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On bipolar fuzzy soft graphs

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ABSTRACT. In this paper, we combine the concepts of bipolar fuzzy soft sets and graph theory. Then we introduce notations of bipolar fuzzy soft graph and strong bipolar fuzzy soft graph. We also present several different types of operations including cartesian product, strong product and composition on bipolar fuzzy soft graphs and investigate some properties of them.

1. INTRODUCTION

The concept of fuzzy graph was firstly introduced by Rosenfeld [19]. Later, Bhattacharya [8] gave some remarks on fuzzy graphs. Mordeson and Peng [17] introduced several notations on fuzzy graphs. Akram et al. [1, 2, 3, 4] have introduced several new concepts on fuzzy graphs.

The concept of bipolar fuzzy sets which is a generalization of fuzzy sets [23] was initiated by Zhang [24]. Bipolar fuzzy sets whose range of membership degree is [-1,1]. If the membership degree of an element is 0, then the element is irrelevant to the corresponding property. If the membership degree of an element is within (0,1], then the element somewhat satisfies the implicit counter property. If the membership degree of an element is within [-1,0), then the element somewhat satisfies the implicit counter property. It is emphasized that positive information gives us what is granted to be possible, while negative information gives us what is considered to be impossible.

The concept of fuzzy soft set theory was firstly introduced by Maji et al. [15] as a new mathematical tool for cope with uncertainties. Fuzzy soft sets have potential applications in various fields. Maji et al. [15] investigated some properties of this notion. Thereafter many researchers have applied this concept on different algebraic structures in mathematics (see, [5, 7, 9, 10, 11, 12, 13, 14, 16, 20, 21, 22]). The concepts of fuzzy bipolar soft sets and bipolar fuzzy soft sets have been introduced by Naz and Shabir [18]. They defined their special union and special intersection and also showed that the both concepts are equivalent. Aslam et al. [6] studied some basic operations on bipolar fuzzy soft sets.

In this paper, we apply concept of bipolar fuzzy soft sets to graph structure. We introduce notations of bipolar fuzzy soft graph and strong bipolar fuzzy soft graph. We also present several different types of operations including cartesian product, strong product and composition on bipolar fuzzy soft graphs and investigate some properties of them.

2. PRELIMINARIES

Definition 2.1. [23] A fuzzy subset μ of X is defined as a map from X to $[0,1]$. A map $\nu : X \times X \to [0,1]$ is called a fuzzy relation on X if $\nu(x,y) \leq min(\mu(x), \mu(y))$ for all $x \in X$.

Definition 2.2. [19] Let $V \neq \emptyset$ be a finite set, μ and ν be fuzzy subset and fuzzy relation on X, respectively. The pair $G = (\mu, \nu)$ is called a fuzzy graph over V if $\nu(x,y) \leq$

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$min(\mu(x), \mu(y))$ for all $x, y \in V$, where μ and ν are called fuzzy vertex and fuzzy edge of G, respectively.

Definition 2.3. [19] Let $H = (\tau, \rho)$ and $G = (\mu, \nu)$ be two fuzzy graphs over the set V. Then H is called the fuzzy subgraph of the fuzzy graph G if $\tau(x) \leq \mu(x)$ and $\rho(x, y) \leq \nu(x, y)$ for all $x, y \in V$.

Definition 2.4. [24] Let $X \neq \emptyset$ be a set. A *bipolar fuzzy set* B on X is given by as $B = \{(x, \mu^+(x), \mu^-(x)) | x \in X\}$ where $\mu^+ : X \to [0,1]$ and $\mu^- : X \to [-1, 0]$. For brevity, we shall use the symbol $B = (\mu^+, \mu^-)$.

Definition 2.5. [24] Let $X \neq \emptyset$ be a set. Then a mapping $A = (\mu_A^+, \mu_A^-) : X \times X \to [0,1] \times [-1, 0]$ is called a bipolar fuzzy relation on X such that $\mu_A^+(x, y) \in [0,1]$ and $\mu_A^-(x, y) \in [-1, 0]$.

Definition 2.6. [24] Let $A = (\mu_A^+, \mu_A^-)$ and $B = (\mu_B^+, \mu_B^-)$ be two bipolar fuzzy sets on a set X. If $A = (\mu_A^+, \mu_A^-)$ is a bipolar fuzzy relation on a set X, then $A = (\mu_A^+, \mu_A^-)$ is a bipolar fuzzy relation on $B = (\mu_B^+, \mu_B^-)$ if $\mu_A^+(x, y) \leq min(\mu_B^+(x), \mu_B^+(y))$ and $\mu_A^-(x, y) \geq max(\mu_B^-(x), \mu_B^-(y))$ for all $x, y \in X$.

Definition 2.7. [1] A bipolar fuzzy graph with an underlying set V is defined to be a pair $G = (A, B)$ where $A = (\mu_A^+, \mu_A^-)$ is a bipolar fuzzy set in V and $B = (\mu_B^+, \mu_B^-)$ is a bipolar fuzzy set in E such that $\mu_B^+(x, y) \leq min(\mu_A^+(x), \mu_A^+(y))$ and $\mu_B^-(x, y) \geq max(\mu_A^-(x), \mu_A^-(y))$ for all $\{x, y\} \in E$.

