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A study in the fixed point theory for a new iterative scheme and a class of generalized mappings

KADRI DOGAN and VATAN KARAKAYA

ABSTRACT. In this study, we introduce a new iteration scheme and prove the strong convergence result for this iteration method. We also compare the rate of convergence with the iterative scheme and the fixed point iteration scheme known as Picard-S due to Gursoy. Then we prove that this new iteration method is equivalent to convergence of the iteration schemes given in the introduction section of the manuscript. Moreover, we show the result of its data dependency.

1. INTRODUCTION AND PRELIMINARIES

Iterative methods are popular tools to approximate fixed points of nonlinear mappings. When studied an iterative procedure, it should be considered two criteria which are the faster and the simplify. In this direction, some of notable studies were conducted by Mann, Ishikawa, Noor, Suantai, Karakaya, Gursoy, Dogan, Yildirim, Karahan, Sainuan, Agarwal, Rhoades and Khan [1, 6, 8, 10–14, 16, 17, 19, 21–23]

In this study, we handle a new iterative process and general mapping which is the class of contraction mapping.

Let E be a real normed spaces and $\wp : E \to E$ be a mapping. A point $p \in E$ is a fixed point of $\wp$ if $\wp(p) = p$. We denote by $F(\wp)$ the set of fixed points of $\wp$.

Now, we will consider some of iterative schemes related to this work. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_0 \in E, \\ x_{n+1} = \wp x_n, \ (n \in \mathbb{N}) \end{cases} \tag{1.1}$$

is called to Picard iterative process.

In 2013, Karakaya et all [13] introduced a new three step iterative process as follows:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\gamma_n - \alpha_n)\, y_n + \alpha_n \wp y_n + \gamma_n \wp z_n \\ y_n = (1-\lambda_n - \beta_n)\, z_n + \lambda_n \wp x_n + \beta_n \wp z_n \\ z_n = (1-\theta_n)\, x_n + \theta_n \wp x_n, \ (n \in \mathbb{N}), \end{cases} \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\lambda_n\}_{n=0}^{\infty}$, $\{\theta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty} \in [0,1]$.

Gürsoy and Karakaya [8] introduced Picard-S iterative process as follows:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = \wp y_n \\ y_n = (1-\alpha_n)\, \wp x_n + \alpha_n \wp z_n \\ z_n = (1-\beta_n)\, x_n + \beta_n \wp x_n, \ (n \in \mathbb{N}), \end{cases} \tag{1.3}$$

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Corresponding author: Kadri Dogan; dogankadri@hotmail.com
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where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \in [0,1]$.

We inspired by the above fixed point iteration schemes and introduce a new fixed point iteration scheme as follows:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)\wp z_n + \alpha_n \wp y_n \\ y_n = (1-\beta_n)\wp x_n + \beta_n \wp z_n \\ z_n = \wp x_n, \ (n \in \mathbb{N}), \end{cases} \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \in [0,1]$.

Osilike [18] proved several stability results using the following contractive definition: for each $x, y \in X$, there exist $\delta \in [0,1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L d(x, Tx). \tag{1.5}$$

Imoru and Olatinwo [9] proved some stability results using the following general contractive definition:

$$d(Tx, Ty) \leq \delta d(x, y) + \phi(d(x, Tx)) \tag{1.6}$$

where $x, y \in X$, $\delta \in [0,1)$ and $\phi : R^+ \to R^+$ a monotone increasing function with $\phi(0) = 0$.

Recently, Bosede and Rhoades [4] made an assumption implied by (1.5) and the one which renders all generalizations of the form (1.6) pointless. That is, if $x = Tp = p$, then (1.5) becomes for each $x, y \in X$, there exist $\delta \in [0,1)$ such that

$$d(p, Ty) \leq \delta d(p, y), \tag{1.7}$$

and in the real normed spaces, for each $x, y \in X$, there exist $\delta \in [0,1)$ such that

$$\|p - Ty\| \leq \delta \|p - y\|. \tag{1.8}$$

Chidume and Olaleru [5] gave several examples to show that (1.8) is more general than that of (1.5) and (1.6), provided the fixed point exists. They proved that every contraction mapping with a fixed point satisfies inequality (1.8).

