CREAT. MATH. INFORM.
Volume **27** (2018), No. 2,
Pages 197 - 206

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2018.02.12

Hadamard's inequality and its extensions for conformable fractional integrals of any order $\alpha > 0$

ERHAN SET[1], AHMET OCAK AKDEMIR[2] and İ. MUMCU[1]

ABSTRACT. Recently the authors Abdeljawad [Abdeljawad, T., *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57-66] and Khalil et al. [Khalil, R., Horani, M. Al., Yousef, A. and Sababheh, M., *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65-70] introduced a new and simple well-behaved concept of fractional integral called conformable fractional integral. In this article, we establish Hermite-Hadamard's inequalities for conformable fractional integral. We also give extensions of Hermite-Hadamard type inequalities for conformable fractional integrals.

1. INTRODUCTION

The following definition has an important place in all fields of mathematics and inequality theory.

A mapping $g : J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$g(\lambda x + (1-\lambda) y) \leq \lambda g(x) + (1-\lambda) g(y)$$

satisfies for all $x, y \in I$ and $\lambda \in [0,1]$ (See [11]).

This definition has been used in the following inequality that is called Hadamard's inequality (see [11]) or Hermite-Hadamard inequality (HH-inequality).

Suppose that $g : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping and $u, v \in J$ with $u < v$, then

$$g\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v g(x)\,dx \leq \frac{g(u)+g(v)}{2}. \tag{1.1}$$

Hadamard's inequality is sensitive in terms of Cauchy Mean-Value Theorem for convex mappings. Because one can find upper and lower bounds for the mean value of a convex mapping with Hadamard's inequality. Many researchers have expended efforts to provide new bounds and estimations by using this inequality. Of all of these, we mention Riemann-Liouville fractional integrals that have beneficial uses.

Definition 1.1. Assume that $g \in L_1[u,v]$. The Riemann-Liouville integrals $J^{\mu}_{u+}g$ and $J^{\mu}_{v-}g$ of order $\mu > 0$ are introduced by

$$J^{\mu}_{u+}g(y) = \frac{1}{\Gamma(\mu)}\int_u^y (y-t)^{\mu-1} g(t)dt, \qquad y > u$$

and

$$J^{\mu}_{v-}g(y) = \frac{1}{\Gamma(\mu)}\int_y^v (t-y)^{\mu-1} g(t)dt, \qquad y < v$$

respectively, where $\Gamma(\mu) = \int_0^{\infty} e^{-t}u^{\mu-1}du$. Here $J^0_{u+}g(y) = J^0_{v-}g(y) = g(y)$.

Received: 13.08.2017. In revised form: 16.03.2018. Accepted: 23.03.2018
2010 *Mathematics Subject Classification.* 26A33, 26A51, 26D07, 26D10, 26D15.
Key words and phrases. *Convex functions, Hermite-Hadamard inequality, conformable fractional integrals.*
Corresponding author: Erhan Set; erhanset@yahoo.com

If we set $\mu = 1$ in Definition 1.1, one can obtain the classical integral.

We consider the Beta function [7, p18]:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \quad a,b>0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}u^{\alpha-1}du$ is Gamma function.

Owing to this definition the inequalities that have been obtained by classical integral and derivative have been improved and generalized. In [2] and [9], authors have proved some HH-type inequalities for Riemann-Liouville fractional integrals.

Recently several Hermite-Hadamard type inequalities were obtained for various classes of functions using fractional integrals; one may refer to such works as (for example) [3, 4, 5, 8, 10].

In spite of its valuable contribitions to mathematical analysis, the Riemann-Liouville Fractional integrals have deficiencies. For example, the solution of the differential equation that is given as

$$y^{(\frac{1}{2})} + y = x^{(\frac{1}{2})} + \frac{2}{\Gamma(2.5)}x^{(\frac{3}{2})}, \quad y(0)=0,$$

where $y^{(\frac{1}{2})}$ is the fractional derivative of y of order $\frac{1}{2}$.

