p_k(x_k - L), t\right) \geq \varepsilon\}\right|.$$

By the assumption, it follows that $S_{\overline{N}_\lambda}^{(\mu,\upsilon)} - \lim x = L$. □

In the following theorems, we establish the relation between $S_{\overline{N}_\lambda}{}^{(\mu,\upsilon)}$- convergence and $\left(\overline{N}_\lambda, p\right)^{(\mu,\upsilon)}$- summability in intuitionistic fuzzy normed linear spaces.

Theorem 3.3. *Let $(X, \mu, \upsilon, *, \circ)$ be an IFNLS, $x = (x_k)$ be a sequence in X and $\frac{P_{\lambda_n}}{\lambda_n} \geq 1$ for all $n \in \mathbb{N}$. If $x = (x_k)$ in X is $\left(\overline{N}_\lambda, p\right)^{(\mu,\upsilon)}$- summable to $L \in X$, then $x = (x_k)$ is $S_{\overline{N}_\lambda}^{(\mu,\upsilon)}$-convergent to $L \in X$.*

Proof. Suppose that $\left(\overline{N}_\lambda, p\right)^{(\mu,\upsilon)} - \lim x = L$. For every $\varepsilon > 0$ and $t > 0$, let

$$K_{P_{\lambda_n}}(\varepsilon) = \{k \leq P_{\lambda_n} : \mu\left(p_k\left(x_k - L\right), t\right) \leq 1-\varepsilon \text{ or } \upsilon\left(p_k\left(x_k - L\right), t\right) \geq \varepsilon\}$$

and

$$K^c_{P_{\lambda_n}}(\varepsilon) = \{k \leq P_{\lambda_n} : \mu\left(p_k\left(x_k - L\right), t\right) > 1-\varepsilon \text{ and } \upsilon\left(p_k\left(x_k - L\right), t\right) < \varepsilon\}.$$

Then,

$$\begin{aligned} \tfrac{1}{P_{\lambda_n}} \sum_{k\in I_n} \mu\left(p_k\left(x_k - L\right), t\right) &= \tfrac{1}{P_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\in K_{P_{\lambda_n}}(\varepsilon)}} \mu\left(p_k\left(x_k - L\right), t\right) \\ &+ \tfrac{1}{P_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}} \mu\left(p_k\left(x_k - L\right), t\right) \\ &\geq \tfrac{1}{P_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}} \mu\left(p_k\left(x_k - L\right), t\right) \\ &> \tfrac{1}{P_n} \left|K^c_{P_{\lambda_n}}(\varepsilon)\right| (1-\varepsilon). \end{aligned} \tag{3.2}$$

By inequality (3.2.), we have $\lim\limits_{n\to\infty} \frac{1}{P_{\lambda_n}} \left|K^c_{P_{\lambda_n}}(\varepsilon)\right| = 1$. On the other hand, for every $\varepsilon > 0$ and $t > 0$,

$$\begin{aligned} \tfrac{1}{P_{\lambda_n}} \sum_{k\in I_n} \upsilon\left(p_k\left(x_k - L\right), t\right) &= \tfrac{1}{P_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\in K_{P_{\lambda_n}}(\varepsilon)}} \upsilon\left(p_k\left(x_k - L\right), t\right) \\ &+ \tfrac{1}{P_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}} \upsilon\left(p_k\left(x_k - L\right), t\right) \\ &\geq \tfrac{1}{P_{\lambda_n}} \sum_{\substack{k\in I_n \\ k\in K_{P_{\lambda_n}}(\varepsilon)}} \upsilon\left(p_k\left(x_k - L\right), t\right) \\ &> \tfrac{1}{P_n} \left|K_{P_{\lambda_n}}(\varepsilon)\right| \varepsilon. \end{aligned} \tag{3.3}$$

By inequality (3.3.), we have $\lim\limits_{n\to\infty} \frac{1}{P_{\lambda_n}} \left|K_{P_{\lambda_n}}(\varepsilon)\right| = 0$. Hence, we obtain $S_{\overline{N}_\lambda}^{(\mu,\upsilon)} - \lim x = L$. □

The following example shows that the converse of Theorem 3.3. is not valid in general.

