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Subconvex sequences revisited

DAN BĂRBOSU

ABSTRACT. The paper is the English variant, with additional remarks, of two earlier papers published by the author in Romanian, many years ago. It is motivated by the fact that in the last years many mathematicians were interested on the subject and they can't found copies of acceptable graphic quality. In the paper we introduce the notions of convex sequence of p order (p being a positive integer) and respectively system of subconvex sequence of first order. One proves that any subconvex sequence of second order and any system of subconvex sequences of first order are convergent.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Problem 365, page 46 in [7], asks: "Prove that any sequence $\{x_n\}_{n\in\mathbb{N}_0}$ having the properties $x_n \geqslant 0, 2x_{n+2} \leqslant x_{n+1} + x_n, (\forall)\, n \in \mathbb{N}_0$ is convergent". It is easy to see that the second relation verified by the terms of $\{x_n\}_{n\in\mathbb{N}_0}$ can be rewritten in the form

$$x_{n+2} \leqslant \frac{1}{2}x_{n+1} + \frac{1}{2}x_n,\ (\forall)\, n \in \mathbb{N}_0. \tag{1.1}$$

From (1.1) it follows that each term of $\{x_n\}_{n\in\mathbb{N}_0}$ starting with the term of rank two is less or at most equal as a convex combination of the two preceding terms. This remark leads us to study the convergence of sequences satisfying the conditions:

$$x_n \geqslant 0,\ x_{n+2} \leqslant \alpha_1 x_{n+1} + \alpha_2 x_n,\ (\forall)\, n \in \mathbb{N}_0, \tag{1.2}$$

where

$$\alpha_1, \alpha_2 \in \left]0, 1\right[,\ \ \alpha_1 + \alpha_2 \leqslant 1. \tag{1.3}$$

In the case when the second inequality in (1.2) is verified with equality, the term of rank $(n+2)$ of $(x_n)_{n\in\mathbb{N}_0}$ is a convex combination of the terms of rank $(n+1)$ and n, respectively.

These considerations can be generalized in the following

Definition 1.1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers and p be a positive integer. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is called *subconvex of order p* if the following inequality

$$x_{n+p} \leqslant \sum_{k=1}^{p} \alpha_k x_{n+p-k},\ (\forall)\, n \in \mathbb{N}_0 \tag{1.4}$$

holds, where

$$\sum_{k=1}^{p} \alpha_k \leqslant 1, \alpha_k \in \left]0, 1\right[,\ (\forall)\, k \in \{1, 2, \ldots, p\}. \tag{1.5}$$

Example 1.1. For $p = 1$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convex of 1st order if $x_n \geqslant 0,\ (\forall)\, n \in \mathbb{N}_0$ and there exist $\alpha_1 \in \left]0, 1\right[$ such that

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$$x_{n+1} \leqslant \alpha_1 x_n,\ (\forall)\, n \in \mathbb{N}_0. \tag{1.6}$$

Clearly, any subconvex sequence $\{x_n\}_{n\in\mathbb{N}}$ of 1-th order is convergent and $\lim_{n\to\infty} x_n = 0$.

2. The convergence of subconvex sequences of second order

For $p = 2$, Definition 1.1 becomes

Definition 2.2. The sequence of non-negatives real numbers $\{x_n\}_{n\in\mathbb{N}}$ is subconvex of second order if there exist $\alpha_1, \alpha_2 \in \,]0,1[\,, \alpha_1 + \alpha_2 \leqslant 1$ such that

$$x_{n+2} \leqslant \alpha_1 x_{n+1} + \alpha_2 x_n,\ (\forall)\, n \in \mathbb{N}_0. \tag{2.7}$$

Regarding any subconvex sequence of second order, we shall prove the following

Theorem 2.1. *Any subconvex sequence of second order is convergent.*

Proof. Define the sequence $\{y_n\}_{n\in\mathbb{N}}$ by

$$y_{n+1} = x_{n+1} + (1-\alpha_1)x_n,\ (\forall)\, n \in \mathbb{N}_0. \tag{2.8}$$

