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Some remarks on the notations and terminology in the ordered set theory

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ABSTRACT. By various examples we illustrate that the ordered set theory is far away from having unitary notations and terminology. Some suggestions for unifiying the terminology are also presented.

1. INTRODUCTION

There are several surprising facts about ordered set theory. Although order itself is one of the mathematical phenomena most easily recognizable in the real world by any human being, as an independent mathematics subject the ordered set theory is a pretty young one, if one judges by the dedicated textbooks or journals, for example.

Another puzzling fact is that in what concerns notations and terminology, ordered set theory is a space of relative disorder. One can easily meet the same concept under several names, as well as different concepts bearing the same name. Therefore each author has to make it clear from the beginning of the book or paper what language there will be spoken in the sequel, by clearly defining and denoting his working space. Nevertheless, this is the case over almost the whole territory of mathematics, as mathematicians very well know.

The aim of this paper is to discuss this and to illustrate it by some examples, mainly in direct connection to fixed point theory, as well as to present some suggestions for unifying the terminology in ordered set theory.

2. EXAMPLES

There are several examples of concepts that bear different names in different sources, as well as different concepts hiding behind the same name. We do not discuss here the reasons why an author chooses one or the other. We shall simply list some of the basic examples, found in the references mentioned at the end of this paper. And there are many more in literature (see for example [1], [3], [5], [6], [11], [12], [16], [21], [23], [25], [26], [29], [30], [31]).

For a clear presentation, we shall also briefly (and therefore not always very rigorously) define the referred concepts. The first name is the one we personally prefer and propose for a common use, while those presented in parentheses are other variants in literature.

Example 2.1. If S is a set and $R \in S \times S$, then R is called a **binary relation (relation)** on S, and S is called the **underlying set (carrier, ground set, base)** of R.

Example 2.2. A relation that is both reflexive and transitive is called **pre-order (quasi-order)**.

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Example 2.3. A relation that is reflexive, antisymmetric and transitive is called **order (partial order)**.

Example 2.4. A set endowed with an order relation is called **ordered set (partially ordered set, poset)**.

Example 2.5. An ordered set such that each two elements can be compared is called **totally ordered set (linearly ordered set, chain, ordered set, simply ordered set, nested set).**

Example 2.6. An operator f defined on an ordered set $(X, \leq)$ such that for every two elements $x, y \in X, x \leq y$ one has $f(x) \leq f(y)$ is called **increasing (order-preserving, isotone, non-decreasing, monotone, ascending, morphism in order category)**.

Example 2.7. An operator f defined on an ordered set $(X, \leq)$ such that for every $x \in X$ one has $x \leq f(x)$ is called **progressive (expansive, expanding, increasing, inflationary)**.

Example 2.8. Given an ordered set $X \neq \emptyset$ and a subset of it, say $Y \subset X$, a lower bound a of Y such that $a \geq x$, for all lower bounds of Y, is called **infimum (greatest lower bound, meet)** of Y.

Example 2.9. If, in Example 2.8, $a \in Y$, then a is called **minimum (least element, first element, smallest element)** of Y.

Dually, we find the next two examples:

Example 2.10. Given an ordered set $X \neq \emptyset$ and a subset of it, say $Y \subset X$, an upper bound a of Y such that $a \leq x$, for all upper bounds of Y, is called **supremum (least upper bound, join)** of Y.

Example 2.11. If, in Example 2.10, $a \in Y$, then a is called **maximum (greatest element, last element, largest element)** of Y.

Example 2.12. An ordered set such that each subset of it has infimum and supremum is called **complete lattice (completely ordered set)**.

If one goes further and pays attention also to the notations, the situation gets even more confusing. A few examples:

Example 2.13. For the infimum of the subset Y of an ordered set, one may use $\inf Y$ or $\bigwedge Y$. Dually for the supremum one may use $\sup Y$ or $\bigvee Y$.

Example 2.14. If $(X, \leq)$ is an ordered set and $x \in X$, then the set $\{y \in X | x \leq y\}$ can be found under various names and notations:

- **terminal segment at** x: $[x, \rightarrow)_{\leq}$, $[x, \rightarrow)$;
- **terminal tail at** x: $T(x)$;
- **right section at** x: $S_{+}(x)$;
- **closed final segment or principal filter of** x **in** X: $(X \geq x)$, $X(x, \leq)$, $[x)$;
- **filter or up-set of** x: $\uparrow x$, $\uparrow_X x$.

The list of examples could continue. As one can see, even choosing the key words of a paper in ordered set theory might be a problem.