Definition 1.1. [2] Let E be a Banach space and $\wp, \widetilde{\wp} : E \to E$ two operators. Then $\widetilde{\wp}$ is an approximate operator of $\wp$ if for all $x \in E$ and for a fixed $\varepsilon > 0$ such that

$$\|\wp x - \widetilde{\wp} x\| \leq \varepsilon. \tag{1.9}$$

Theorem 1.1. [24] *Let E be a real Banach space, let $B \subset E$ be a nonempty convex and closed set, and let $\varepsilon > 0$ be a fixed number. If $\wp : B \to B$ is a contractive-like operator with the fixed point p and $\widetilde{\wp} : B \to B$ is an operator with a fixed point q, and if the following relation is satisfied* (1.9), *then*

$$\|p - q\| \leq \frac{\varepsilon}{1-\delta}.$$

Definition 1.2. [2] Let $\{k_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$ be two sequences of real numbers which converge to k and t, respectively. Also, assume that

$$\lim_{n\to\infty} \left| \frac{k_n - k}{t_n - t} \right| = l.$$

- If $l = 0$, then it can be said that $\{k_n\}_{n=0}^{\infty}$ converges faster to k than $\{t_n\}_{n=0}^{\infty}$ to t.
- If $0 < l < \infty$, then it can be said that $\{k_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 1.3. [2] Let $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be two fixed point iteration procedures both converging to the same fixed point p with the error estimates $\|x_n - p\| \leq k_n$ and $\|u_n - p\| \leq t_n$, where $\{k_n\}_{n=0}^{\infty}$ and $\{t_n\}_{n=0}^{\infty}$ be two sequences of real numbers converging to 0. Then, $\{x_n\}_{n=0}^{\infty}$ converges faster than $\{u_n\}_{n=0}^{\infty}$ to p, if $\{k_n\}_{n=0}^{\infty}$ converges faster than $\{t_n\}_{n=0}^{\infty}$.

Lemma 1.1. [25] *If ρ is a real number satisfying $0 \leq \rho < 1$ and $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $lim_{n\to\infty}\epsilon_n = 0$, then for any sequence of positive numbers $(\epsilon_n)_{n\in\mathbb{N}}$ satisfying*

$$a_{n+1} \leq \rho a_n + \epsilon_n, n = 1, 2, ...,$$

one has

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 1.2. [24] *Let $\{\psi_n\}$ be a nonnegative sequence for which one supposes there exists $n_0 \in \mathbb{N}$,*

such that for all $n \geq n_0$ one has satisfied the following inequality:

$$\psi_{n+1} \leq (1 - \lambda_n)\,\psi_n + \lambda_n \phi_n$$

where $\lambda_n \in (0, 1), \forall n \in \mathbb{N}, \sum\limits_{n=1}^{\infty} \lambda_n = \infty$ and $\phi_n \geq 0, \forall n \in \mathbb{N}$. Then

$$0 \leq \lim_{n\to\infty} \sup \psi_n \leq \lim_{n\to\infty} \sup \phi_n.$$

2. Main result

Theorem 2.2. *Let $(E, \|\cdot\|)$ be a real normed linear space and $T : E \to E$ be a map with a fixed point p satisfying the condition (1.8). Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by (1.4), where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in$ [0,1] and $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n = \infty$. Then the iterative scheme $\{x_n\}_{n=0}^{\infty}$ converges to the fixed point of T.*

Proof.