Example 3.1. Let $(\mathbb{R}, |.|)$ denote the space of real numbers with the usual norm and let $a * b = ab$ and $a \circ b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider $\mu(x,t) := \frac{t}{t+|x|}$ and $\upsilon(x,t) := \frac{|x|}{t+|x|}$. In this case, $(\mathbb{R}, \mu, \upsilon, *, \circ)$ is an IFNLS. Consider $\lambda_n = n$ for all $n \in \mathbb{N}$ and $p_k = \frac{1}{k+1}$ for all $k \in \mathbb{N}$ and define a sequence $x = (x_k)$ whose terms are given by

$$x_k = \begin{cases} \sqrt{k}, & \text{if } k = m^8 \, (m \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Then, for every $0 < \varepsilon < 1$ and for every $t > 0$, let

$$K_{P_n}(\varepsilon) = \{k \le P_n : \mu(p_k x_k, t) \le 1 - \varepsilon \quad \text{or} \quad \upsilon(p_k x_k, t) \ge \varepsilon\}.$$

Since

$$\begin{aligned} K_{P_n}(\varepsilon) &= \left\{k \le P_n : \frac{t}{t + \frac{|x_k|}{k+1}} \le 1 - \varepsilon \quad \text{or} \quad \frac{\frac{|x_k|}{k+1}}{t + \frac{|x_k|}{k+1}} \ge \varepsilon \right\} \\ &= \left\{k \le P_n : \ |x_k| \ge \frac{(k+1)\varepsilon t}{(1-\varepsilon)} \right\} \subseteq \left\{k \le P_n : x_k = \sqrt{k}\right\}, \end{aligned}$$

we have

$$\lim_{n\to\infty} \frac{1}{P_n} |K_{P_n}(\varepsilon)| = 0.$$

So, by definition 3.10. we obtain $S_{\overline{N}_\lambda}^{(\mu,\upsilon)} - \lim x = 0$. However, since as $n \to \infty$,

$$\frac{1}{P_n} \sum_{k=1}^{n} \mu(p_k x_k, t) \to \infty$$

and

$$\frac{1}{P_n} \sum_{k=1}^{n} \upsilon(p_k x_k, t) \to \infty,$$

$x = (x_k)$ is not $(\overline{N}_\lambda, p)$-summable to 0 with respect to intuitionistic fuzzy norm (μ, υ).

Theorem 3.4. *Let $(X, \mu, \upsilon, *, \circ)$ be an IFNLS, $x = (x_k)$ in X and $\frac{P_{\lambda_n}}{\lambda_n} \ge 1$ for all $n \in \mathbb{N}$. If $S_{\overline{N}_\lambda}^{(\mu,\upsilon)} - \lim x = L$, $\mu(p_k(x_k - L), t) \ge 1 - M$ and $\upsilon(p_k(x_k - L), t) \le M$ for all $k \in \mathbb{N}$, then $(\overline{N}_\lambda, p)^{(\mu,\upsilon)} - \lim x = L$.*

Proof. Suppose that $S_{\overline{N}_\lambda}^{(\mu,\upsilon)} - \lim x = L$. Then, for every $\varepsilon > 0$ and $t > 0$,

$$K_{P_{\lambda_n}}(\varepsilon) = \{k \le P_{\lambda_n} : \mu(p_k(x_k - L), t) \le 1 - \varepsilon \quad \text{or} \quad \upsilon(p_k(x_k - L), t) \ge \varepsilon\}$$

and

$$K_{P_{\lambda_n}}^c(\varepsilon) = \{k \le P_{\lambda_n} : \mu(p_k(x_k - L), t) > 1 - \varepsilon \quad \text{and} \quad \upsilon(p_k(x_k - L), t) < \varepsilon\}.$$