A and B are called bipolar fuzzy vertex and bipolar fuzzy edge of the bipolar fuzzy graph $G = (A, B)$, respectively. We use the notation xy for an element of E.

Thus, $G = (A, B)$ is a bipolar fuzzy graph of $G^* = (V, E)$ if $\mu_B^+(xy) \leq min(\mu_A^+(x), \mu_A^+(y))$ and $\mu_B^-(xy) \geq max(\mu_A^-(x), \mu_A^-(y))$ for all $\{xy\} \in E$.

Definition 2.8. [15] Let X be an initial universe, E be set of parameters and $\mathcal{F}(X)$ be collection of all fuzzy subsets of X. If F is a mapping given by $F : A \to \mathcal{F}(X)$, then the pair (F, A) is called a *fuzzy soft set* over X.

Definition 2.9. [6] Let X be an initial universe and $A \subseteq E$ be a set of parameters. Let $BF(X)$ denotes the set of all bipolar fuzzy subset of X. If ϕ is a mapping given by $\phi : A \to BF(X)$, then the pair (ϕ, A) is called a *bipolar fuzzy soft set* over X. For any $a \in A$, $\phi(a)$ is referred to as the set of *a*-approximate elements of (ϕ, A) and can be defined as $\phi(a) = \{(\mu^+\phi_{(a)}(x), \mu^-\phi_{(a)}(x)) | x \in X, a \in A\}$, where $\mu^+\phi_{(a)}(x)$ represents the degree of x keeping the parameter a, $\mu^-\phi_{(a)}(x)$ represents the degree of x keeping the non-parameter a.

Definition 2.10. [6] Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over X. We say that (ϕ, A) is a bipolar fuzzy soft subset of (ψ, B) and write $(\phi, A) \widetilde{\preceq} (\psi, B)$ if $A \subseteq B$ and $\phi(a) \subseteq \psi(a)$ for all $a \in A$.

3. Bipolar fuzzy soft graphs

Definition 3.11. A bipolar fuzzy soft graph with underlying set V is an ordered 4-tuple $\widetilde{G} = (G^*, F, K, A)$ such that

i. A is non-empty set of parameters

ii. (F, A) is a bipolar fuzzy soft set over V

iii. (K, A) is a bipolar fuzzy soft set over E

iv. $(F(e), K(e))$ is a bipolar fuzzy graph for all $e \in A$. That is $\mu_{K(e)}^+(xy) \leq min(\mu_{F(e)}^+(x), \mu_{F(e)}^+(y))$ and $\mu_{K(e)}^-(xy) \geq max(\mu_{F(e)}^-(x), \mu_{F(e)}^-(y))$ for all $\{xy\} \in E$.

Note that (F, A) is called a bipolar fuzzy soft vertex and (K, A) is called a bipolar fuzzy soft edge.

Throughout this paper, we denote $G^* = (V, E)$ a crisp graph, $H(e) = (F(e), K(e))$ a bipolar fuzzy graph, and $\widetilde{G} = (G^*, F, K, A) = ((F, A), (K, A))$ a bipolar fuzzy soft graph.

Example 3.1. Let $G^* = (V, E)$ such that

$$V = \{x_1, x_2, x_3, x_4\} \text{ and } E = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}.$$

Let $A = \{e_1, e_2\}$ be a parameter set and let (F, A) be a bipolar fuzzy soft set over V defined by

$$\begin{aligned} F(e_1) &= \{(x_1, 0.8, -0.2), (x_2, 0.4, -0.5), (x_3, 0.5, -0.5)\} \\ F(e_2) &= \{(x_1, 0.5, -0.2), (x_2, 0.7, -0.3), (x_3, 0.4, -0.3), (x_4, 0.7, -0.3)\} \end{aligned}$$

Now let (K, A) be a bipolar fuzzy soft set over E defined by

$$\begin{aligned} K(e_1) &= \{(x_1x_2, 0.3, -0.1), (x_2x_3, 0.1, -0.4), (x_3x_1, 0.4, -0.1)\} \\ K(e_2) &= \{(x_1x_2, 0.5, -0.1), (x_2x_3, 0.4, -0.3), (x_3x_4, 0.4, -0.2), (x_4x_1, 0.3, -0.1)\} \end{aligned}$$

It is clearly seen that $H(e_1) = (F(e_1), K(e_1))$ and $H(e_2) = (F(e_2), K(e_2))$ are bipolar fuzzy graphs corresponding to the parameters e_1 and e_2, respectively, as shown in Figure 1. Hence $\widetilde{G} = (G^*, F, K, A)$ is a bipolar fuzzy soft graph.

