On ideal invariant convergence of double sequences and some properties

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ABSTRACT. In this paper, we study the concepts of invariant convergence, $p$-strongly invariant convergence ([$V_2^p$]), $I_2$-invariant convergence ($I_2^*$), $I_2$-invariant convergence ($I_2^*$) of double sequences and investigate the relationships among invariant convergence, $[V_2^p]$, $I_2^*$ and $I_2^*$. Also, we introduce the concepts of $I_2^*$-Cauchy double sequence and $I_2^*$-Cauchy double sequence.

1. INTRODUCTION AND BACKGROUND

The idea of $I$-convergence was introduced by Kostyrko et al. [6] as a generalization of statistical convergence which is based on the structure of the ideal $I$ of subset of the set of natural numbers $\mathbb{N}$. $I$-convergence of double sequences in a metric space and some properties of this convergence and similar concepts which are noted following can be seen in [2, 4, 7].

A family of sets $I \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in I$, (ii) For each $A, B \in I$ we have $A \cup B \in I$, (iii) For each $A \in I$ and each $B \subseteq A$ we have $B \in I$.

An ideal is called non-trivial if $\mathbb{N} \notin I$ and non-trivial ideal is called admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Throughout the paper we take $I$ as an admissible ideal in $\mathbb{N}$.

A family of sets $F \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

(i) $\emptyset \notin F$, (ii) For each $A, B \in F$ we have $A \cap B \in F$, (iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$.

For any ideal there is a filter $F(I)$ corresponding with $I$, given by

$$F(I) = \{M \subseteq N : (\exists A \in I)(M = N\setminus A)\}.$$  

A nontrivial ideal $I_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $I_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $I_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

$${I_2}^0 = \{ A \subseteq N \times N : (\exists m(A) \in N)(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}.$$ Then $I_2^0$ is a strongly admissible ideal and clearly an ideal $I_2$ is strongly admissible if and only if $I_2^0 \subset I_2$.

A double sequence $x = (x_{kj})_{k,j \in \mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim’s sense if for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \epsilon$, whenever $k, j > N_\epsilon$. In this case, we write $P - \lim_{k,j \to \infty} x_{kj} = L$ or $\lim_{k,j \to \infty} x_{kj} = L$.

A double sequence $x = (x_{kj})$ is said to be bounded if $\sup_{k,j} x_{kj} < \infty$. The set of all bounded double sequences of sets will be denoted by $\ell^2_\infty$.  

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Let \((X, \rho)\) be a metric space. A sequence \(x = (x_{mn})\) in \(X\) is said to be \(\mathcal{I}_2\)-convergent to \(L \in X\), if for any \(\varepsilon > 0\),
\[
A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \subseteq \mathcal{I}_2.
\]
In this case, we write \(\mathcal{I}_2 - \lim_{m,n \to \infty} x_{mn} = L\).

A double sequence \(x = (x_{kj})\) is \(\mathcal{I}_2^*\)-convergent to \(L\) if there exists a set \(M_2 \in \mathcal{F}(\mathcal{I}_2)\) (i.e., \(\mathbb{N} \times \mathbb{N} \setminus M_2 = H \in \mathcal{I}_2\)) such that
\[
\lim_{k,j \to \infty} x_{kj} = L.
\]
In this case, we write \(\mathcal{I}_2 - \lim_{k,j \to \infty} x_{kj} = L\).

A double sequence \(x = (x_{kj})\) is \(\mathcal{I}_2\)-Cauchy sequence if for every \(\varepsilon > 0\), there exists \((r, s)\) in \(\mathbb{N} \times \mathbb{N}\) such that
\[
\{(k, j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - x_{rs}| \geq \varepsilon\} \subseteq \mathcal{I}_2.
\]

A double sequence \(x = (x_{kj})\) is \(\mathcal{I}_2^*\)-Cauchy if there exists a set \(M_2 \in \mathcal{F}(\mathcal{I}_2)\) \((\mathbb{N} \times \mathbb{N} \setminus M_2 = H \in \mathcal{I}_2)\) such that,
\[
\lim_{k,j,r,s \to \infty} |x_{kj} - x_{rs}| = 0,
\]
for \((k, j), (r, s) \in M_2\).