This problem has caused to imagine a new and simple representation of the definition of fractional derivative. In [6], Khalil et al. gave a new definition that is called "Conformable fractional derivative". They not only proved further properties of this definition but also gave the differences with the other fractional derivatives. Besides, another considerable study have presented by Abdeljawad to discuss the basic concepts of fractional calculus. In [1], Abdeljawad gave the following definitions of Right-Left fractional integrals:

Definition 1.2. Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of any order $\alpha > 0$ is defined by

$$(I_\alpha^a f)(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx$$

Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$({}^b I_\alpha f)(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$ and hence $(I_\alpha^a f)(t) = (J_{n+1}^a f)(t)$.

We think that a new evaluation is necessary to explain the deficiencies of the previous results obtained via Riemann-Liouville fractional integrals.

Let us consider the mapping f defined as $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x^2 e^x$ which is convex. If we choose this function to provide applications by the previous inequalities that have been obtained by Riemann-Liouville fractional inequalities, we can see that the inequalities do not hold for $f(x)$. Because, the Riemann-Liouville derivatives are not valid for product of two mappings. The results which are obtained by using the conformable fractional integrals have a wide range of validity.

The aim of this paper is to prove new Hadamard's type inequalities that are valid for all elements of the class of convex mappings via conformable fractional integrals. We also obtain extensions of Hadamard's inequality by using the conformable fractional integrals.

2. HH-INEQUALITY FOR CONFORMABLE FRACTIONAL INTEGRALS

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex on $[a,b]$, then one can obtains the following inequalities for conformable fractional integrals:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I^a_\alpha f)(b) + ({}^bI_\alpha f)(a)] \leq \frac{f(a)+f(b)}{2} \tag{2.2}$$

with $\alpha \in (n, n+1]$.

Proof. Let $x, y \in [a,b]$. If f is a convex function on $[a,b]$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

i.e., with $x = ta + (1-t)b$, $y = (1-t)a + tb$,

$$2f\left(\frac{a+b}{2}\right) \leq f(ta+(1-t)b) + f((1-t)a+tb). \tag{2.3}$$

Multiplying both sides of (2.3) by $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}
\frac{2}{n!}f(\frac{a+b}{2})\int_0^1 t^n(1-t)^{\alpha-n-1}dt &\leq \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}f(ta+(1-t)b)dt \\
&+ \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}f((1-t)a+tb)dt \\
&= \frac{1}{n!}\int_a^b (\frac{b-u}{b-a})^n(\frac{u-a}{b-a})^{\alpha-n-1}f(u)\frac{du}{a-b} \\
&+ \frac{1}{n!}\int_a^b (\frac{u-a}{b-a})^n(\frac{b-u}{b-a})^{\alpha-n-1}f(u)\frac{du}{b-a} \\
&= \frac{1}{(b-a)^\alpha}[I^a_\alpha f(b) +^b I_\alpha f(a)].
\end{aligned}$$

Note we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[I^a_\alpha f(b) +^b I_\alpha f(a)]$$

where

$$\int_0^1 t^n(1-t)^{\alpha-n-1}dt = B(n+1,\alpha-n) = \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}$$

and the first part of the inequality in (2.2) is proved.

Since f is a convex, we have the following inequalities:

$$\begin{aligned}
f(ta+(1-t)b) &\leq tf(a)+(1-t)f(b) \\
f((1-t)a+tb) &\leq (1-t)f(a)+tf(b).
\end{aligned}$$

Adding these two inequalities, we get

$$f(ta+(1-t)b) + f((1-t)a+tb) \leq f(a)+f(b).$$

Multiplying both sides of the resulting inequality by $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$ and integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}\frac{1}{(b-a)^\alpha}[I_\alpha^a f(b) +^b I_\alpha f(a)] &\leq \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}[f(a)+f(b)]dt \\ &\leq \frac{1}{n!}B(n+1,\alpha-n)[f(a)+f(b)] \\ &\leq \frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}[f(a)+f(b)].\end{aligned}$$

So we get

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}[(I_\alpha^a f)(b)+({}^bI_\alpha f)(a)] \leq f(a)+f(b).$$

□

Remark 2.1. In Theorem 2.1, if we take $\alpha = n+1$, then from the the inequality (2.2) we get the inequalities in Theorem 2 of [9] and we don't suppose that f is a positive function, a condition which is required in Theorem 2 of [9].