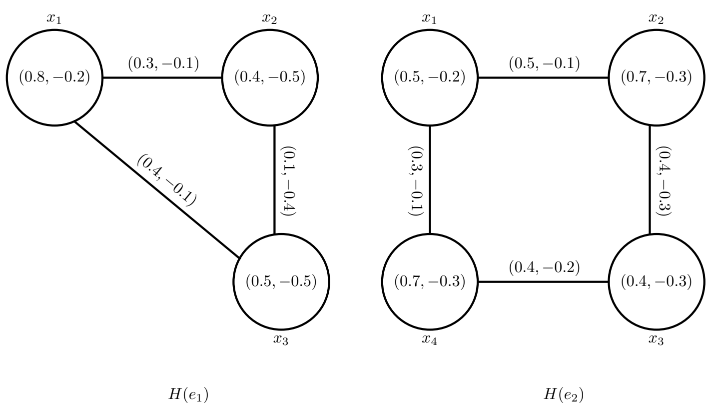

FIGURE 1. Bipolar fuzzy soft graph $\widetilde{G}$

Definition 3.12. Let $\widetilde{G_1} = (G^*, F_1, K_1, A)$ and $\widetilde{G_2} = (G^*, F_2, K_2, B)$ be two bipolar fuzzy soft graphs of G^*. Then $\widetilde{G_1}$ is called bipolar fuzzy soft subgraph of $\widetilde{G_2}$ if

i. $A \subseteq B$

ii. $H_1(a) = (F_1(a), K_1(a))$ is a bipolar fuzzy subgraph of $H_2(a) = (F_2(a), K_2(a))$ for all $a \in A$.

Example 3.2. Consider the bipolar fuzzy soft graph $\widetilde{G} = (G^*, F, K, A)$ as taken in Example 3.2. Let $B = \{e_1, e_2\}$ be a parameter set, (F_1, B) be a bipolar fuzzy soft set over V and (K_1, B) be a bipolar fuzzy soft set on E defined by

$$\begin{aligned} F_1(e_1) &= \{(x_2, 0.2, -0.1), (x_3, 0.4, -0.2) \\ K_1(e_1) &= \{(x_2x_3, 0.1, -0.1)\} \\ F_1(e_2) &= \{(x_1, 0.4, -0.1), (x_2, 0.4, -0.3), (x_3, 0.3, -0.1)\} \\ K_1(e_2) &= \{(x_1x_2, 0.4, -0.1), (x_2x_3, 0.3, -0.1)\} \end{aligned}$$

It is clearly seen that $H_1(e_1) = (F_1(e_1), K_1(e_1))$ and $H_1(e_2) = (F_1(e_2), K_1(e_2))$ are bipolar fuzzy graphs corresponding to the parameters e_1 and e_2, respectively, as shown in Figure 2.

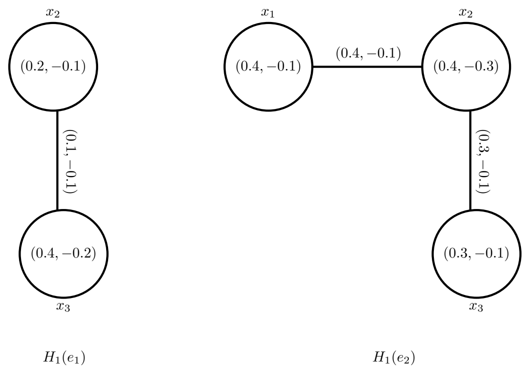

FIGURE 2. Bipolar fuzzy soft graph $\widetilde{G_1}$

Also $\widetilde{G_1} = (G^*, F_1, K_1, B)$ is a bipolar fuzzy soft graph. Hence $\widetilde{G_1}$ is a bipolar fuzzy soft subgraph of $\widetilde{G}$.

Definition 3.13. A bipolar fuzzy soft graph $\widetilde{G} = (G^*, F, K, A)$ is called strong bipolar fuzzy soft graph if

$$\mu^+_{K(e)}(xy) = \min(\mu^+_{F(e)}(x), \mu^+_{F(e)}(y)) \qquad \mu^-_{K(e)}(xy) = \max(\mu^-_{F(e)}(x), \mu^-_{F(e)}(y))$$

for all $e \in A$ and $xy \in E$.

Example 3.3. Let $G^* = (V, E)$ such that $V = \{x_1, x_2, x_3\}$ and $E = \{x_1x_2, x_2x_3, x_3x_1\}$. Let $A = \{e_1, e_2\}$ be a set of parameters and let (F, A) be a bipolar fuzzy soft set over V defined by

$$\begin{aligned} F(e_1) &= \{(x_1, 0.7, -0.4), (x_2, 0.5, -0.6), (x_3, 0.6, -0.5)\} \\ F(e_2) &= \{(x_1, 0.4, -0.3), (x_2, 0.8, -0.5), (x_3, 0.3, -0.4)\} \end{aligned}$$

Now let (K, A) be a bipolar fuzzy soft set over E defined by

$$\begin{aligned} K(e_1) &= \{(x_1x_2, 0.5, -0.4), (x_2x_3, 0.5, -0.5), (x_3x_1, 0.6, -0.4)\} \\ K(e_2) &= \{(x_1x_2, 0.4, -0.3), (x_2x_3, 0.3, -0.4), (x_3x_1, 0.3, -0.3)\} \end{aligned}$$

Obviously, $H(e_1) = (F(e_1), K(e_1))$ and $H(e_2) = (F(e_2), K(e_2))$ are bipolar fuzzy graphs. Hence $\widetilde{G} = (G^*, F, K, A)$ is strong bipolar fuzzy soft graph of G^* as shown in Figure 3.