Hence,

$$\begin{aligned}\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}\mu\left(p_k\left(x_k-L\right),t\right) &= \frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{P_{\lambda_n}}(\varepsilon)}}\mu\left(p_k\left(x_k-L\right),t\right)\\ &+\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}}\mu\left(p_k\left(x_k-L\right),t\right)\\ &= S_1\left(n\right)+S_2\left(n\right)\end{aligned}$$

where

$$S_1\left(n\right)=\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{P_{\lambda_n}}(\varepsilon)}}\mu\left(p_k\left(x_k-L\right),t\right) \tag{3.4}$$

and

$$S_2\left(n\right)=\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}}\mu\left(p_k\left(x_k-L\right),t\right). \tag{3.5}$$

If $k\in K_{P_{\lambda_n}}\left(\varepsilon\right)$, then

$$S_1\left(n\right)=\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{P_{\lambda_n}}(\varepsilon)}}\mu\left(p_k\left(x_k-L\right),t\right)\geq\frac{\left|K_{P_{\lambda_n}}\left(\varepsilon\right)\right|}{P_{\lambda_n}}\left(1-M\right). \tag{3.6}$$

Since $S^{(\mu,\upsilon)}_{\overline{N}_\lambda}-\lim x=L$, we have

$$\lim_{n\to\infty}S_1\left(n\right)\geq 0. \tag{3.7}$$

If $k\in K^c_{P_{\lambda_n}}\left(\varepsilon\right)$, then we have

$$S_2\left(n\right)=\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}}\mu\left(p_k\left(x_k-L\right),t\right)>\frac{\left|K^c_{P_{\lambda_n}}\left(\varepsilon\right)\right|}{P_{\lambda_n}}\left(1-\varepsilon\right) \tag{3.8}$$

which yields that

$$\lim_{n\to\infty}S_2\left(n\right)>\left(1-\varepsilon\right). \tag{3.9}$$

Using equalities (3.4.)-(3.5.) and inequalities (3.6.)-(3.9.), we get

$$\lim_{n\to\infty}\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}\mu\left(p_k\left(x_k-L\right),t\right)=1. \tag{3.10}$$

Similarly,

$$\begin{aligned}\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}\upsilon\left(p_k\left(x_k-L\right),t\right) &= \frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{P_{\lambda_n}}(\varepsilon)}}\upsilon\left(p_k\left(x_k-L\right),t\right)\\ &+\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}}\upsilon\left(p_k\left(x_k-L\right),t\right)\\ &= S_3\left(n\right)+S_4\left(n\right)\end{aligned}$$

where

$$S_3\left(n\right)=\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{P_{\lambda_n}}(\varepsilon)}}\upsilon\left(p_k\left(x_k-L\right),t\right)$$

and

$$S_4\left(n\right)=\frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}}\upsilon\left(p_k\left(x_k-L\right),t\right).$$

If $k\in K_{P_{\lambda_n}}\left(\varepsilon\right)$, then as $n\to\infty$

$$\begin{aligned}S_3\left(n\right) &= \frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K_{P_{\lambda_n}}(\varepsilon)}}\upsilon\left(p_k\left(x_k-L\right),t\right)\\ &\le \frac{1}{P_{\lambda_n}}\left|K_{P_{\lambda_n}}\left(\varepsilon\right)\right|M\to 0\end{aligned}\tag{3.11}$$

since $\lim\limits_{n\to\infty}\frac{1}{P_{\lambda_n}}\left|K_{P_{\lambda_n}}\left(\varepsilon\right)\right|=0.$

If $k\in K^c_{P_{\lambda_n}}\left(\varepsilon\right)$,

$$\begin{aligned}S_4\left(n\right) &= \frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}}\upsilon\left(p_k\left(x_k-L\right),t\right)\\ &< \frac{1}{P_{\lambda_n}}\sum_{\substack{k\in I_n\\ k\in K^c_{P_{\lambda_n}}(\varepsilon)}}\varepsilon=\frac{1}{P_{\lambda_n}}\left|K^c_{P_{\lambda_n}}\left(\varepsilon\right)\right|\varepsilon\end{aligned}\tag{3.12}$$

which yields that

$$\lim_{n\to\infty}S_4\left(n\right)<\varepsilon.\tag{3.13}$$

By inequalities (3.11.)-(3.13.), we get

$$\lim_{n\to\infty}\frac{1}{P_{\lambda_n}}\sum_{k\in I_n}\upsilon\left(p_k\left(x_k-L\right),t\right)=0.\tag{3.14}$$

