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On ideal invariant convergence of double sequences and some properties

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ABSTRACT. In this paper, we study the concepts of invariant convergence, p-strongly invariant convergence $\big([V_\sigma^2]_p\big)$, $\mathcal{I}_2$-invariant convergence $\big(\mathcal{I}_2^\sigma\big)$, $\mathcal{I}_2^*$-invariant convergence $\big(\mathcal{I}_2^{\sigma*}\big)$ of double sequences and investigate the relationships among invariant convergence, $[V_\sigma^2]_p$, $\mathcal{I}_2^\sigma$ and $\mathcal{I}_2^{\sigma*}$. Also, we introduce the concepts of $\mathcal{I}_2^\sigma$-Cauchy double sequence and $\mathcal{I}_2^{\sigma*}$-Cauchy double sequence.

1. INTRODUCTION AND BACKGROUND

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [6] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. $\mathcal{I}$-convergence of double sequences in a metric space and some properties of this convergence and similar concepts which are noted following can be seen in [2,4,7].

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout the paper we take $\mathcal{I}$ as an admissible ideal in $\mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

(i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

For any ideal there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$, given by

$$\mathcal{F}(\mathcal{I}) = \big\{M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N}\backslash A)\big\}.$$

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

$\mathcal{I}_2^0 = \big\{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\big\}$. Then $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

A double sequence $x = (x_{kj})_{k,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case, we write $P - \lim\limits_{k,j\to\infty} x_{kj} = L$ *or* $\lim\limits_{k,j\to\infty} x_{kj} = L$.

A double sequence $x = (x_{kj})$ is said to be bounded if $\sup\limits_{k,j} x_{kj} < \infty$. The set of all bounded double sequences of sets will be denoted by ℓ_∞^2.

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Let (X, ρ) be a metric space. A sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$,

$$A(\varepsilon) = \big\{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\big\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} x_{mn} = L$.

A double sequence $x = (x_{kj})$ is $\mathcal{I}_2^*$-convergent to L if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2$) such that

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} x_{kj} = L.$$

In this case, we write $\mathcal{I}_2^* - \lim\limits_{k,j\to\infty} x_{kj} = L$.

A double sequence $x = (x_{kj})$ is $\mathcal{I}_2$-Cauchy sequence if for every $\varepsilon > 0$, there exists (r,s) in $\mathbb{N} \times \mathbb{N}$ such that

$$\big\{(k,j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - x_{rs}| \geq \varepsilon\big\} \in \mathcal{I}_2.$$

A double sequence $x = (x_{kj})$ is $\mathcal{I}_2^*$-Cauchy if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ ($\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2$) such that,

$$\lim_{k,j,r,s\to\infty} |x_{kj} - x_{rs}| = 0,$$

for $(k,j), (r,s) \in M_2$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that $E_j \Delta F_j \in \mathcal{I}_2^0$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2$ (hence $F_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Several authors have studied convergence, invariant convergence and Cauchy sequences (see, [1,3,5,8–10,12–17]).

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1,1,1,...)$ and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and the space V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$.

It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \;\; \text{uniformly in } n \right\}.$$

The concept of strongly σ-convergence was defined by Mursaleen in [8]:

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0,$$

uniformly in n. It is denoted by $x_k \to L[V_\sigma]$.

By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences. In the case $\sigma(n) = n+1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

The concept of strongly σ-convergence was generalized by Savaş [14] as below:

$$[V_\sigma]_p = \left\{ x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0, \text{ uniformly in } n \right\},$$

where $0 < p < \infty$. If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.

Recently, the concepts of σ-uniform density of the set $A \subseteq \mathbb{N}$, $\mathcal{I}_\sigma$-convergence and $\mathcal{I}_\sigma^*$-convergence of sequences of real numbers were defined by Nuray et al. [12]. Also, the concept of σ-convergence of double sequences was studied by Çakan et al. [1] and the concept of σ-uniform density of $A \subseteq \mathbb{N}\times\mathbb{N}$ was defined by Tortop and Dündar [17].