3. Extensions of HH- Inequality

Theorem 3.2. *Assume that $f : [a,b] \to \mathbb{R}$ is a twice differentiable function with $a < b$ and $f \in L_1[a,b]$. If $f^{''}$ is bounded in $[a,b]$, then we have*

$$\begin{aligned} &\frac{m\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2 \\ &\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx \\ \leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}[(I_\alpha^a f)(b)+\ ({}^bI_\alpha f)(a)] - f\left(\frac{a+b}{2}\right) \\ \leq\ &\frac{M\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2, \\ &\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\frac{-M\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}(x-a)(b-x) \\ &\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx \\ \leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}[(I_\alpha^a f)(b)+\ ({}^bI_\alpha f)(a)] - \frac{f(a)+f(b)}{2} \\ \leq\ &\frac{-m\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}(x-a)(b-x) \\ &\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx \end{aligned} \tag{3.5}$$

with $\alpha \in (n, n+1]$, where $m = inf_{t\in[a,b]}f^{''}(t),\ M = sup_{t\in[a,b]}f^{''}(t)$.

Proof. We will prove (3.4), firstly."

$$
\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a}f(b)+\ {}^{b}I_{\alpha}f(a)] \\
= & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\Big[\frac{1}{n!}\int_{a}^{b}(b-x)^{n}(x-a)^{\alpha-n-1}f(x)dx \\
& +\frac{1}{n!}\int_{a}^{b}(x-a)^{n}(b-x)^{\alpha-n-1}f(x)dx\Big] \\
= & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!} \\
& \times\int_{a}^{b}f(x)[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx \\
= & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!} \\
& \times\int_{a}^{b}f(a+b-x)[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx
\end{aligned}
$$

So

$$
\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a}f(b)+\ {}^{b}I_{\alpha}f(a)] \\
= & \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{b}[f(x)+f(a+b-x)] \\
& \times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx
\end{aligned}
\tag{3.6}
$$

Then, we get

$$
\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+\ ({}^{b}I_{\alpha}f)(a)]-f\left(\frac{a+b}{2}\right) \\
= & \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{b}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right] \\
& \times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx.
\end{aligned}
$$

Since

$$
\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right][(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]
$$

is symmetric about $x=\frac{a+b}{2}$, we have

$$
\begin{aligned}
& \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{b}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right] \\
& \times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx \\
= & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right] \\
& \times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx.
\end{aligned}
\tag{3.7}
$$

As consequence, we have

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+\ ({}^{b}I_{\alpha}f)(a)]-f\left(\frac{a+b}{2}\right) \\ = \ & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right] \\ & \times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx. \end{aligned}$$

Since

$$f(a+b-x)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f^{'}(t)dt$$

and

$$f\left(\frac{a+b}{2}\right)-f(x)=\int_{x}^{\frac{a+b}{2}}f^{'}(t)dt,$$

we get

$$\begin{aligned} f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right) &= \int_{\frac{a+b}{2}}^{a+b-x}f^{'}(t)dt-\int_{x}^{\frac{a+b}{2}}f^{'}(t)dt \qquad (3.8) \\ &= \int_{x}^{\frac{a+b}{2}}f^{'}(a+b-t)dt-\int_{x}^{\frac{a+b}{2}}f^{'}(t)dt \\ &= \int_{x}^{\frac{a+b}{2}}[f^{'}(a+b-t)-f^{'}(t)]dt. \end{aligned}$$