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)Tz_n + \alpha_n Ty_n - p\| \\ &\leq (1-\alpha_n)\|Tz_n - p\| + \alpha_n \|Ty_n - p\| \\ &\leq (1-\alpha_n)\delta\|z_n - p\| + \alpha_n\delta\|y_n - p\| \end{aligned} \tag{2.10}$$

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| \\ &\leq (1-\beta_n)\|Tx_n - p\| + \beta_n \|Tz_n - p\| \\ &\leq (1-\beta_n)\delta\|x_n - p\| + \beta_n\delta\|z_n - p\| \end{aligned} \tag{2.11}$$

$$\begin{aligned} \|z_n - p\| &= \|Tx_n - p\| \\ &\leq \delta\|x_n - p\| \end{aligned} \tag{2.12}$$

Substituting (2.12) in (2.11), we have

$$\begin{aligned} \|y_n - p\| &\leq (1-\beta_n)\delta\|x_n - p\| + \beta_n\delta^2\|x_n - p\| \\ &= \delta(1 - \beta_n(1-\delta))\|x_n - p\|. \end{aligned} \tag{2.13}$$

Substituting (2.12) and (2.13) in (2.10), we have

$$\begin{aligned} \|x_{n+1}-p\| &\leq (1-\alpha_n)\delta^2\|x_n-p\|+\alpha_n\delta^2(1-\beta_n(1-\delta))\|x_n-p\| \\ &= \delta^2(1-\alpha_n+\alpha_n(1-\beta_n(1-\delta)))\|x_n-p\| \\ &= \delta^2(1-\alpha_n\beta_n(1-\delta))\|x_n-p\|. \end{aligned}$$

By using Lemmma 1.2, we obtain

$$\lim_{n\to\infty}\|x_n-p\|=0.$$

□

Theorem 2.3. *Let* $(E,\|\cdot\|)$ *be a real normed linear space and* $\wp: E\to E$ *be a map with a fixed point* p *satisfying the condition* (1.8). *Let* $\{x_n\}_{n=0}^{\infty}$ *and* $\{u_n\}_{n=0}^{\infty}$ *be defined by* (1.4) *and* (1.2), *respectively, with the real sequences* $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\lambda_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\theta_n\}_{n=0}^{\infty}\in[0,1]$ *and* $\sum_{n=0}^{\infty}\alpha_n\beta_n=\infty$. *Then the following claims are equivalent:*

- $\{x_n\}_{n=0}^{\infty}$ *strongly converges to the unique fixed point* p *of* $\wp$;
- $\{u_n\}_{n=0}^{\infty}$ *strongly converges to the unique fixed point* p *of* $\wp$.

Proof. We will show that $(i)\Rightarrow(ii)$. Suppose that the iteration scheme (1.4) converges to p. we wil show that the iteration scheme(1.2) is convergence to fixed point p of $\wp$. Thus, by using the iterative schemes (1.2), (1.4) and a generalized mapping (1.8), we have

$$\begin{aligned} \|u_{n+1}-p\| &= \left\| \begin{array}{c} (1-\gamma_n-\alpha_n)v_n+\alpha_n\wp v_n+\gamma_n\wp w_n+(1-\gamma_n-\alpha_n)\wp z_n \\ +\alpha_n\wp y_n+\gamma_n\wp z_n-(1-\gamma_n-\alpha_n)\wp z_n-\alpha_n\wp y_n-\gamma_n\wp z_n-p \end{array} \right\| \\ &= \left\| \begin{array}{c} (1-\gamma_n-a_n)v_n+a_n\wp v_n+\gamma_n\wp w_n \\ -(1-\gamma_n-\alpha_n)\wp z_n-\alpha_n\wp y_n-\gamma_n\wp z_n \end{array} \right\| \\ &\quad +\|(1-\gamma_n-\alpha_n)\wp z_n+\alpha_n\wp y_n+\gamma_n\wp z_n-p\| \\ &\leq (1-\gamma_n-\alpha_n)\|\wp z_n-v_n\|+\alpha_n\|\wp y_n-\wp v_n\|+\gamma_n\|\wp z_n-\wp w_n\| \\ &\quad +(1-\gamma_n-\alpha_n)\|\wp z_n-p\|+\alpha_n\|\wp y_n-p\|+\gamma_n\|\wp z_n-p\| \\ &\leq (1-\gamma_n-\alpha_n)\delta\|z_n-p\|+(1-\gamma_n-\alpha_n)\|p-v_n\| \\ &\quad +\alpha_n\delta\|y_n-p\|+\alpha_n\delta\|p-v_n\|+\gamma_n\delta\|z_n-p\|+\gamma_n\delta\|p-w_n\| \\ &\quad +(1-\alpha_n)\delta\|z_n-p\|+\alpha_n\delta\|y_n-p\| \end{aligned}$$