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| A \cap \{\sigma(n), \sigma^2(n), \cdots, \sigma^m(n)\} \right|$$

and

$$S_m = \max_n \left| A \cap \{\sigma(n), \sigma^2(n), \cdots, \sigma^m(n)\} \right|.$$

If the following limits exist

$$\underline{V}(A) = \lim_{m\to\infty} \frac{s_m}{m}, \ \overline{V}(A) = \lim_{m\to\infty} \frac{S_m}{m}$$

then they are called a lower and upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathcal{I}_\sigma$-convergent to the number L if for every $\varepsilon > 0$ $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}_\sigma$; i.e., $V(A_\varepsilon) = 0$. In this case, we write $\mathcal{I}_\sigma - \lim_k = L$.

The set of all $\mathcal{I}_\sigma$-convergent sequences will be deneted by $\mathfrak{I}_\sigma$.

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathcal{I}_\sigma^*$-convergent to the number L if there exists a set $M = \{m_1 < m_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that $\lim_{k\to\infty} x_{m_k} = L$. In this case, we write $\mathcal{I}_\sigma^* - \lim_k = L$.

A bounded double sequences $x = (x_{kj})$ of real numbers is said to be σ-convergent to a limit L if

$$\lim_{mn} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} x_{\sigma^k(s), \sigma^j(t)} = L$$

uniformly in s, t. In this case, we write $\sigma_2 - \lim x = L$.

Let $A \subseteq \mathbb{N}\times\mathbb{N}$ and

$$s_{mn} := \min_{k,j} \left| A \cap \left\{ \big(\sigma(k), \sigma(j)\big), \big(\sigma^2(k), \sigma^2(j)\big), ..., \big(\sigma^m(k), \sigma^n(j)\big) \right\} \right|$$

and

$$S_{mn} := \max_{k,j} \left| A \cap \left\{ \big(\sigma(k), \sigma(j)\big), \big(\sigma^2(k), \sigma^2(j)\big), ..., \big(\sigma^m(k), \sigma^n(j)\big) \right\} \right|.$$

If the following limits exists

$$\underline{V_2}(A) := \lim_{m,n\to\infty} \frac{s_{mn}}{mn}, \quad \overline{V_2}(A) := \lim_{m,n\to\infty} \frac{S_{mn}}{mn}$$

then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V_2}(A) = \overline{V_2}(A)$, then $V_2(A) = \underline{V_2}(A) = \overline{V_2}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_2^\sigma$ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(A) = 0$.

Throughout the paper we let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal

2. $\mathcal{I}_2$-INVARIANT CONVERGENCE

In this section, we introduce the concepts of strongly invariant convergence $\big([V_\sigma^2]\big)$, p-strongly invariant convergence $\big([V_\sigma^2]_p\big)$, $\mathcal{I}_2$-invariant convergence $\big(\mathcal{I}_2^\sigma\big)$ of double sequences and investigate the relationships among invariant convergence, $[V_\sigma^2]_p$ and $\mathcal{I}_2^\sigma$.

Definition 2.1. A double sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_2^\sigma$-convergent to L, if for every $\varepsilon > 0$

$$A(\varepsilon) = \big\{(k,j) : |x_{kj} - L| \geq \varepsilon\big\} \in \mathcal{I}_2^\sigma$$

that is, $V_2\big(A(\varepsilon)\big) = 0$. In this case, we write $\mathcal{I}_2^\sigma - \lim x = L$ or $x_{kj} \to L(\mathcal{I}_2^\sigma)$.

The set of all $\mathcal{I}_2$-invariant convergent double sequences will be denoted by $\mathfrak{I}_2^\sigma$.

Theorem 2.1. *If $\mathcal{I}_2^\sigma - \lim x_{kj} = L_1$ and $\mathcal{I}_2^\sigma - \lim y_{kj} = L_2$, then*

(i) $\mathcal{I}_2^\sigma - \lim(x_{kj} + y_{kj}) = L_1 + L_2$

(ii) $\mathcal{I}_2^\sigma - \lim \alpha x_{kj} = \alpha L_1$ (α is a constant).