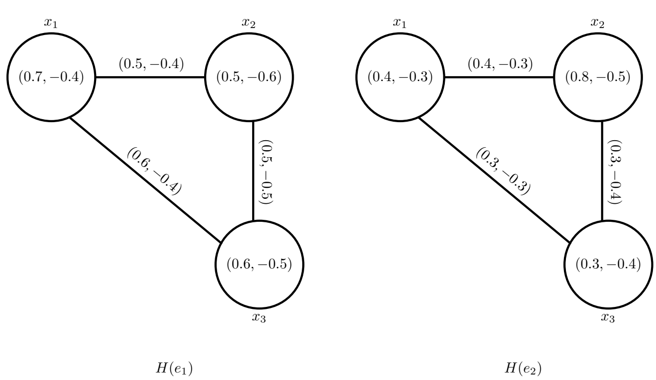

FIGURE 3. Strong bipolar fuzzy soft graph $\widetilde{G}$

Definition 3.14. Let $\widetilde{G_1} = (G_1^*, F_1, K_1, A)$ and $\widetilde{G_2} = (G_2^*, F_2, K_2, B)$ be two bipolar fuzzy soft graphs of simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The cartesian product of $\widetilde{G_1}$ and $\widetilde{G_2}$ is denoted by $\widetilde{G_1} \times \widetilde{G_2} = (G^*, F, K, A \times B)$, where $G^* = (V_1 \times V_2, E_1 \times E_2)$, and is defined by

$$\begin{cases} \begin{cases} (\mu^+_{F_1(e_1)} \times \mu^+_{F_2(e_2)})_{(x_1,x_2)} = \min(\mu^+_{F_1(e_1)}(x_1), \mu^+_{F_2(e_2)}(x_2)) \\ (\mu^-_{F_1(e_1)} \times \mu^-_{F_2(e_2)})_{(x_1,x_2)} = \max(\mu^-_{F_1(e_1)}(x_1), \mu^-_{F_2(e_2)}(x_2)) \end{cases} \\ \text{for all} \quad (x_1, x_2) \in V_1 \times V_2 \\ \\ \begin{cases} (\mu^+_{K_1(e_1)} \times \mu^+_{K_2(e_2)})_{((x,x_2)(x,y_2))} = \min(\mu^+_{F_1(e_1)}(x), \mu^+_{K_2(e_2)}(x_2, y_2)) \\ (\mu^-_{K_1(e_1)} \times \mu^-_{K_2(e_2)})_{((x,x_2)(x,y_2))} = \max(\mu^-_{F_1(e_1)}(x), \mu^-_{K_2(e_2)}(x_2, y_2)) \end{cases} \\ \text{for all} \quad x \in V_1 \quad and \quad (x_2, y_2) \in E_2 \\ \\ \begin{cases} (\mu^+_{K_1(e_1)} \times \mu^+_{K_2(e_2)})_{((x_1,z)(y_1,z))} = \min(\mu^+_{K_1(e_1)}(x_1, y_1), \mu^+_{F_2(e_2)}(z)) \\ (\mu^-_{K_1(e_1)} \times \mu^-_{K_2(e_2)})_{((x_1,z)(y_1,z))} = \max(\mu^-_{K_1(e_1)}(x_1, y_1), \mu^-_{F_2(e_2)}(z)) \end{cases} \\ \text{for all} \quad z \in V_2 \quad and \quad (x_1, y_1) \in E_1 \end{cases}$$

for all $e_1 \in A$ and $e_2 \in B$.

Example 3.4. Consider two graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that

$$V_1 = \{x_1, y_1, z_1, t_1\},\ E_1 = \{x_1y_1, z_1t_1\} \text{ and } V_2 = \{x_2, y_2, z_2\},\ E_2 = \{x_2y_2, y_2z_2\}$$

Let $A = \{e_1\}$ be a set of parameters, and let (F_1, A) and (K_1, A) be two bipolar fuzzy soft sets over V_1 and E_1, respectively, defined by

$$\begin{aligned} F_1(e_1) &= \{(x_1, 0.5, -0.2), (y_1, 0.7, -0.3), (z_1, 0.4, -0.3), (t_1, 0.7, -0.3)\} \\ K_1(e_1) &= \{(x_1y_1, 0.5, -0.2), (z_1t_1, 0.4, -0.3)\} \end{aligned}$$

Now let $B = \{e_2\}$ be a set of parameters, and let (F_2, B) and (K_2, B) be two bipolar fuzzy soft sets over V_2 and E_2, respectively, defined by

$$\begin{aligned} F_2(e_2) &= \{(x_2, 0.2, -0.2), (y_2, 0.3, -0.3), (z_2, 0.5, -0.4)\} \\ K_2(e_2) &= \{(x_2y_2, 0.2, -0.2), (y_2z_2, 0.3, -0.3)\} \end{aligned}$$

It is easy to see that, $H(e_1) = (F_1(e_1), K_1(e_1))$ and $H(e_2) = (F_2(e_2), K_2(e_2))$ are bipolar fuzzy graphs. Hence $\widetilde{G_1} = (G_1^*, F_1, K_1, A)$ and $\widetilde{G_2} = (G_2^*, F_2, K_2, B)$ are bipolar fuzzy soft graphs of G_1^* and G_2^*, respectively, as shown in Figure 4.