$$\begin{aligned} \|u_{n+1}-p\| &\leq 2(1-\alpha_n)\delta^2\|x_n-p\|+2\alpha_n\delta^2(1-\beta_n)\|x_n-p\|+2\alpha_n\delta^3\beta_n\|x_n-p\| \\ &\quad +[(1-\gamma_n-\alpha_n)+\alpha_n\delta](1-\lambda_n-\beta_n)\|p-w_n\| \\ &\quad +[(1-\gamma_n-\alpha_n)+\alpha_n\delta]\lambda_n\delta\|p-u_n\|+[(1-\gamma_n-\alpha_n)+\alpha_n\delta]\beta_n\delta\|p-w_n\| \\ &\quad +\gamma_n\delta(1-\theta_n)\|p-u\|+\gamma_n\delta^2\theta_n\|p-u_n\|. \end{aligned}$$

Again using (1.2), (1.4) and (1.8), we have

$$\begin{aligned} \|u_{n+1}-p\| &\leq \left[2(1-\alpha_n)\delta^2+2\alpha_n\delta^2(1-\beta_n)+2\alpha_n\delta^3\beta_n\right]\|x_n-p\| \\ &\quad +\left[\begin{array}{c} [(1-\gamma_n-\alpha_n)+\alpha_n\delta](1-\lambda_n-\beta_n) \\ +[(1-\gamma_n-\alpha_n)+\alpha_n\delta]\beta_n\delta \end{array}\right]\|p-w_n\| \\ &\quad +\delta\left[\gamma_n(1-\theta_n(1-\delta))+[(1-\gamma_n-\alpha_n)+\alpha_n\delta]\lambda_n\right]\|p-u_n\| \\ &\leq \left[2(1-\alpha_n)\delta^2+2\alpha_n\delta^2(1-\beta_n)+2\alpha_n\delta^3\beta_n\right]\|x_n-p\| \\ &\quad +(1-\gamma_n-\alpha_n(1-\delta))(1-\beta_n(1-\delta)-\lambda_n)\|p-(1-\theta_n)u_n-\theta_n\wp u_n\| \\ &\quad +\delta\left[\gamma_n(1-\theta_n(1-\delta))+[(1-\gamma_n-\alpha_n)+\alpha_n\delta]\lambda_n\right]\|p-u_n\|. \end{aligned}$$

$$\begin{aligned}\|u_{n+1}-p\| &\leq \left[2(1-\alpha_n)\delta^2+2\alpha_n\delta^2(1-\beta_n)+2\alpha_n\delta^3\beta_n\right]\|x_n-p\| \\ &+(1-\gamma_n-\alpha_n(1-\delta))(1-\beta_n(1-\delta)-\lambda_n)(1-\theta_n(1-\delta))\|p-u_n\| \\ &+\delta\left[\gamma_n(1-\theta_n(1-\delta))+\left[(1-\gamma_n-\alpha_n)+\alpha_n\delta\right]\lambda_n\right]\|p-u_n\|.\end{aligned}$$

If we do some simplifications in the coefficients of the inequality, we obtain

$$\begin{aligned}\|u_{n+1}-p\| &\leq 2\delta^2\left[1-\alpha_n\beta_n(1-\delta)\right]\|x_n-p\| \\ &+\left[\begin{array}{c}(1-\gamma_n-\alpha_n(1-\delta))(1-\beta_n(1-\delta)-\lambda_n)(1-\theta_n(1-\delta)) \\ +\delta\left[\gamma_n(1-\theta_n(1-\delta))+(1-\gamma_n-\alpha_n(1-\delta))\lambda_n\right]\end{array}\right]\|p-u_n\| \\ &\leq 2\delta^2\left[1-\alpha_n\beta_n(1-\delta)\right]\|x_n-p\| \\ &+(1-[\gamma_n+\alpha_n](1-\delta))(1-[\lambda_n+\beta_n](1-\delta))\|p-u_n\|.\end{aligned}$$