Proof. The proof is clear so we omit it. □

Theorem 2.2. *Suppose that $x = (x_{kj})$ is a bounded double sequence. If $x = (x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent to L, then $x = (x_{kj})$ is invariant convergent to L.*

Proof. Let $m, n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. We estimate

$$u(m,n,s,t) = \left| \frac{1}{mn} \sum_{k,j=1,1}^{m,n} x_{\sigma^k(s),\sigma^j(t)} - L \right|.$$

Then, we have

$$u(m,n,s,t) \leq u^1(m,n,s,t) + u^2(m,n,s,t)$$

where

$$u^1(m,n,s,t) = \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon}}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|$$

and

$$u^2(m,n,s,t) = \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L| < \varepsilon}}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|.$$

Therefore, we have

$$u^2(m,n,s,t) < \varepsilon,$$

for every $s, t = 1, 2, \ldots$. The boundedness of (x_{kj}) implies that there exists $K > 0$ such that

$$|x_{\sigma^k(s),\sigma^j(t)} - L| \leq K, \quad (k,j,s,t = 1,2,\ldots),$$

then this implies that

$$u^1(m,n,s,t) \le \frac{K}{mn}\Big|\big\{1 \le k \le m, 1 \le j \le n : |x_{\sigma^k(s),\sigma^j(t)} - L| \ge \varepsilon\big\}\Big|$$
$$\le K\frac{\max_{s,t}\Big|\big\{1 \le k \le m, 1 \le j \le n : |x_{\sigma^k(s),\sigma^j(t)} - L| \ge \varepsilon\big\}\Big|}{mn}$$
$$= K\frac{S_{mn}}{mn}.$$

Hence, (x_{kj}) is invariant convergent to L. □

The converse of Theorem 2.2 does not hold. For example, $x = (x_{kj})$ is the double sequence defined by following;

$$x_{kj} := \begin{cases} 1 \quad , & \text{if } k{+}j \text{ is even integer,} \\ 0 \quad , & \text{if } k{+}j \text{ is odd integer.} \end{cases}$$

When $\sigma(s) = s+1$ and $\sigma(t) = t+1$, this sequence is invariant convergent to $\frac{1}{2}$ but it is not $\mathcal{I}_2^\sigma$-convergent.

In [12], Nuray et al. gave some inclusion relations between $[V_\sigma]_p$-convergence and $\mathcal{I}$-invariant convergence, and showed that these are equivalent for bounded sequences. Now, we shall give analogous theorems which states inclusion relations between $[V_2^\sigma]_p$-convergence and $\mathcal{I}_2$-invariant convergence, and show that these are equivalent for bounded double sequences.

Definition 2.2. A double sequence $x = (x_{kj})$ is said to be strongly invariant convergent to L, if

$$\lim_{m,n\to\infty} \frac{1}{mn}\sum_{k,j=1,1}^{m,n} \big|x_{\sigma^k(s),\sigma^j(t)} - L\big| = 0,$$

uniformly in s,t. In this case, we write $x_{kj} \to L\big([V_\sigma^2]\big)$.

Definition 2.3. A double sequence $x = (x_{kj})$ is said to be p-strongly invariant convergent to L, if

$$\lim_{m,n\to\infty} \frac{1}{mn}\sum_{k,j=1,1}^{m,n} \big|x_{\sigma^k(s),\sigma^j(t)} - L\big|^p = 0,$$

uniformly in s,t, where $0 < p < \infty$. In this case, we write $x_{kj} \to L\big([V_\sigma^2]_p\big)$.

The set of all p-strongly invariant convergent double sequences will be denoted by $[V_\sigma^2]_p$.

Theorem 2.3. *Let* $0 < p < \infty$.

(i) *If* $x_{kj} \to L\big([V_\sigma^2]_p\big)$, *then* $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$.

(ii) *If* $(x_{kj}) \in \ell_\infty^2$ *and* $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$, *then* $x_{kj} \to L\big([V_\sigma^2]_p\big)$.

(iii) *If* $(x_{kj}) \in \ell_\infty^2$, *then* $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$ *if and only if* $x_{kj} \to L\big([V_\sigma^2]_p\big)$.

Proof. (i) : Assume that $x_{kj} \to L\big([V_\sigma^2]_p\big)$. Then, for every $\varepsilon > 0$, we can write

$$\begin{aligned}
\sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right|^p &\geq \sum_{\substack{k,j=1,1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon}}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p \\
&\geq \varepsilon^p \left|\left\{k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon\right\}\right| \\
&\geq \varepsilon^p \max_{s,t} \left|\left\{k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon\right\}\right|
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right|^p &\geq \varepsilon^p \frac{\max_{s,t} \left|\left\{k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon\right\}\right|}{mn} \\
&= \varepsilon^p \frac{S_{mn}}{mn}
\end{aligned}$$

for every $s, t = 1, 2, \ldots$. This implies

$$\lim_{m,n \to \infty} \frac{S_{mn}}{mn} = 0$$

and so (x_{kj}) is $\mathcal{I}_2^\sigma$-convergent to L.