Using equalities (3.10.) and (3.14.), we have

$$\left(\overline{N}_\lambda,p\right)^{(\mu,\upsilon)}-\lim x=L.\tag{3.15}$$

□

4. Conclusion

In this paper, we defined a new variant of summability method, called as $(\overline{N}_\lambda, p)$-summability in IFNLS and used this summability method to introduce a new variant of statistical convergence, called as weighted λ-statistical convergence in IFNLS. Moreover, we investigate some relations between these concepts. In establishing most of the proofs, we have benefited from the properties of intuitionistic fuzzy norm. The obtained results and relations in this paper are more general than the corresponding results for normed linear spaces.

References

[1] Alghamdi, M. A., Alotaibi, A., Lahani, Q. M. D. and Mursaleen, M., *Statistical limit superior and limit inferior in intuitionistic fuzzy normed spaces*, J. Inequal. Appl., **96** (2012), 1–12

[2] Altundağ, S. and Kamber, E., *Weighted statistical convergence in intuitionistic fuzzy normed linear spaces*, J. Inequal. Spec. Funct., **8** (2017), 113–124

[3] Atanassov, K. T., *Intuitionistic fuzzy sets*, Fuzzy Set. Syst., **20** (1986), 87–96

[4] Atanassov, K., Pasi, G. and Yager, R., *Intuitionistic fuzzy interpretations of multi-person multicriteria decision making*, in: Proceedings of 2002 First International IEEE Symposium Intelligent Systems, **1** (2002), 115–119

[5] Barros, L. C., Bassanezi, R. C. and Tonelli, P. A., *Fuzzy modelling in population dynamics*, Ecol. Model., **128** (2000), 27–33

[6] Belen, C. and Mohiuddine, S. A., *Generalized weighted statistical convergence and application*, Appl. Math. Comput., **219** (2013) 9821–9826

[7] Connor, J. S., *The statistical and strong p-Cesaro convergence of sequences*, Analysis, **8** (1988), 47–63

[8] El Naschie, M. S., *On the unification of heterotic strings, $m-$theory and e^{∞}-theory*, Chaos Soliton. Fractal., **11** (2000), 2397–2408

[9] Erceg, M. A., *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl., **69** (1979), 205–230

[10] Fast, H., *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244

[11] Fradkov, A. L. and Evans, R. J., *Control of chaos: Methods and applications in engineering*, Chaos Soliton. Fractal., **29** (2005), 33–56

[12] Fridy, J. A., *On statistical convergence*, Analysis, **5** (1985), 301–313

[13] Fridy, J. A. and Miller, H. I., *A matrix characterization of statistical convergence*, Analysis, 11(1991), 59–66

[14] George, A. and Veeramani, P., *On some result in fuzzy metric space*, Fuzzy Set. Syst., **64** (1994), 395–399

[15] Hardy, G. H., *Divergent Series*, Oxford University Press, London (1949)

[16] Hazarika, B., Kumar, V. and Lafuerza Guillénc, B., *Generalized ideal convergence in intuitionistic fuzzy normed linear spaces*, Filomat, **27** (2013), 811–820

[17] Hong, L. and Sun, J. Q., *Bifurcations of fuzzy nonlinear dynamical systems*, Commun. Nonlinear Sci. Numer. Simul., **1** (2006), 1–12

[18] Kaleva, O. and Seikkala, S., *On fuzzy metric spaces*, Fuzzy Set. Syst., **12** (1984), 215–229

[19] Karakaya, V. and Chishti, T. A., *Weighted statistical convergence*, Iran. J. Sci. Technol. Trans. A, **33** (2009), 219–223