Since $p=\wp(p)\in F(\wp)$ and

$$\lim_{n\to\infty}\|x_n-p\|=0.$$

It follows from Lemma 1.2 that

$$\lim_{n\to\infty}\|u_n-p\|=0.$$

Conversely, We will show that $(ii)\Rightarrow(i)$. Suppose that the iteration scheme (1.2) converges to p. we wil show that the iteration scheme(1.4) is convergence to fixed point p of $\wp$. Thus, by using the iterative schemes (1.2), (1.4) and a generalized mapping (1.8), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \left\|\begin{array}{c}(1-\gamma_n-\alpha_n)\wp z_n+\alpha_n\wp y_n+\gamma_n\wp z_n+(1-\gamma_n-\alpha_n)v_n+\alpha_n\wp v_n \\ +\gamma_n\wp w_n-(1-\gamma_n-\alpha_n)v_n-\alpha_n\wp v_n-\gamma_n\wp w_n-p\end{array}\right\| \\ &\leq \left\|\begin{array}{c}(1-\gamma_n-\alpha_n)\wp z_n+\alpha_n\wp y_n+\gamma_n\wp z_n \\ -(1-\gamma_n-\alpha_n)v_n-\alpha_n\wp v_n-\gamma_n\wp w_n\end{array}\right\| \\ &+\|(1-\gamma_n-\alpha_n)v_n+\alpha_n\wp v_n+\gamma_n\wp w_n-p\|\end{aligned}$$

Again using (1.2), (1.4) and (1.8), we have

$$\begin{aligned}&\leq (1-\gamma_n-\alpha_n)\left[\|\wp z_n-p\|+\|p-v_n\|\right]+\alpha_n\left[\|\wp y_n-p\|+\|p-\wp v_n\|\right] \\ &+\gamma_n\left[\|\wp z_n-p\|+\|p-\wp w_n\|\right] \\ &+(1-\gamma_n-\alpha_n)\|v_n-p\|+\alpha_n\delta\|v_n-p\|+\gamma_n\delta\|w_n-p\| \\ &\leq (1-\gamma_n-\alpha_n)\delta\|z_n-p\|+(1-\gamma_n-\alpha_n)\|p-v_n\|+\alpha_n\delta\|y_n-p\|+\alpha_n\delta\|p-v_n\| \\ &+\gamma_n\delta\|z_n-p\|+\gamma_n\delta\|p-w_n\| \\ &+(1-\gamma_n-\alpha_n)\|v_n-p\|+\alpha_n\delta\|v_n-p\|+\gamma_n\delta\|w_n-p\|.\end{aligned}$$

$$\begin{aligned}&\leq (1-\gamma_n(1-\delta)-\alpha_n)\delta^2\|x_n-p\|+\alpha_n\delta^2(1-\beta_n(1-\delta))\|x_n-p\| \\ &+2\left[(1-\gamma_n-\alpha_n(1-\delta))(1-\beta_n(1-\delta)-\lambda_n)+\gamma_n\delta\right]\|p-w_n\| \\ &+2\delta(1-\gamma_n-\alpha_n(1-\delta))\lambda_n\|p-u_n\|\end{aligned}$$

If we do some simplifications in the coefficients of the inequality, we obtain

$$\begin{aligned}\|x_{n+1}-p\| &\leq \delta^2(1-\gamma_n(1-\delta))\|x_n-p\| \\ &+\left[\begin{array}{c}2\left[\begin{array}{c}(1-\gamma_n-\alpha_n(1-\delta)) \\ \times(1-\beta_n(1-\delta)-\lambda_n) \\ +\gamma_n\delta\end{array}\right](1-\theta_n(1-\delta)) \\ +2\delta(1-\gamma_n-\alpha_n(1-\delta))\lambda_n\end{array}\right]\|p-u_n\|.\end{aligned}$$