(ii) : Suppose that $(x_{kj}) \in \ell_\infty^2$ and $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$. Let $0 < p < \infty$ and $\varepsilon > 0$. By assumption we have $V_2\big(A(\varepsilon)\big) = 0$. Since (x_{kj}) is bounded, (x_{kj}) implies that there exists $K > 0$ such that

$$|x_{\sigma^k(s),\sigma^j(t)} - L| \leq K,$$

for all k, j, s, t. Then, we have

$$\begin{aligned}
\frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right|^p &= \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon}}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p \\
&\quad + \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L| < \varepsilon}}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p \\
&\leq K \frac{\max_{s,t} \left|\left\{k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon\right\}\right|}{mn} + \varepsilon^p \\
&\leq K \frac{S_{mn}}{mn} + \varepsilon^p.
\end{aligned}$$

Hence, we obtain

$$\lim_{m,n \to \infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right|^p = 0,$$

uniformly in s, t.

(iii) : This is immediate consequence of (i) and (ii). $\square$

Now, we introduce $\mathcal{I}_2^*$-invariant convergence concept, $\mathcal{I}_2$-invariant Cauchy double sequence and $\mathcal{I}_2^*$-invariant Cauchy double sequence concepts and give the relationships among these concepts and relationships with $\mathcal{I}_2$-invariant convergence concept.

Definition 2.4. A double sequence (x_{kj}) is $\mathcal{I}_2^*$-invariant convergent or $\mathcal{I}_2^{\sigma*}$-convergent to L if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ ($\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^\sigma$) such that

$$\lim_{\substack{k,j\to\infty\\(k,j)\in M_2}} x_{kj} = L.$$

In this case, we write $\mathcal{I}_2^{\sigma*} - \lim_{k,j\to\infty} x_{kj} = L$ or $x_{kj} \to L(\mathcal{I}_2^{\sigma*})$.

Theorem 2.4. *If a double sequence (x_{kj}) is $\mathcal{I}_2^{\sigma*}$-convergent to L, then this sequence is $\mathcal{I}_2^\sigma$-convergent to L.*

Proof. Since $\mathcal{I}_2^{\sigma*} - \lim_{k,j\to\infty} x_{kj} = L$, there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ ($\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^\sigma$) such that

$$\lim_{\substack{k,j\to\infty\\(k,j)\in M_2}} x_{kj} = L.$$

Let $\varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that

$$|x_{kj} - L| < \varepsilon,$$

for all $(k,j) \in M_2$ and $k, j \geq k_0$. Hence, for every $\varepsilon > 0$, we have

$$\begin{aligned} T(\varepsilon) &= \big\{(k,j)\in\mathbb{N}\times\mathbb{N} : |x_{kj}-L| \geq \varepsilon\big\} \\ &\subset H \cup \Big(M_2 \cap \big((\{1,2,...,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,...,(k_0-1)\})\big)\Big). \end{aligned}$$

Since $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal,

$$H \cup \Big(M_2 \cap \big((\{1,2,...,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,...,(k_0-1)\})\big)\Big) \in \mathcal{I}_2^\sigma,$$

so we have $T(\varepsilon) \in \mathcal{I}_2^\sigma$ that is $V_2\big(T(\varepsilon)\big) = 0$. Hence,

$$\mathcal{I}_2^\sigma - \lim_{k,j\to\infty} x_{kj} = L.$$

□

Theorem 2.5. *Let $\mathcal{I}_2^\sigma$ has property $(AP2)$. If (x_{kj}) is $\mathcal{I}_2^\sigma$-convergent to L, then (x_{kj}) is $\mathcal{I}_2^{\sigma*}$-convergent to L.*