Because $x_n \geqslant 0, (\forall)\, n \in \mathbb{N}_0$ and $\alpha_1 \in \,]0,1[$ it follows

$$y_n \geqslant 0, (\forall)\, n \in \mathbb{N}. \tag{2.9}$$

From Definition 2.2 , we get successively:

$$\begin{aligned} y_{n+2} &= x_{n+2} + (1-\alpha_1)x_{n+1} \leqslant \alpha_1 x_{n+1} + \alpha_2 x_n + (1-\alpha_1)x_{n+1} = \\ &= x_{n+1} + \alpha_2 x_n \leqslant x_{n+1} + (1-\alpha_1)x_n = y_{n+1},\ (\forall)\, n \in \mathbb{N}_0. \end{aligned}$$

It follows that $(y_n)_{n\in\mathbb{N}_0}$ is decreasing, and taking (2.9) into account, $(y_n)_{n\in\mathbb{N}_0}$ it is convergent. Let $l = \lim_{n\to\infty} y_n, l \in \mathbb{R}$. Using Definition 2.2, it follows that $(\forall)\, \varepsilon > 0$ there exists a rank $N = N(\varepsilon)$ such that

$$1 - \varepsilon - (1-\alpha_1)x_n \leqslant x_{n+1} \leqslant 1 + \varepsilon - (1-\alpha_1)x_n,\ (\forall)\, n \in \mathbb{N}_0. \tag{2.10}$$

Using (2.10), we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is convergent and we will compute $l_1 = \lim_{n\to\infty} x_n$. Let m be a positive integer. Applying the right hand side of (2.10), we get

$$\begin{aligned} x_{N+m+1} &\leqslant 1 + \varepsilon - (1-\alpha_1)x_{N+m} \leqslant 1 + \varepsilon - (1-\alpha_1)\{-(l-\varepsilon) + (1+\alpha_1)x_{N+m-1}\} \\ &= l\{1 - (1-\alpha_1)\} + \varepsilon\{1 + (1-\alpha_1)\} + (1-\alpha_1)^2 x_{N+m-1} \leqslant \dots \\ &\cdots \leqslant l\{1 - (1-\alpha_1) + (1-\alpha_1)^2 + \dots + (-1)^m(1-\alpha_1)^m\} + \\ &+ \varepsilon\{1 + (1-\alpha_1) + \dots + (1-\alpha_1)^m\} + (-1)^{m+1}(1-\alpha_1)^{m+1}x_N = \\ &= l\frac{1 - (-1)^{m+1}(1-\alpha_1)^{m+1}}{2-\alpha_1} + \varepsilon\frac{1-(1-\alpha_1)^{m+1}}{\alpha_1} + (-1)^{m+1}(1-\alpha_1^{m+1})x_N. \end{aligned}$$

Taking into account that

$$\lim_{n\to\infty} \left|(-1)^{m+1}(1-\alpha_1)^{m+1}\right| = 0,$$

for any $\varepsilon_1 > 0$, there exists $m_1 = m_1(\varepsilon)$ such that for any $m \geq m_1$ the following holds

$$\mid (-1)^{m+1}(1-\alpha_1)^{m+1} \mid < \varepsilon_1. \tag{2.11}$$

Comming back, we get that for any $\varepsilon_1 > 0$ there exists $m_1 = m_1(\varepsilon_1)$ such that for any $m \geq m_1$ the following holds

$$x_{N+m+1} \leqslant \frac{l}{2-\alpha_1} + \frac{\varepsilon_1}{2-\alpha_1} + \varepsilon \cdot \frac{\varepsilon_1}{\alpha_1} + \varepsilon_1 \cdot x_N \tag{2.12}$$