[20] Karakaya, V., Şimşek, N., Ertürk, M. and Gürsoy, F., *λ-statistical convergence of sequences of functions in intuitionistic fuzzy normed spaces*, J. Funct. Space. Appl., **2012** (2012), 1–14

[21] Karakaya, V., Şimşek, N., Ertürk, M. and Gürsoy, F., *Statistical convergence of sequences of functions in intuitionistic fuzzy normed spaces*, Abstr. Appl. Anal., **2012** (2012), 1–19

[22] Karakuş, S., Demirci, K. and Duman, O., *Statistical convergence on intuitionistic fuzzy normed spaces*, Chaos Soliton. Fractal., **35** (2008), 763–769

[23] Kumar, V. and Mursaleen, M., *On (λ, μ)-statistical convergence of double sequences on intuitionistic fuzzy normed spaces*, Filomat, **25** (2011), 109–120

[24] Leindler, L., *Uber die de la Vallee-Pousinsche summierbarkeit allgemeiner orthogonalreihen*, Acta Math. Acad. Sci. Hungar., **16** (1965), 375–387

[25] Madore, J., *Fuzzy physics*, Ann. Phys., **219** (1992), 187–198

[26] Mohiuddine, S. A. and Danish Lohani, Q. M., *On generalized statistical convergence in intuitionistic fuzzy normed space*, Chaos Soliton. Fractal., **42** (2009), 1731–1737

[27] Mursaleen, M., Karakaya, V., Ertürk, M., Gürsoy, F., *Weighted statistical convergence and its application to Korovkin type approximation theorem*, Appl. Math. Comput., **218** (2012), 9132–9137

[28] Mursaleen, M. and Mohiuddine, S. A., *Statistical convergence of double sequences on intuitionistic fuzzy normed spaces*, Chaos Soliton. Fractal., **41** (2009), 2414–2421

[29] Mursaleen, M., *λ-statistical convergence*, Math. Slovaca, **50** (2000), 111–115

[30] Park, J. H., *Intuitionistic fuzzy metric spaces*, Chaos Soliton. Fractal., **22** (2004), 1039–1046
[31] Saadati, R., Mansour Vaezpour, S., Cho, Y. J., *Quicksort algorithm: Application of a fixed point theorem in intuitionistic fuzzy quasi-metric spaces at a domain of words*, J. Comput. Appl. Math., **228** (2009), No. 1, 219–225
[32] Saadati, R. and Park, J. H., *On the intuitionistic fuzzy topological spaces*, Chaos Soliton. Fractal., **27** (2006), 331–344
[33] Saadati, R. and Park, J. H., *Intuitionistic fuzzy euclidean normed spaces*, Commun. Math. Anal., **12** (2006), 85–90
[34] Salat, T., *On statistical convergence of real numbers*, Math. Slovaca, **30** (1980), 139–150
[35] Samanta, T. K. and Jebril, Iqbal H., *Finite Dimensional Intuitionistic Fuzzy Normed Linear Space*, Int. J. Open Problems Compt. Math., **2** (2009), 574–591
[36] Savaş, E., *λ-statistical convergence in intuitionistic fuzzy* 2*-normed space*, Appl. Math. Inf. Sci., **9** (2015), 501–505
[37] Savaş, E. and Gürdal, M., *A generalized statistical convergence in intuitionistic fuzzy normed spaces*, ScienceAsia, **41** (2015), 289–294
[38] Schoenberg, I. J., *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375
[39] Schweizer, B. and Sklar, A., *Statistical metric spaces*, Pacific J. Math., **10** (1960), 313–334
[40] Steinhaus, H., *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73–74
[41] Zadeh, L. A., *Fuzzy Sets*, Inform. Cont., **8** (1965)

DEPARTMENT OF MATHEMATICS
SAKARYA UNIVERSITY
TURKEY
Email address: scaylan@sakarya.edu.tr
Email address: e.burdurlu87@gmail.com