Since $p=\wp(p)\in F(\wp)$ and

$$\lim_{n\to\infty}\|u_n-p\|=0.$$

It follows from Lemma 1.2 that

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

□

Corollary 2.1. *Let $(E, \|\cdot\|)$ be a real normed linear space and $\wp : E \to E$ be a map with a fixed point p satisfying the condition (1.8) . Let $\{x_n\}_{n=0}^{\infty}$ be a sequence built with the operator $\wp$ and the real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\lambda_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\theta_n\}_{n=0}^{\infty} \in [0,1]$. Then, in the iterative process (1.2):*

1: *If taken $\alpha_n = \lambda_n = \gamma_n = \beta_n = 0$, then the iterative scheme (1.2) reduce to the Mann iterative scheme [16]. Thus the Mann iterative scheme converges to fixed point p of $\wp$.*
2: *If taken $\alpha_n + \gamma_n = 1$ and $\beta_n = \theta_n = 0$, then the iterative scheme (1.2) reduce to the Agarwal iterative scheme [1]. Thus the Agarwal iterative scheme converges to fixed point p of $\wp$.*
3: *If taken $\gamma_n = \beta_n = 0$, then the iterative scheme (1.2) reduce to the SP iterative scheme [19]. Thus the SP iterative scheme converges to fixed point p of $\wp$.*
4: *If taken $\lambda_n = \gamma_n = \beta_n = 0$ and $\alpha_n = 1$, then the iterative scheme (1.2) reduce to the Picard-Mann iterative scheme [14]. Thus the PM iterative scheme converges to fixed point p of $\wp$.*
5: *If taken $\lambda_n = \gamma_n = \beta_n = \theta_n = 0$, then the iterative scheme (1.2) reduce to the MP iterative scheme [7]. Thus the MP iterative scheme converges to fixed point p of $\wp$.*
6: *If taken $\gamma_n = 0$, $\lambda_n + \beta_n = 1$ and $\alpha_n = 1$, then the iterative scheme (1.2) reduce to the iterative scheme (1.3). Thus the iterative scheme (1.3) converges to fixed point p of $\wp$.*
7: *If taken $\alpha_n + \gamma_n = 1$ and $\beta_n = 0$, then the iterative scheme (1.2) reduce to the Sainuan iterative scheme [23]. Thus the Sainuan iterative scheme converges to fixed point p of $\wp$.*

Theorem 2.4. *Let $(E, \|\cdot\|)$ be a real normed linear space and $\wp : E \to E$ be a map with a fixed point p satisfying the condition (1.8). Let $\{k_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$ be defined by (1.4) and (1.3), respectively, with the real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\lambda_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\theta_n\}_{n=0}^{\infty} \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$. Then the iteration schemes $\{k_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$ have the same rate of convergence to p of $\wp$.*

Proof. The following equality was obtained by the using mapping (1.8) and fixed point iterative schemes (1.3) and (1.4).

From the proof of Theorem 2.2, we have the following estimates:

$$\|l_{n+1} - p\| \leq \delta^{2n+2} \|l_0 - p\| \prod_{k=0}^{\infty} \left(1 - \alpha_k\beta_k \left(1 - \delta\right)\right),$$

$$\|k_{n+1} - p\| \leq \delta^{2n+2} \|k_0 - p\| \prod_{k=0}^{\infty} \left(1 - \alpha_k\beta_k \left(1 - \delta\right)\right).$$

for all $n \geq 0$.

Define

$$a_n = \delta^{2n+2} \|l_0 - p\| \prod_{k=0}^{\infty} \left(1 - \alpha_k\beta_k \left(1 - \delta\right)\right),$$

$$b_n = \delta^{2n+2} \|k_0 - p\| \prod_{k=0}^{\infty} \left(1 - \alpha_k\beta_k \left(1 - \delta\right)\right).$$

Since $k_0 = l_0$ and $\sum_{n=0}^{\infty} \alpha_n\beta_n = \infty$, it is clear that $a_n \to 0$ and $b_n \to 0$ for all $n \geq 0$.