An admissible ideal \(\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}\) satisfies the property (AP2) if for every countable family of mutually disjoint sets \(\{E_1, E_2, \ldots\}\) belonging to \(\mathcal{I}_2\), there exists a countable family of sets \(\{F_1, F_2, \ldots\}\) such that \(E_j \Delta F_j \in \mathcal{I}_2^0\), i.e., \(E_j \Delta F_j\) is included in the finite union of rows and columns in \(\mathbb{N} \times \mathbb{N}\) for each \(j \in \mathbb{N}\) and \(F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2\) (hence \(F_j \in \mathcal{I}_2\) for each \(j \in \mathbb{N}\)).

Several authors have studied convergence, invariant convergence and Cauchy sequences (see, [1, 3, 5, 8, 10, 12–17]).

Let \(\sigma\) be a mapping of the positive integers into themselves. A continuous linear functional \(\phi\) on \(\ell_\infty\), the space of real bounded sequences, is said to be an invariant mean or a \(\sigma\)-mean if it satisfies following conditions:

1. \(\phi(x) \geq 0\), when the sequence \(x = (x_n)\) has \(x_n \geq 0\) for all \(n\),
2. \(\phi(e) = 1\), where \(e = (1, 1, 1, \ldots)\) and
3. \(\phi(x_{\sigma(n)}) = \phi(x_n)\) for all \(x \in \ell_\infty\).

The mappings \(\sigma\) are assumed to be one-to-one and such that \(\sigma^m(n) \neq n\) for all positive integers \(n\) and \(m\), where \(\sigma^m(n)\) denotes the \(m\) th iterate of the mapping \(\sigma\) at \(n\). Thus, \(\phi\) extends the limit functional on \(c\), the space of convergent sequences, in the sense that \(\phi(x) = \lim x\) for all \(x \in c\).

In the case \(\sigma\) is translation mappings \(\sigma(n) = n + 1\), the \(\sigma\)-mean is often called a Banach limit and the space \(V_\sigma\), the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences \(\hat{c}\).

It can be shown that
\[
V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } n \right\}.
\]

The concept of strongly \(\sigma\)-convergence was defined by Mursaleen in [8]:

A bounded sequence \(x = (x_k)\) is said to be strongly \(\sigma\)-convergent to \(L\) if
\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0,
\]
uniformly in \(n\). It is denoted by \(x_k \to L[V_\sigma]\).
By $[V_σ]$, we denote the set of all strongly $σ$-convergent sequences. In the case $σ(n) = n + 1$, the space $[V_σ]$ is the set of strongly almost convergent sequences $[c]$.

The concept of strongly $σ$-convergence was generalized by Savas $[14]$ as below:

$$[V_σ]_p = \left\{ x = (x_k) : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^m |x_{σ^k(n)} - L|^p = 0, \text{ uniformly in } n \right\},$$

where $0 < p < \infty$. If $p = 1$, then $[V_σ]_p = [V_σ]$. It is known that $[V_σ]_p \subset \ell_∞$.

Recently, the concepts of $σ$-uniform density of the set $A \subseteq \mathbb{N}$, $I_σ$-convergence and $I_σ^*$-convergence of sequences of real numbers were defined by Nuray et al. $[12]$. Also, the concept of $σ$-convergence of double sequences was studied by Çakan et al. $[1]$ and the concept of $σ$-uniform density of $A \subseteq \mathbb{N} \times \mathbb{N}$ was defined by Tortop and Dündar $[17]$.

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n |A \cap \{σ(n), σ^2(n), \ldots, σ^m(n)\}|$$

and

$$S_m = \max_n |A \cap \{σ(n), σ^2(n), \ldots, σ^m(n)\}|.$$

If the following limits exist

$$V(A) = \lim_{m \to \infty} \frac{s_m}{m}, \quad \overline{V}(A) = \lim_{m \to \infty} \frac{S_m}{m},$$

then they are called a lower and upper $σ$-uniform density of the set $A$, respectively. If $V(A) = \overline{V}(A)$, then $V(A) = \overline{V}(A) = \overline{V}(A)$ is called $σ$-uniform density of $A$.

Denote by $I_σ$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Let $I_σ \subset 2^\mathbb{N}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $I_σ$-convergent to the number $L$ if for every $ε > 0$ $A_ε = \{k : |x_k - L| \geq ε\} \in I_σ$; i.e., $V(A_ε) = 0$. In this case, we write $I_σ - \lim_k x = L$.