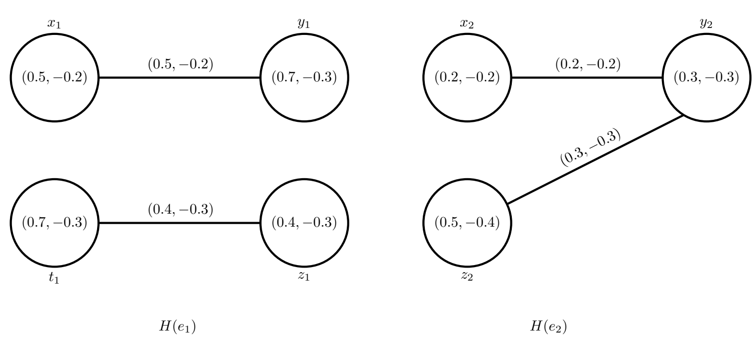

FIGURE 4. Bipolar fuzzy soft graphs $\widetilde{G_1}$ and $\widetilde{G_2}$

The cartesian product of $\widetilde{G_1}$ and $\widetilde{G_2}$ is as shown in Figure 5.

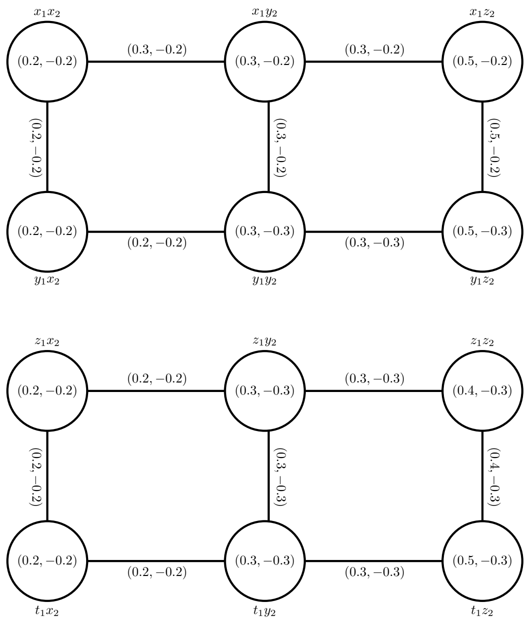

FIGURE 5. Cartesian product of bipolar fuzzy soft graphs $\widetilde{G_1}$ and $\widetilde{G_2}$

Theorem 3.1. *If $\widetilde{G_1}$ and $\widetilde{G_2}$ are two bipolar fuzzy soft graphs, then so is $\widetilde{G_1} \times \widetilde{G_2}$.*

Proof. Let $\widetilde{G_1} = (G_1^*, F_1, K_1, A)$ and $\widetilde{G_2} = (G_2^*, F_2, K_2, B)$ be two bipolar fuzzy soft graphs of simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively.

From Definition 3.14, for all $e_1 \in A$ and $e_2 \in B$, there are three cases.

Case (i) If $x_1 \in V_1$ and $x_2 \in V_2$, then

$$\begin{aligned}(\mu^+_{K_1(e_1)} \times \mu^+_{K_2(e_2)})_{(x_1,x_2)} &= \min(\mu^+_{F_1(e_1)}(x_1), \mu^+_{F_2(e_2)}(x_2)) \\ &= \min[(\mu^+_{F_1(e_1)} \times \mu^+_{F_2(e_2)})_{(x_1)}, (\mu^+_{F_1(e_1)} \times \mu^+_{F_2(e_2)})_{(x_2)}]\end{aligned}$$

$$\begin{aligned}(\mu^-_{K_1(e_1)} \times \mu^-_{K_2(e_2)})_{(x_1,x_2)} &= \max(\mu^-_{F_1(e_1)}(x_1), \mu^-_{F_2(e_2)}(x_2)) \\ &= \max[(\mu^-_{F_1(e_1)} \times \mu^-_{F_2(e_2)})_{(x_1)}, (\mu^-_{F_1(e_1)} \times \mu^-_{F_2(e_2)})_{(x_2)}]\end{aligned}$$

Case (ii) If $x \in V_1$ and $(x_2, y_2) \in E_2$, then

$$\begin{aligned}&(\mu^+_{K_1(e_1)} \times \mu^+_{K_2(e_2)})_{((x,x_2)(x,y_2))} \\ =\ & \min(\mu^+_{F_1(e_1)}(x), \mu^+_{K_2(e_2)}(x_2, y_2)) \\ \leq\ & \min[\mu^+_{F_1(e_1)}(x), \min(\mu^+_{F_2(e_2)}(x_2), \mu^+_{F_2(e_2)}(y_2))] \\ =\ & \min[\min(\mu^+_{F_1(e_1)}(x), \mu^+_{F_2(e_2)}(x_2)), \min(\mu^+_{F_1(e_1)}(x), \mu^+_{F_2(e_2)}(y_2))] \\ =\ & \min[(\mu^+_{F_1(e_1)} \times \mu^+_{F_2(e_2)})_{(x,x_2)}, (\mu^+_{F_1(e_1)} \times \mu^+_{F_2(e_2)})_{(x,y_2)}]\end{aligned}$$