Hence

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\delta^{2n+2}\|l_0-p\|\prod_{k=0}^{\infty}\left(1-\alpha_k\beta_k\left(1-\delta\right)\right)}{\delta^{2n+2}\|k_0-p\|\prod_{k=0}^{\infty}\left(1-\alpha_k\beta_k\left(1-\delta\right)\right)} = 1.$$

it is implies that $\{k_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$ fixed point iterative schemes have the same rate of convergence. □

Example 2.1. Let $E=[0,\infty]$ and $\wp : E\to E$ be a map defined by

$$\wp x = \frac{x-\ln(x+1)}{2}.$$

It is clear that the operator $\wp$ satisfies condition (1.8) The following table and figures show that iterative schemes (1.3) and (1.4) have the same rate of converge to 0 for $k_0 = l_0 = 1$, $\delta\in(0,1)$ and $\beta_n=\alpha_n=\frac{n+1}{n+2}$.

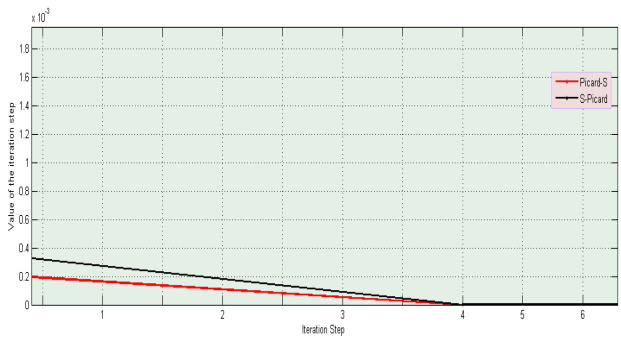

FIGURE 1. Graphical presentations of Picard-S and S-Picard iterative schemes

In the following table shows that Picard-S iteration scheme and S-Picard iteration scheme reaches the fixed point at the 4^{th} step. That is, they have same of the rate of convergence.

TABLE 1. Comparison rate of convergence between two iteration schemes

x_n	Picard-S	S-Picard
x_1	1	1
x_2	0,001368288539945	0,002262870819999
x_3	0,000000000000005	0,000000000000121
x_4	0,000000000000000	0,000000000000000

Example 2.2. Let $\widetilde{\wp}$ be a approximate operator of $\wp$ and $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (1.4) for $\wp$. Let $E=[0,0.5]$ and $\widetilde{\wp},\wp : E\to E$ be two maps defined by

$$\wp x = \frac{x-\ln(x+1)}{2}$$

$$\widetilde{\wp}x = \frac{3x^4}{2(\frac{\cos x}{11} + x)} + \frac{1}{100}.$$

It is clear that the operators $\wp$ and $\widetilde{\wp}$ satisfy the condition (1.8) for $x_0 = 0.005,$, $\delta \in [0.0002, 1)$ and $\beta_n = \alpha_n = \frac{n+1}{n+2}$. Therefore, $\wp$ and $\widetilde{\wp}$ have a unique fixed point $p = 0$ and $q = 0,010000148659176$, respectively. The following figure shows the location of operators $\wp$, $\widetilde{\wp}$ and $y = x$.