Since

$$f^{'}(a+b-t)-f^{'}(t)=\int_{t}^{a+b-t}f^{''}(y)dy,$$

then for $t\in\left[a,\frac{a+b}{2}\right]$, we get

$$m(a+b-2t)\leq f^{'}(a+b-t)-f^{'}(t)\leq M(a+b-2t).$$

So

$$\begin{aligned} \int_{x}^{\frac{a+b}{2}}m(a+b-2t)dt &\leq f(x)+f(a+b-x)-2f(\frac{a+b}{2}) \\ &\leq \int_{x}^{\frac{a+b}{2}}M(a+b-2t)dt. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} m\left(\frac{a+b}{2}-x\right)^{2} &\leq f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right) \\ &\leq M\left(\frac{a+b}{2}-x\right)^{2}. \end{aligned}$$

Then

$$\begin{aligned}
&\frac{m\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2 \\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx \\
\leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}[(I_\alpha^a f)(b)+\ ({}^bI_\alpha f)(a)]-f\left(\frac{a+b}{2}\right) \\
\leq\ &\frac{M\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2 \\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx,
\end{aligned}$$

which completes the proof of (3.4).

Now we prove the second inequality. From (3.7), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}[(I_\alpha^a f)(b)+\ ({}^bI_\alpha f)(a)]-\frac{f(a)+f(b)}{2} \\
=\ &\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^b[f(x)+f(a+b-x)-(f(a)+f(b))] \\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx.
\end{aligned}$$

By using

$$[f(x)+f(a+b-x)-(f(a)+f(b))]\,[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]$$

is symetric about $x=\frac{a+b}{2}$, we get

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}[(I_\alpha^a f)(b)+\ ({}^bI_\alpha f)(a)]-\frac{f(a)+f(b)}{2} \\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}[f(x)+f(a+b-x)-(f(a)+f(b))] \\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx.
\end{aligned}\tag{3.9}$$

Since

$$f(b)-f(a+b-x)=\int_{a+b-x}^b f'(t)dt$$

and

$$f(x)-f(a)=\int_a^x f'(t)dt,$$

we obtain

$$\begin{aligned}
&f(x)+f(a+b-x)-(f(a)+f(b)) \\
=\ &\int_a^x f'(t)dt-\int_{a+b-x}^b f'(t)dt \\
=\ &\int_a^x f'(t)dt-\int_a^x f'(a+b-t)dt \\
=\ &-\int_a^x[f'(a+b-t)-f'(t)]dt.
\end{aligned}\tag{3.10}$$

We also have

$$f'(a+b-t)-f'(t)=\int_t^{a+b-t} f''(y)dy.$$

Then for $t \in [a, \frac{a+b}{2}]$, we get

$$m(a+b-2t) \leq f^{'}(a+b-t) - f^{'}(t) \leq M(a+b-2t)$$

Hence

$$\begin{aligned} -\int_a^x M(a+b-2t)dt &\leq f(x) + f(a+b-x) - (f(a)+f(b)) \\ &\leq -\int_a^x m(a+b-2t)dt. \end{aligned}$$

Namely,

$$\begin{aligned} -M(x-a)(b-x) &\leq f(x) + f(a+b-x) - (f(a)+f(b)) \\ &\leq -m(x-a)(b-x) \end{aligned}$$

and

$$\begin{aligned} & \frac{-M\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}(x-a)(b-x) \\ & \times[(b-x)^n(x-a)^{\alpha-n-1} + (x-a)^n(b-x)^{\alpha-n-1}]dx \\ \leq \; & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b) + ({}^{b}I_{\alpha}f)(a)] - \frac{f(a)+f(b)}{2} \\ \leq \; & \frac{-m\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}(x-a)(b-x) \\ & \times[(b-x)^n(x-a)^{\alpha-n-1} + (x-a)^n(b-x)^{\alpha-n-1}]dx. \end{aligned}$$

□

Remark 3.2. If the function $f : [a, b] \to \mathbb{R}$ is differentiable with a nondecreasing derivative, then f is convex. In particular, if f is twice differentiable and $f^{''} \geq 0$, then the function is convex. In Theorem (3.2), if $f^{''} \geq 0$, then we obtain inequality (2.2). Moreover if $f^{''} \geq 0$, $\alpha = n+1$ and $n = 0$, we obtain inequality (1.1).