$$\begin{aligned}&(\mu^-_{K_1(e_1)} \times \mu^-_{K_2(e_2)})_{((x,x_2)(x,y_2))} \\ =\ & \max(\mu^-_{F_1(e_1)}(x), \mu^-_{K_2(e_2)}(x_2, y_2)) \\ \geq\ & \max[\mu^-_{F_1(e_1)}(x), \max(\mu^-_{F_2(e_2)}(x_2), \mu^-_{F_2(e_2)}(y_2))] \\ =\ & \max[\max(\mu^-_{F_1(e_1)}(x), \mu^-_{F_2(e_2)}(x_2)), \max(\mu^-_{F_1(e_1)}(x), \mu^-_{F_2(e_2)}(y_2))] \\ =\ & \max[(\mu^-_{F_1(e_1)} \times \mu^-_{F_2(e_2)})_{(x,x_2)}, (\mu^-_{F_1(e_1)} \times \mu^-_{F_2(e_2)})_{(x,y_2)}]\end{aligned}$$

Case (iii) If $x \in V_2$ and $(x_1, y_1) \in E_1$, then

$$\begin{aligned}&(\mu^+_{K_1(e_1)} \times \mu^+_{K_2(e_2)})_{((x_1,z)(y_1,z))} \\ =\ & \min(\mu^+_{K_1(e_1)}(x_1, y_1), \mu^+_{F_2(e_2)}(z)) \\ \leq\ & \min[\mu^+_{F_2(e_2)}(z), \min(\mu^+_{F_1(e_1)}(x_1), \mu^+_{F_1(e_1)}(y_1))] \\ =\ & \min[\min(\mu^+_{F_1(e_1)}(x_1), \mu^+_{F_2(e_2)}(z)), \min(\mu^+_{F_1(e_1)}(y_1), \mu^+_{F_2(e_2)}(z))] \\ =\ & \min[(\mu^+_{F_1(e_1)} \times \mu^+_{F_2(e_2)})_{(x_1,z)}, (\mu^+_{F_1(e_1)} \times \mu^+_{F_2(e_2)})_{(y_1,z)}]\end{aligned}$$

$$\begin{aligned}&(\mu^-_{K_1(e_1)} \times \mu^-_{K_2(e_2)})_{((x_1,z)(y_1,z))} \\ =\ & \max(\mu^-_{K_1(e_1)}(x_1, y_1), \mu^-_{F_2(e_2)}(z)) \\ \geq\ & \max[\mu^-_{F_2(e_2)}(z), \max(\mu^-_{F_1(e_1)}(x_1), \mu^-_{F_1(e_1)}(y_1))] \\ =\ & \max[\max(\mu^-_{F_1(e_1)}(x_1), \mu^-_{F_2(e_2)}(z)), \max(\mu^-_{F_1(e_1)}(y_1), \mu^-_{F_2(e_2)}(z))] \\ =\ & \max[(\mu^-_{F_1(e_1)} \times \mu^-_{F_2(e_2)})_{(x_1,z)}, (\mu^-_{F_1(e_1)} \times \mu^-_{F_2(e_2)})_{(y_1,z)}]\end{aligned}$$

□

Definition 3.15. Let $\widetilde{G_1} = (G_1^*, F_1, K_1, A)$ and $\widetilde{G_2} = (G_2^*, F_2, K_2, B)$ be two bipolar fuzzy soft graphs of simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The strong product of $\widetilde{G_1}$ and $\widetilde{G_2}$ is denoted by $\widetilde{G_1} \otimes \widetilde{G_2} = (G^*, F, K, A \times B)$ and is defined by

$$\begin{cases} \begin{cases} (\mu^+_{F_1(e_1)} \otimes \mu^+_{F_2(e_2)})_{(x_1,x_2)} = \min(\mu^+_{F_1(e_1)}(x_1), \mu^+_{F_2(e_2)}(x_2)) \\ (\mu^-_{F_1(e_1)} \otimes \mu^-_{F_2(e_2)})_{(x_1,x_2)} = \max(\mu^-_{F_1(e_1)}(x_1), \mu^-_{F_2(e_2)}(x_2)) \end{cases} \\ \text{for all} \quad (x_1,x_2) \in V_1 \times V_2 \\ \\ \begin{cases} (\mu^+_{K_1(e_1)} \otimes \mu^+_{K_2(e_2)})_{((x,x_2)(x,y_2))} = \min(\mu^+_{F_1(e_1)}(x), \mu^+_{K_2(e_2)}(x_2,y_2)) \\ (\mu^-_{K_1(e_1)} \otimes \mu^-_{K_2(e_2)})_{((x,x_2)(x,y_2))} = \max(\mu^-_{F_1(e_1)}(x), \mu^-_{K_2(e_2)}(x_2,y_2)) \end{cases} \\ \text{for all} \quad x \in V_1 \quad and \quad (x_2,y_2) \in E_2 \end{cases}$$