The set of all $I_σ$-convergent sequences will be denoted by $I_σ$.

Let $I_σ^* \subset 2^\mathbb{N}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $I_σ^*$-convergent to the number $L$ if there exists a set $M = \{m_1 < m_2 < \ldots\} \in \mathcal{F}(I_σ)$ such that $\lim_{k \to \infty} x_{mk} = L$. In this case, we write $I_σ^* - \lim_k x = L$.

A bounded double sequences $x = (x_{kj})$ of real numbers is said to be $σ$-convergent to a limit $L$ if

$$\lim_{mn} \frac{1}{mn} \sum_{k=0}^m \sum_{j=0}^n x_{σ^k(s), σ^j(t)} = L$$

uniformly in $s, t$. In this case, we write $σ_2 - \lim x = L$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mn} := \min_{k,j} |A \cap \{(σ(k), σ(j)), (σ^2(k), σ^2(j)), \ldots, (σ^m(k), σ^m(j))\}|$$

and

$$S_{mn} := \max_{k,j} |A \cap \{(σ(k), σ(j)), (σ^2(k), σ^2(j)), \ldots, (σ^m(k), σ^m(j))\}|.$$

If the following limits exist

$$V_2(A) := \lim_{m,n \to \infty} \frac{s_{mn}}{mn}, \quad \overline{V}_2(A) := \lim_{m,n \to \infty} \frac{S_{mn}}{mn},$$

then they are called a lower and an upper $σ$-uniform density of the set $A$, respectively. If $V_2(A) = \overline{V}_2(A)$, then $V_2(A) = \overline{V}_2(A) = \overline{V}_2(A)$ is called the $σ$-uniform density of $A$. 

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Denote by $\mathcal{I}_2^\sigma$ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(A) = 0$.
Throughout the paper we let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal.

2. $\mathcal{I}_2$-INVARIANT CONVERGENCE

In this section, we introduce the concepts of strongly invariant convergence ($[V_2^\sigma]$), $p$-strongly invariant convergence ($[V_2^\sigma]_p$), $\mathcal{I}_2$-invariant convergence ($\mathcal{I}_2^\sigma$) of double sequences and investigate the relationships among invariant convergence, $[V_2^\sigma]$ and $\mathcal{I}_2^\sigma$.

**Definition 2.1.** A double sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2^\sigma$-invariant convergent or $\mathcal{I}_2^\sigma$-convergent to $L$, if for every $\varepsilon > 0$

$$A(\varepsilon) = \{(k, j) : |x_{kj} - L| \geq \varepsilon\} \in \mathcal{I}_2^\sigma$$

that is, $V_2(A(\varepsilon)) = 0$. In this case, we write $\mathcal{I}_2^\sigma - \lim x = L$ or $x_{kj} \to L(\mathcal{I}_2^\sigma)$.

The set of all $\mathcal{I}_2$-invariant convergent double sequences will be denoted by $\mathcal{I}_2^\sigma$.

**Theorem 2.1.** If $\mathcal{I}_2^\sigma - \lim x_{kj} = L_1$ and $\mathcal{I}_2^\sigma - \lim y_{kj} = L_2$, then

(i) $\mathcal{I}_2^\sigma - \lim (x_{kj} + y_{kj}) = L_1 + L_2$

(ii) $\mathcal{I}_2^\sigma - \lim \alpha x_{kj} = \alpha L_1$ ($\alpha$ is a constant).

**Proof.** The proof is clear so we omit it. $\square$

**Theorem 2.2.** Suppose that $x = (x_{kj})$ is a bounded double sequence. If $x = (x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent to $L$, then $x = (x_{kj})$ is invariant convergent to $L$.