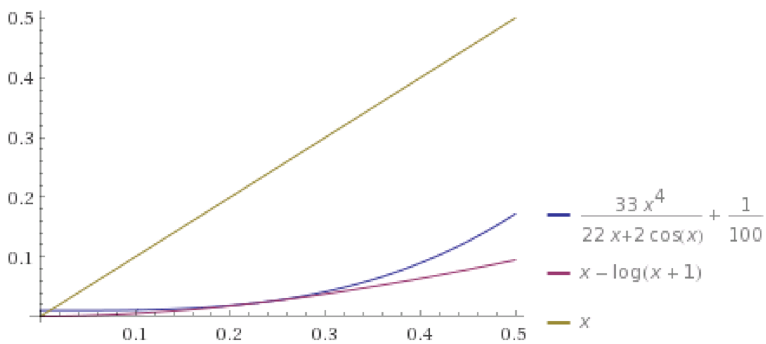

FIGURE 2. Graphical presentations of the operators $\wp$ and $\widetilde{\wp}$

By using Wolfram Mathematica 9 software package, we obtain

$$\max_{x\in[0,0.5]} \{|\wp x - \widetilde{\wp}x|\} = 0.0099995 = \varepsilon.$$

Hence

$$|\wp x - \widetilde{\wp}x| \leq 0.0099995 = \varepsilon$$

for all $x \in [0, 0.5]$. It implies that the operator $\widetilde{\wp}$ is approximate operator of $\wp$. Also, The iterative scheme (1.4) can generated by $\widetilde{\wp}x = \frac{3x^4}{2(\frac{\cos x}{11}+x)} + \frac{1}{100}$ as follows:

$$\begin{cases} x_0 \in E, \\ u_{n+1} = (1 - \frac{n+1}{n+2})\left(\frac{3w_n^4}{2(\frac{\cos w_n}{11}+w_n)} + \frac{1}{100}\right) + \left(\frac{n+1}{n+2}\right)\left(\frac{3v_n^4}{2(\frac{\cos v_n}{11}+v_n)} + \frac{1}{100}\right) \\ v_n = (1 - \frac{n+1}{n+2})\left(\frac{3u_n^4}{2(\frac{\cos u_n}{11}+u_n)} + \frac{1}{100}\right) + \left(\frac{n+1}{n+2}\right)\left(\frac{3w_n^4}{2(\frac{\cos w_n}{11}+w_n)} + \frac{1}{100}\right) \\ w_n = \frac{3u_n^4}{2(\frac{\cos u_n}{11}+u_n)} + \frac{1}{100}, (n \in \mathbb{N}). \end{cases} \tag{2.14}$$

The following table shows that the iterative scheme (2.14) converges to the fixed point $q = 0,010000148663966$ of $\widetilde{\wp}$.

TABLE 2. Comparison between $\wp$ and $\widetilde{\wp}$

x_n	S-Picard generated by $\widetilde{\wp}$	S-Picard generated by $\wp$
x_1	$0,000500000000000$	$0,000500000000000$
x_2	0,010000148659176	0,000000000000000
x_3	0,010000148663966	0,000000000000000
x_4	0,010000148663966	0,000000000000000

It is clear that $\lim_{n\to\infty} u_n = q = 0,010000148663966$.

Thus

$$\begin{aligned} |p-q| &= |0-0,010000148663966| \\ &= 0,010000148663966. \end{aligned}$$

Since $\delta \in [0.0002, 1)$, we can take $\delta = 0.0002$. Then

$$|p-q| \leq \frac{0.0099995}{1-(0.0002)} = \frac{\varepsilon}{1-\delta} = 0.100015003.$$

It implies that

$$|p-q| \leq \frac{\varepsilon}{1-\delta}.$$

3. Conclusion

In this study, a new iterative scheme was defined and this iteration scheme was compared with the Picard-S iteration scheme. It is evident that these two iterations have the same convergence rate as they are completely independent of each other. Since it is Picard-S iterative scheme is the fastest among three-step fixed point iterative schemes, S-Picard iteration scheme proved to be fastest three-step iteration method. This work has brought a new perspective to defining the iteration method.

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DEPARTMENT OF COMPUTER ENGINEERING
ARTVIN CORUH UNIVERSITY
ARTVIN, TURKEY
Email address: `dogankadri@hotmail.com`

DEPARTMENT OF MATHEMATICAL ENGINEERING
YILDIZ TECHNICAL UNIVERSITY
DAVUTPASA CAMPUS
ESENLER, 34210 ISTANBUL,TURKEY
Email address: `vkkaya@yahoo.com`