Remark 2.1. For the terminology of ordered set theory in the Romanian language see [8], [36], [24], [27], [32].

3. Hypostases of an order relation

The order relation appears in many hypostases. For example, we present the following (see [3], [5], [10], [13], [16], [19], [20], [21], [26], [30]).

Example 3.15. An ordered relation on a set X as a subset $R \subset X \times X$, which is reflexive, antisymmetric and transitive. In this case

$$x \leq_R y \Leftrightarrow (x, y) \in R.$$

Example 3.16. An order relation on a set X as a multivalued operator $S : X \to \mathcal{P}(X)$ which satisfies the following conditions:

(a) $x \in S(x), \forall x \in X$;
(b) $y \in S(x), z \in S(y) \Rightarrow z \in S(x)$;
(c) $y \in S(x), x \in S(y) \Rightarrow x = y$.

In this case, we have that:

- $x \leq_S y \Leftrightarrow y \in S(x)$;
- $x \in X$ is a maximal element iff x is a strict fixed point of S, i.e., $S(x) = \{x\}$.

Example 3.17. Let $(X, +, \mathbb{R})$ be a real linear space. Let $K \subset X$ be a cone in X, i.e., K satisfies the following conditions:

(a) $x, y \in K \Rightarrow x + y \in K$;
(b) $\lambda \in \mathbb{R}_+, x \in K \Rightarrow \lambda x \in K$;
(c) $K \cap (-K) = \{\theta_X\}$.

In this case we have that

$$x \leq_K y \Leftrightarrow y - x \in K.$$

So, an order relation on a linear space X can be viewed as a cone in X.

Example 3.18. Let (X, d) be a metric space and $\varphi : X \to \mathbb{R}$ be a functional. The binary relation defined by

$$x \leq_{d,\varphi} \Leftrightarrow d(x, y) \leq \varphi(x) - \varphi(y)$$

is an order relation on X.

Remark 3.2. For an order relation as a graph see [3], [4], [20], [13] and others.

Remark 3.3. For an order relation as a category and for the category of ordered sets see [22], [2], [20] and others.

4. Impact on the understanding of results

Mathematicians admit that seeing a concept or a results from different points of view assures (or is even a condition of) a better understanding of it and of its possible areas of application.

As we have seen, the language of ordered set theory is very rich in synonyms, and below we shall present an example of situation when using different names for the same concept obviously leads to apparently different results, which are actually a single one expressed in different forms. This is not a single case.

Theorem 4.1 (Turinici, 1982, see [33], [34], [35]**).** *Let (X, d) be a complete metric space and $(X, \leq)$ be an ordered set. We assume that:*

(a) *$[x, \to)_{\leq}$ is closed in (X, d), $\forall x \in X$;*
(b) *for each increasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset (X, \leq)$ we have that*

$$d(x_n, x_{n+1}) \longrightarrow 0 \textit{ as } n \longrightarrow \infty.$$

Then $Max(X, \leq) \cap [x, \rightarrow)_{\leq} \neq \emptyset, \forall x \in X$.

Theorem 4.2 (**Dancs-Hegedüs-Medvegyev**, 1983, [10]). *Let* (X, d) *be a complete metric space and* $S : X \longrightarrow \mathcal{P}(X)$ *defined like in Example 3.16. We assume that:*

(a) $S(x)$ *is closed in* (X, d), $\forall x \in X$;

(b) *for each sequence* $\{x_n\}_{n\in\mathbb{N}}$ *with* $x_{n+1} \in S(x_n)$, $\forall n \in \mathbb{N}$, *we have that*

$$d(x_n, x_{n+1}) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Then $(SF)_S \cap S(x) \neq \emptyset, \forall x \in X$, *where* $(SF)_S = \{y \in X | S(y) = \{y\}\}$.

Theorem 4.3 (**Granas-Horvath**, 2000, [14]). *Let* (X, d) *be a complete metric space and* $(X, \leq)$ *be a poset. We assume that:*

(a) *the terminal tails* $T(x)$ *are closed in* (X, d), $\forall x \in X$;

(b) *for each tail* $T(x)$ *and any* $\varepsilon > 0$, *there exists a subtail* $T(y) \subset T(x)$ *with diameter* $\delta_d(T(y)) \leq \varepsilon$.

Then $(SF)_T \cap T(x) \neq \emptyset, \forall x \in X$.

It is clear that the three results presented above are actually the same, but expressed each in a different language.

Remark 4.4. For other results of this type see [20].

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