**Proof.** Let $m, n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. We estimate

$$u(m, n, s, t) = \left| \frac{1}{mn} \sum_{k,j=1,1}^{m,n} x_{\sigma^k(s),\sigma^j(t)} - L \right|.$$ 

Then, we have

$$u(m, n, s, t) \leq u^1(m, n, s, t) + u^2(m, n, s, t)$$

where

$$u^1(m, n, s, t) = \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|$$

for $|x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon$

and

$$u^2(m, n, s, t) = \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|$$

for $|x_{\sigma^k(s),\sigma^j(t)} - L| < \varepsilon$

Therefore, we have

$$u^2(m, n, s, t) < \varepsilon,$$

for every $s, t = 1, 2, \ldots$. The boundedness of $(x_{kj})$ implies that there exists $K > 0$ such that

$$|x_{\sigma^k(s),\sigma^j(t)} - L| \leq K, \quad (k, j, s, t = 1, 2, \ldots).$$
then this implies that
\[
\begin{aligned}
u^1(m, n, s, t) &\leq \frac{K}{mn} \left| \left\{ 1 \leq k \leq m, 1 \leq j \leq n : |x_{\sigma^k(s), \sigma^j(t)} - L| \geq \varepsilon \right\} \right| \\
&\leq \frac{1}{mn} \max_{s, t} \left\{ 1 \leq k \leq m, 1 \leq j \leq n : |x_{\sigma^k(s), \sigma^j(t)} - L| \geq \varepsilon \right\} \\
&= K \frac{S_{mn}}{mn}
\end{aligned}
\]

Hence, \((x_{kj})\) is invariant convergent to \(L\). 

The converse of Theorem 2.2 does not hold. For example, \(x = (x_{kj})\) is the double sequence defined by following;

\[
x_{kj} := \begin{cases} 
1 &, \text{if } k+j \text{ is even integer,} \\
0 &, \text{if } k+j \text{ is odd integer.}
\end{cases}
\]

When \(\sigma(s) = s + 1\) and \(\sigma(t) = t + 1\), this sequence is invariant convergent to \(\frac{1}{2}\) but it is not \(\mathcal{I}_2^\sigma\)-convergent.

In [12], Nuray et al. gave some inclusion relations between \([V_\sigma]_p\)-convergence and \(\mathcal{I}\)-invariant convergence, and showed that these are equivalent for bounded sequences. Now, we shall give analogous theorems which states inclusion relations between \([V_\sigma^2]_p\)-convergence and \(\mathcal{I}_2\)-invariant convergence, and show that these are equivalent for bounded double sequences.

**Definition 2.2.** A double sequence \(x = (x_{kj})\) is said to be strongly invariant convergent to \(L\), if

\[
\lim_{m, n \to \infty} \frac{1}{mn} \sum_{k, j=1}^{m, n} |x_{\sigma^k(s), \sigma^j(t)} - L| = 0,
\]

uniformly in \(s, t\). In this case, we write \(x_{kj} \to L([V_\sigma^2])\).

**Definition 2.3.** A double sequence \(x = (x_{kj})\) is said to be \(p\)-strongly invariant convergent to \(L\), if

\[
\lim_{m, n \to \infty} \frac{1}{mn} \sum_{k, j=1}^{m, n} |x_{\sigma^k(s), \sigma^j(t)} - L|^p = 0,
\]

uniformly in \(s, t\), where \(0 < p < \infty\). In this case, we write \(x_{kj} \to L([V_\sigma^2]_p)\).

The set of all \(p\)-strongly invariant convergent double sequences will be denoted by \([V_\sigma^2]_p\).

**Theorem 2.3.** Let \(0 < p < \infty\).

(i) If \(x_{kj} \to L([V_\sigma^2]_p)\), then \(x_{kj} \to L(\mathcal{I}_2^\sigma)\).

(ii) If \((x_{kj}) \in \ell_\infty^2\) and \(x_{kj} \to L(\mathcal{I}_2^\sigma)\), then \(x_{kj} \to L([V_\sigma^2]_p)\).

(iii) If \((x_{kj}) \in \ell_\infty^2\), then \(x_{kj} \to L(\mathcal{I}_2^\sigma)\) if and only if \(x_{kj} \to L([V_\sigma^2]_p)\).
Proof. (i) : Assume that $x_{kj} \to L([V^2_\sigma])$. Then, for every $\varepsilon > 0$, we can write

$$\sum_{k,j=1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p \geq \sum_{k,j=1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p$$

$$\geq \varepsilon^p \{ k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon \}$$

$$\geq \varepsilon^p \max_{s,t} \{ k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon \}$$

and

$$\frac{1}{mn} \sum_{k,j=1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p \geq \varepsilon^p \frac{\max_{s,t} \{ k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon \} \frac{1}{mn}}{\{ k \leq m, j \leq n : \}}$$

$$\leq \frac{\varepsilon^p S_{mn}}{mn}$$

for every $s, t = 1, 2, \ldots$. This implies

$$\lim_{m,n \to \infty} \frac{S_{mn}}{mn} = 0$$

and so $(x_{kj})$ is $\mathcal{T}^2_\sigma$-convergent to $L$.