Proof. Suppose that $\mathcal{I}_2^\sigma$ satisfies property $(AP2)$. Let (x_{kj}) is $\mathcal{I}_2^\sigma$-convergent to L. Then,

$$T(\varepsilon) = \big\{(k,j)\in\mathbb{N}\times\mathbb{N} : |x_{kj}-L| \geq \varepsilon\big\} \in \mathcal{I}_2^\sigma \tag{2.1}$$

for every $\varepsilon > 0$. Put

$$T_1 = \big\{(k,j)\in\mathbb{N}\times\mathbb{N} : |x_{kj}-L| \geq 1\big\}$$

and

$$T_v = \left\{(k,j)\in\mathbb{N}\times\mathbb{N} : \frac{1}{v} \leq |x_{kj}-L| < \frac{1}{v-1}\right\}$$

for $v \geq 2$ and $v \in \mathbb{N}$. Obviously $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}_2^\sigma$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exits a sequence of sets $\{E_v\}_{v\in\mathbb{N}}$ such that $T_i \Delta E_i$ is included in finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each i and $E = \bigcup_{i=1}^\infty E_i \in \mathcal{I}_2^\sigma$.

We shall prove that for $M_2 = \mathbb{N}\times\mathbb{N}\backslash E$ we have

$$\lim_{\substack{k,j\to\infty\\(k,j)\in M_2}} x_{kj} = L.$$

Let $\eta > 0$ be given. Choose $v \in \mathbb{N}$ such that $\frac{1}{v} < \eta$. Then,

$$\{(k,j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - L| \geq \eta\} \subset \bigcup_{i=1}^{v} T_i.$$

Since $T_i \Delta E_i$, $i = 1, 2, ...$ are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{v} T_i\right) \cap \{(k,j) : k \geq n_0 \wedge j \geq n_0\} = \left(\bigcup_{i=1}^{v} E_i\right) \cap \{(k,j) : k \geq n_0 \wedge j \geq n_0\}. \qquad (2.2)$$

If $k, j > n_0$ and $(k,j) \notin E$, then

$$(k,j) \notin \bigcup_{i=1}^{v} E_i \text{ and } (k,j) \notin \bigcup_{i=1}^{v} T_i.$$

This implies that

$$|x_{kj} - L| < \frac{1}{v} < \eta.$$

Hence, we have

$$\lim_{\substack{k,j \to \infty \\ (k,j) \in M_2}} x_{kj} = L.$$

□

Finally, we define the concepts of $\mathcal{I}_2^{\sigma}$-Cauchy and $\mathcal{I}_2^{\sigma *}$-Cauchy double sequences.

Definition 2.5. A double sequence (x_{kj}) is said to be $\mathcal{I}_2$-invariant Cauchy or $\mathcal{I}_2^{\sigma}$-Cauchy sequence, if for every $\varepsilon > 0$, there exist numbers $r = r(\varepsilon), s = s(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(k,j) : |x_{kj} - x_{rs}| \geq \varepsilon\} \in \mathcal{I}_2^{\sigma},$$

that is, $V_2(A(\varepsilon)) = 0$.

Definition 2.6. A double sequence (x_{kj}) is $\mathcal{I}_2^{*}$-invariant Cauchy or $\mathcal{I}_2^{\sigma *}$-Cauchy sequence if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma})$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2^{\sigma}$) such that for every $(k,j), (r,s) \in M_2$

$$\lim_{k,j,r,s \to \infty} |x_{kj} - x_{rs}| = 0.$$

We give following theorems which show relationships between $\mathcal{I}_2^{\sigma}$-convergence, $\mathcal{I}_2^{\sigma}$-Cauchy double sequence and $\mathcal{I}_2^{\sigma *}$-Cauchy double sequence. The proof of them are similar to the proof of Theorems in [3, 4, 11], so we omit them.

Theorem 2.6. *If a double sequence (x_{kj}) is $\mathcal{I}_2^{\sigma}$-convergent, then (x_{kj}) is an $\mathcal{I}_2^{\sigma}$-Cauchy double sequence.*

Theorem 2.7. *If a double sequence (x_{kj}) is $\mathcal{I}_2^{\sigma *}$-Cauchy double sequence, then (x_{kj}) is $\mathcal{I}_2^{\sigma}$-Cauchy double sequence.*

Theorem 2.8. *Let $\mathcal{I}_2^{\sigma}$ has property $(AP2)$. Then, the concepts $\mathcal{I}_2^{\sigma}$-Cauchy double sequence and $\mathcal{I}_2^{\sigma *}$-Cauchy double sequence coincides.*

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