It is obvious that $f^{''} \geq 0$ implies that $f^{'}$ non-decreasing. Therefore

$$f^{'}(a+b-x) \geq f^{'}(x). \tag{3.11}$$

holds for all $x \in [a, \frac{a+b}{2}]$. So, we can establish the following theorem using inequality (3.11).

Theorem 3.3. *Let $f : [a, b] \to \mathbb{R}$ be a positive, differentiable function with $a < b$ and $f \in L_1[a, b]$. If $f^{'}(a+b-x) \geq f^{'}(x)$ for all $x \in [a, \frac{a+b}{2}]$. Then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b) + ({}^{b}I_{\alpha}f)(a)] \leq \frac{f\,(a)+f\,(b)}{2}. \tag{3.12}$$

Proof. From (3.8) and (3.9), one has

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a}f(b)+\ {}^{b}I_{\alpha}f(a)]-f\left(\frac{a+b}{2}\right)\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[\int_{a}^{\frac{a+b}{2}}[f^{'}(a+b-t)-f^{'}(t)]dt\right]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx\\
\geq\ &0.
\end{aligned}$$

Similarly, from (3.10) and (3.11), one gets

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}[(I_{\alpha}^{a}f)(b)+\ ({}^{b}I_{\alpha}f)(a)]-\frac{f(a)+f(b)}{2}\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[-\int_{a}^{x}[f^{'}(a+b-t)-f^{'}(t)]dt\right]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx\\
\leq\ &0.
\end{aligned}$$

□

Remark 3.3. It is easy to see that inequality (3.12) is a new refinement of (1.1). Also, inequality (2.2) is the generalization of inequality (1.1).

REFERENCES

[1] Abdeljawad, T., *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66

[2] Chen, F., *Extensions of the Hermite-Hadamard Inequality for convex functions via fractional integrals*, J. Math. Ineq., **10** (2016), No. 1, 75–81

[3] Dahmani, Z., *New inequalities in fractional integrals*, Int. J. Nonlinear Sci., **9** (2010), No. 4, 493–497

[4] Dahmani, Z., *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal., **1** (2010), No. 1, 51–58

[5] Dahmani, Z., Tabharit, L. and Taf, S., *Some fractional integral inequalities*, Nonlinear. Sci. Lett. A, **1** (2010), No. 2, 155–160

[6] Khalil, R., Al Horani, M., Yousef, A. and Sababheh, M., *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70

[7] Rainville, E. D., *Special Functions*, The Mcmillan Company, New York, 1960

[8] Noor, M. A., Noor, K. I. and Awan, M. O., *New fractional estimates of Hermite-Hadamard inequalities and applications to means*, Stud. Univ. Babes Bolyai, **61** (2016), No. 1, 3–15

[9] Sarıkaya, M. Z., Set, E., Yaldız, H. and Başak, N., *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling, **57** (2013), 2403–2407

[10] Sarıkaya M. Z. and Ogunmez, H., *On new inequalities via Riemann-Liouville fractional integration*, Abstract Appl. Anal., **2012** (2012), Article ID 428983, 10 Pages

[11] Pečarić, J., Proschan, F. and Tong, Y. L., *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, 1992

[1] DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ORDU UNIVERSITY
ORDU, TURKEY
Email address: erhanset@yahoo.com
Email address: mumcuilker@msn.com

[2] DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND LETTERS
AĞRI İBRAHIM ÇEÇEN UNIVERSITY
AĞRI,TURKEY
Email address: ahmetakdemir@agri.edu.tr