(ii) : Suppose that $(x_{kj}) \in \ell^p_\infty$ and $x_{kj} \to L(\mathcal{T}^2_\sigma)$. Let $0 < p < \infty$ and $\varepsilon > 0$. By assumption we have $V_2(A(\varepsilon)) = 0$. Since $(x_{kj})$ is bounded, $(x_{kj})$ implies that there exists $K > 0$ such that

$$|x_{\sigma^k(s),\sigma^j(t)} - L| \leq K,$$

for all $k, j, s, t$. Then, we have

$$\frac{1}{mn} \sum_{k,j=1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p$$

$$= \frac{1}{mn} \sum_{k,j=1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p$$

$$+ \frac{1}{mn} \sum_{k,j=1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p$$

$$\leq K \frac{\max_{s,t} \{ k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon \} \frac{1}{mn}}{\{ k \leq m, j \leq n : \}} + \varepsilon^p$$

$$\leq K \frac{S_{mn}}{mn} + \varepsilon^p.$$

Hence, we obtain

$$\lim_{m,n \to \infty} \frac{1}{mn} \sum_{k,j=1}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p = 0,$$

uniformly in $s, t$.

(iii) : This is immediate consequence of (i) and (ii).
Now, we introduce $I_2^\sigma$-invariant convergence concept, $I_2$-invariant Cauchy double sequence and $I_2^\sigma$-invariant Cauchy double sequence concepts and give the relationships among these concepts and relationships with $I_2$-invariant convergence concept.

**Definition 2.4.** A double sequence $(x_{k,j})$ is $I_2^\sigma$-invariant convergent or $I_2^\sigma^*$-convergent to $L$ if and only if there exists a set $M_2 \in \mathcal{F}(I_2^\sigma)\ (\mathbb{N} \times \mathbb{N}\ \backslash M_2 = H \in I_2^\sigma)$ such that

$$\lim_{k,j \to \infty} x_{k,j} = L.$$

In this case, we write $I_2^\sigma^* - \lim_{k,j \to \infty} x_{k,j} = L$ or $x_{k,j} \to L(I_2^\sigma^*)$.

**Theorem 2.4.** If a double sequence $(x_{k,j})$ is $I_2^\sigma^*$-convergent to $L$, then this sequence is $I_2^\sigma$-convergent to $L$.

**Proof.** Since $I_2^\sigma^* - \lim_{k,j \to \infty} x_{k,j} = L$, there exists a set $M_2 \in \mathcal{F}(I_2^\sigma)\ (\mathbb{N} \times \mathbb{N}\ \backslash M_2 = H \in I_2^\sigma)$ such that

$$\lim_{k,j \to \infty} x_{k,j} = L.$$

Let $\varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that

$$|x_{k,j} - L| < \varepsilon,$$

for all $(k, j) \in M_2$ and $k, j \geq k_0$. Hence, for every $\varepsilon > 0$, we have

$$T(\varepsilon) = \{(k, j) \in \mathbb{N} \times \mathbb{N} : |x_{k,j} - L| \geq \varepsilon\} \subset H \cup \left(M_2 \cap (\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\{1, 2, ..., (k_0 - 1)\})\right).$$

Since $I_2^\sigma \subset 2^{\mathbb{N} \times \mathbb{N}}$ is a strongly admissible ideal,

$$H \cup \left(M_2 \cap (\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\{1, 2, ..., (k_0 - 1)\})\right) \in I_2^\sigma,$$

so we have $T(\varepsilon) \in I_2^\sigma$ that is $V_2(T(\varepsilon)) = 0$. Hence,

$$I_2^\sigma^* - \lim_{k,j \to \infty} x_{k,j} = L.$$

$\square$

**Theorem 2.5.** Let $I_2^\sigma$ has property $(AP2)$. If $(x_{k,j})$ is $I_2^\sigma$-convergent to $L$, then $(x_{k,j})$ is $I_2^\sigma^*$-convergent to $L$.