$$\begin{cases} \begin{cases} (\mu^+_{K_1(e_1)} \otimes \mu^+_{K_2(e_2)})_{((x_1,z)(y_1,z))} = \min(\mu^+_{K_1(e_1)}(x_1,y_1), \mu^+_{F_2(e_2)}(z)) \\ (\mu^-_{K_1(e_1)} \otimes \mu^-_{K_2(e_2)})_{((x_1,z)(y_1,z))} = \max(\mu^-_{K_1(e_1)}(x_1,y_1), \mu^-_{F_2(e_2)}(z)) \end{cases} \\ \text{for all} \quad z \in V_2 \quad and \quad (x_1,y_1) \in E_1 \\ \\ \begin{cases} (\mu^+_{K_1(e_1)} \otimes \mu^+_{K_2(e_2)})_{((x_1,z_1)(y_1,z_2))} = \min(\mu^+_{K_1(e_1)}(x_1,y_1), \mu^+_{K_2(e_2)}(z_1,z_2)) \\ (\mu^-_{K_1(e_1)} \otimes \mu^-_{K_2(e_2)})_{((x_1,z_1)(y_1,z_2))} = \max(\mu^-_{K_1(e_1)}(x_1,y_1), \mu^-_{K_2(e_2)}(z_1,z_2)) \end{cases} \\ \text{for all} \quad (x_1,y_1) \in E_1 \quad and \quad (z_1,z_2) \in E_2 \end{cases}$$

for all $e_1 \in A$ and $e_2 \in B$.

Example 3.5. Consider two bipolar fuzzy soft graphs $\widetilde{G_1} = (G_1^*, F_1, K_1, A)$ and $\widetilde{G_2} = (G_2^*, F_2, K_2, B)$ as taken in Example 3.4. The strong product of $\widetilde{G_1}$ and $\widetilde{G_2}$ is as shown in Figure 6.

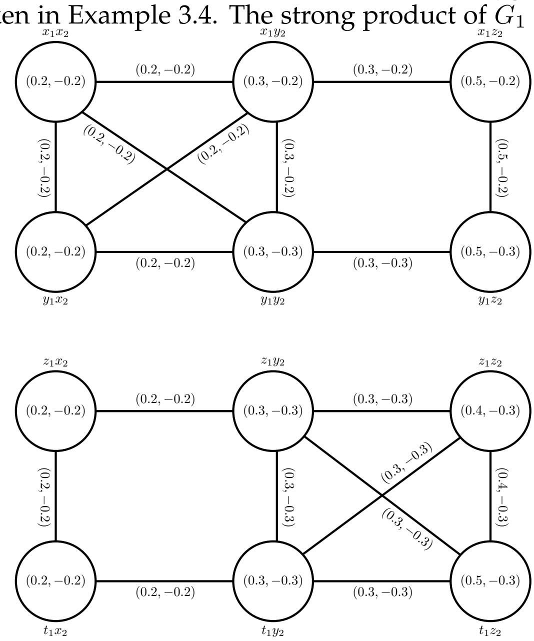

FIGURE 6. Strong product of bipolar fuzzy soft graphs $\widetilde{G_1}$ and $\widetilde{G_2}$

Theorem 3.2. *If $\widetilde{G_1}$ and $\widetilde{G_2}$ are two bipolar fuzzy soft graphs, then so is $\widetilde{G_1} \otimes \widetilde{G_2}$.*

Proof. By using Definition 3.15, it can be shown in a similar way to proof of Theorem 3.1. □

Definition 3.16. Let $\widetilde{G_1} = (G_1^*, F_1, K_1, A)$ and $\widetilde{G_2} = (G_2^*, F_2, K_2, B)$ be two bipolar fuzzy soft graphs of simple graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The composition of $\widetilde{G_1}$ and $\widetilde{G_2}$ is denoted by $\widetilde{G_1} \circ \widetilde{G_2} = (G^*, F, K, A \times B)$ and is defined by

$$\begin{cases} \begin{cases} (\mu^+_{F_1(e_1)} \circ \mu^+_{F_2(e_2)})_{(x_1,x_2)} = \min(\mu^+_{F_1(e_1)}(x_1), \mu^+_{F_2(e_2)}(x_2)) \\ (\mu^-_{F_1(e_1)} \circ \mu^-_{F_2(e_2)})_{(x_1,x_2)} = \max(\mu^-_{F_1(e_1)}(x_1), \mu^-_{F_2(e_2)}(x_2)) \end{cases} \\ \text{for all} \quad (x_1,x_2) \in V_1 \times V_2 \end{cases}$$