Denoting $\varepsilon_2 = \frac{\varepsilon_1}{2-\alpha_1} + \varepsilon \cdot \frac{\varepsilon_1}{\alpha_1} + \varepsilon_1 \cdot x_N$, we get that there exists $m_2 = m_2(\varepsilon_2), m_2 = \max\{N(\varepsilon), m_1(\varepsilon_1)\}$ such that for any $m \geq m_2$, we have

$$x_{N+m+1} \leqslant \frac{l}{2-\alpha_1} + \varepsilon_2. \tag{2.13}$$

In a similar way, using the left side of (2.10), one proves the inequality

$$x_{N+m+1} \geqslant \frac{l}{2-\alpha_1} - \varepsilon_2. \tag{2.14}$$

From (2.13) and (2.14) it follows that for any $\varepsilon_2 > 0$ there exists $m_2 = m_2(\varepsilon_2)$ such that for any $m \geq m_2$ we have

$$\left| x_{N+m+1} - \frac{l}{2-\alpha_1} \right| < \varepsilon_2. \tag{2.15}$$

using (2.15), we get $\lim\limits_{m \longrightarrow \infty} x_n = \dfrac{l}{2-\alpha_1}$. □

Remark 2.1. Another proof was given in [6]. Regarding the case of subconvex sequences of p-order with $p > 2$, in [4] was given a sufficient condition for convergence. The problem was completely solved in [10].

3. Systems of subconvex sequences of first order

Definition 3.3. Suppose $A = [a_{ij}] \in \mathcal{M}_2(\mathbb{R})$ satisfies the conditions

(1) $TrA \in]0, \frac{1}{2}[$.
(2) $detA \in] -\frac{1}{2}, 0, [$.

The sequences of non-negative real numbers $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ form a system of subconvex sequences of order 1 if, in the hypotheses (1) and (2), the following inequalities

$$\begin{cases} x_{n+1} \leqslant a_{11}x_n + a_{12}y_n \\ y_{n+1} \leqslant a_{21}x_n + a_{22}y_n \end{cases} \tag{3.16}$$

hold, for any $n \in \mathbb{N}_0$.

We shall prove

Theorem 3.2. *If $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ form a system of subconvex sequences of 1st order, then they are both convergent and have the share limit.*

Proof. From the first inequality (3.16) for $n := n+1$, we get

$$x_{n+2} \leqslant a_{11}x_{n+1} + a_{12}y_{n+1}. \tag{3.17}$$

Using (3.17) and the second inequality (3.16), it follows

$$\begin{aligned} x_{n+2} &\leqslant a_{11}x_{n+1} + a_{12}(a_{21}x_n + a_{22}y_n) = \\ &= a_{11}x_{n+1} + a_{12}a_{21}x_n - a_{22}(-a_{12}y_n). \end{aligned} \tag{3.18}$$

From the first inequality (3.16), one obtains

$$-a_{12}y_n \leqslant a_{11}x_n - x_{n+1}. \tag{3.19}$$

Combining (3.18) and (3.19) one arrives to

$$x_{n+2} \leqslant (TrA)x_{n+1} - (detA)x_n, \ (\forall)\, n \in \mathbb{N}_0. \tag{3.20}$$

Relations (3.20) prove that the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ is subconvex of second order, with $\alpha_1 = TrA$ and $\alpha_2 = -detA$.

In a similar way, one proves

$$y_{n+2} \leqslant (TrA)y_{n+1} - (detA)y_n,\ (\forall)\, n \in \mathbb{N}_0. \tag{3.21}$$

which means that $\{y_n\}_{n\in\mathbb{N}_0}$ is a subconvex sequence of second order.

By virtue of Theorem 2.1, $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ are convergent and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \frac{l}{2 - TrA}$$

where

$$l = \lim_{n\to\infty} t_n,$$

with $t_{n+1} = x_{n+1} + (1 - \alpha_1)x_n$. □

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NORTH UNIVERSITY CENTER AT BAIA MARE
TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
DEPARTMENT OF MATHEMATICS AND INFORMATICS
VICTORIEI 76, 430122 BAIA MARE, ROMANIA
Email address: barbosudan@yahoo.com