**Proof.** Suppose that $I_2^\sigma$ satisfies property $(AP2)$. Let $(x_{k,j})$ is $I_2^\sigma$-convergent to $L$. Then,

$$T(\varepsilon) = \{(k, j) \in \mathbb{N} \times \mathbb{N} : |x_{k,j} - L| \geq \varepsilon\} \in I_2^\sigma$$

for every $\varepsilon > 0$. Put

$$T_1 = \{(k, j) \in \mathbb{N} \times \mathbb{N} : |x_{k,j} - L| \geq 1\}$$

and

$$T_v = \left\{(k, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{v} \leq |x_{k,j} - L| < \frac{1}{v - 1}\right\}$$

for $v \geq 2$ and $v \in \mathbb{N}$. Obviously $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in I_2^\sigma$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exits a sequence of sets $\{E_v\}_{v \in \mathbb{N}}$ such that $T_i \Delta E_i$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $i \in \mathbb{N}$ and $E = \bigcup_{i=1}^{\infty} E_i \in I_2^\sigma$.

We shall prove that for $M_2 = \mathbb{N} \times \mathbb{N}\ \backslash E$ we have

$$\lim_{k,j \to \infty} x_{k,j} = L.$$
Let $\eta > 0$ be given. Choose $v \in \mathbb{N}$ such that $\frac{1}{v} < \eta$. Then,

$$\{(k, j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - L| \geq \eta\} \subset \bigcup_{i=1}^{v} T_i.$$ 

Since $T_i \Delta E_i, i = 1, 2, ...$ are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$$\left( \bigcup_{i=1}^{v} T_i \right) \cap \{(k, j) : k \geq n_0 \land j \geq n_0\} = \left( \bigcup_{i=1}^{v} E_i \right) \cap \{(k, j) : k \geq n_0 \land j \geq n_0\}. \tag{2.2}$$

If $k, j > n_0$ and $(k, j) \notin E$, then

$$(k, j) \notin \bigcup_{i=1}^{v} E_i \land (k, j) \notin \bigcup_{i=1}^{v} T_i.$$ 

This implies that

$$|x_{kj} - L| < \frac{1}{v} < \eta.$$ 

Hence, we have

$$\lim_{k,j \to \infty, (k,j) \in M_2} x_{kj} = L.$$ 

Finally, we define the concepts of $I_2^g$-Cauchy and $I_2^{g*}$-Cauchy double sequences.

**Definition 2.5.** A double sequence $(x_{kj})$ is said to be $I_2^g$-invariant Cauchy or $I_2^{g*}$-Cauchy sequence, if for every $\varepsilon > 0$, there exist numbers $r = r(\varepsilon), s = s(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(k, j) : |x_{kj} - x_{rs}| \geq \varepsilon\} \in I_2^g,$$

that is, $V_2(A(\varepsilon)) = 0$.

**Definition 2.6.** A double sequence $(x_{kj})$ is $I_2^g$-invariant Cauchy or $I_2^{g*}$-Cauchy sequence if there exists a set $M_2 \in \mathcal{F}(I_2^g)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M_2 = H \in I_2^g$) such that for every $(k, j), (r, s) \in M_2$

$$\lim_{k,j,r,s \to \infty} |x_{kj} - x_{rs}| = 0.$$ 

We give following theorems which show relationships between $I_2^g$-convergence, $I_2^g$-Cauchy double sequence and $I_2^{g*}$-Cauchy double sequence. The proof of them are similar to the proof of Theorems in [3, 4, 11], so we omit them.

**Theorem 2.6.** If a double sequence $(x_{kj})$ is $I_2^g$-convergent, then $(x_{kj})$ is an $I_2^g$-Cauchy double sequence.

**Theorem 2.7.** If a double sequence $(x_{kj})$ is $I_2^{g*}$-Cauchy double sequence, then $(x_{kj})$ is $I_2^g$-Cauchy double sequence.

**Theorem 2.8.** Let $I_2^g$ has property $(AP2)$. Then, the concepts $I_2^g$-Cauchy double sequence and $I_2^{g*}$-Cauchy double sequence coincides.

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