$$\begin{cases} \begin{cases} (\mu^+_{K_1(e_1)} \circ \mu^+_{K_2(e_2)})_{((x,x_2)(x,y_2))} = \min(\mu^+_{F_1(e_1)}(x), \mu^+_{K_2(e_2)}(x_2,y_2)) \\ (\mu^-_{K_1(e_1)} \circ \mu^-_{K_2(e_2)})_{((x,x_2)(x,y_2))} = \max(\mu^-_{F_1(e_1)}(x), \mu^-_{K_2(e_2)}(x_2,y_2)) \end{cases} \\ \text{for all} \quad x \in V_1 \quad and \quad (x_2,y_2) \in E_2 \\ \\ \begin{cases} (\mu^+_{K_1(e_1)} \circ \mu^+_{K_2(e_2)})_{((x_1,z)(y_1,z))} = \min(\mu^+_{K_1(e_1)}(x_1,y_1), \mu^+_{F_2(e_2)}(z)) \\ (\mu^-_{K_1(e_1)} \circ \mu^-_{K_2(e_2)})_{((x_1,z)(y_1,z))} = \max(\mu^-_{K_1(e_1)}(x_1,y_1), \mu^-_{F_2(e_2)}(z)) \end{cases} \\ \text{for all} \quad z \in V_2 \quad and \quad (x_1,y_1) \in E_1 \\ \\ \begin{cases} (\mu^+_{K_1(e_1)} \circ \mu^+_{K_2(e_2)})_{((x_1,z_1)(y_1,z_2))} = \min(\mu^+_{K_1(e_1)}(x_1,y_1), \mu^+_{K_2(e_2)}(z_1,z_2)) \\ (\mu^-_{K_1(e_1)} \circ \mu^-_{K_2(e_2)})_{((x_1,z_1)(y_1,z_2))} = \max(\mu^-_{K_1(e_1)}(x_1,y_1), \mu^-_{K_2(e_2)}(z_1,z_2)) \end{cases} \\ \text{for all} \quad (x_1,y_1) \in E_1 \quad and \quad (z_1,z_2) \in E_2 \\ \\ \begin{cases} (\mu^+_{K_1(e_1)} \circ \mu^+_{K_2(e_2)})_{((x_1,z_1)(y_1,z_2))} = \min(\mu^+_{F_2(e_2)}(z_1), \mu^+_{F_2(e_2)}(z_2), \mu^+_{K_1(e_1)}(x_1,y_1)) \\ (\mu^-_{K_1(e)} \circ \mu^-_{K_2(e)})_{((x_1,z_1)(y_1,z_2))} = \max(\mu^-_{F_2(e_2)}(z_1), \mu^+_{F_2(e_2)}(z_2), \mu^+_{K_1(e_1)}(x_1,y_1)) \end{cases} \\ \text{for all} \quad (x_1,y_1) \in E_1 \quad and \quad z_1,z_2 \in V_2 \quad such\,that \quad z_1 \neq z_2 \end{cases}$$

for all $e_1 \in A$ and $e_2 \in B$.

Example 3.6. Consider two bipolar fuzzy soft graphs $\widetilde{G_1} = (G_1^*, F_1, K_1, A)$ and $\widetilde{G_2} = (G_2^*, F_2, K_2, B)$ as taken in Example 3.4. The composition of $\widetilde{G_1}$ and $\widetilde{G_2}$ is as shown in Figure 7.

Theorem 3.3. *If $\widetilde{G_1}$ and $\widetilde{G_2}$ are two bipolar fuzzy soft graphs, then so is $\widetilde{G_1} \circ \widetilde{G_2}$.*

Proof. By using Definition 3.16, it can be shown in a similar way to proof of Theorem 3.1. □

4. CONCLUSIONS

Graph theory is an extremely useful mathematical tool to solve the complicated problems in different areas including geometry, algebra, number theory, topology, operations research, optimization and computer science. However, in many cases, some aspects of a graph-theoretical problem may be vague or uncertain. It is natural to deal with the vagueness and uncertainty using the methods of fuzzy sets or fuzzy soft sets. The bipolar fuzzy soft sets constitute a generalization of Maji's fuzzy soft set theory [15]. Since bipolar fuzzy soft set give more sensitive, flexibility and conformity to the systems as compared to the fuzzy set, we have applied the concept of bipolar fuzzy soft sets to graph structures. Then we have introduced the concept of bipolar fuzzy soft graph structures and described method of their construction. The concept of bipolar fuzzy soft graphs can be applied in various areas of engineering, computer science: database theory, expert systems, artificial intelligence, pattern recognition, robotics, computer networks, and medical diagnosis. We want to make, in near future, some applications of bipolar fuzzy soft graphs in database theory, neural networks and geographical information system.

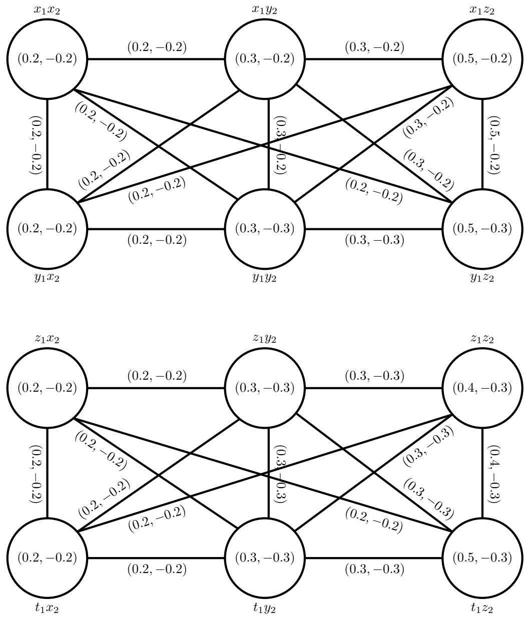

FIGURE 7. Composition of bipolar fuzzy soft graphs $\widetilde{G_1}$ and $\widetilde{G_2}$

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