CREAT. MATH. INFORM.
Volume **27** (2018), No. 2,
Pages 215 - 220

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2018.02.14

Asymptotically ideal invariant equivalence

UĞUR ULUSU

ABSTRACT. In this paper, the concepts of asymptotically $\mathcal{I}_\sigma$-equivalence, σ-asymptotically equivalence, strongly σ-asymptotically equivalence and strongly σ-asymptotically p-equivalence for real number sequences are defined. Also, we give relationships among these new type equivalence concepts and the concept of S_σ-asymptotically equivalence which is studied in [Savaş, E. and Patterson, R. F., *σ-asymptotically lacunary statistical equivalent sequences*, Cent. Eur. J. Math., **4** (2006), No. 4, 648–655]

1. INTRODUCTION AND BACKGROUND

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and the space V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$.

It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \ \text{ uniformly in } n \right\}.$$

Several authors have studied invariant convergent sequences (see, [5–9, 12–14, 16, 18]).

The concept of strongly σ-convergence was defined by Mursaleen in [6] as follows:

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0,$$

uniformly in n. It is denoted by $x_k \to L[V_\sigma]$.

By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences. In the case $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

Received: 12.06.2017. In revised form: 13.01.2018. Accepted: 20.01.2018
2010 *Mathematics Subject Classification.* 34C41, 40A35.
Key words and phrases. *Asymptotically equivalence, statistical convergence, $\mathcal{I}$-convergence, invariant convergence.*

The concept of strongly σ-convergence was generalized by Savaş [13] as below:

$$[V_\sigma]_p = \left\{ x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0, \text{ uniformly in } n \right\},$$

where $0 < p < \infty$.

If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.

The idea of statistical convergence was introduced by Fast [1] and studied by many authors.

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \Big| \big\{ k \le n : |x_k - L| \ge \varepsilon \big\} \Big| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

The concept of σ-statistically convergent sequence was introduced by Savaş and Nuray in [16] as follows:

A sequence $x = (x_k)$ is σ-statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{m\to\infty} \frac{1}{m} \Big| \big\{ k \le m : |x_{\sigma^k(n)} - L| \ge \varepsilon \big\} \Big| = 0,$$

uniformly in n. It is denoted by $S_\sigma - \lim x = L$ or $x_k \to L(S_\sigma)$.

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [3] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Similar concepts can be seen in [2,9].

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

All ideals in this paper are assumed to be admissible.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

For any ideal there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$, given by

$$\mathcal{F}(\mathcal{I}) = \big\{ M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N} \backslash A) \big\}.$$

A sequence $x = (x_k)$ is said to be $\mathcal{I}$-convergent to L if for every $\varepsilon > 0$, the set

$$A(\varepsilon) = \big\{ k \in \mathbb{N} : |x_k - L| \ge \varepsilon \big\},$$

belongs to $\mathcal{I}$. If $x = (x_k)$ is $\mathcal{I}$-convergent to L, then we write $\mathcal{I} - \lim x = L$.

Recently, the concepts of σ-uniform density of subset A of the set $\mathbb{N}$ of positive integers and corresponding $\mathcal{I}_\sigma$-convergence for real number sequences was introduced by Nuray et al. [9].

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| A \cap \left\{ \sigma(n), \sigma^2(n), ..., \sigma^m(n) \right\} \right| \text{ and } S_m = \max_n \left| A \cap \left\{ \sigma(n), \sigma^2(n), ..., \sigma^m(n) \right\} \right|.$$

If the following limits exists

$$\underline{V}(A) = \lim_{m\to\infty} \frac{s_m}{m}, \quad \overline{V}(A) = \lim_{m\to\infty} \frac{S_m}{m},$$

then they are called a lower σ-uniform density and an upper σ-uniform density of the set A, respectively.

If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Throughout the paper we take $\mathcal{I}_\sigma$ as an admissible ideal in $\mathbb{N}$.

A sequence $x = (x_k)$ is said to be $\mathcal{I}_\sigma$-convergent to L if for every $\varepsilon > 0$, the set

$$A_\varepsilon = \left\{ k : |x_k - L| \geq \varepsilon \right\},$$

belongs to $\mathcal{I}_\sigma$; i.e., $V(A_\varepsilon) = 0$. It is denoted by $\mathcal{I}_\sigma - \lim x_k = L$.

Marouf [4] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the concept of asymptotically equivalence has been developed by many other researchers (see, [10,11,15,17]).

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1.$$

It is denoted by $x \sim y$.

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are S_σ-asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| = 0,$$

uniformly in $m = 1, 2, \ldots$, (denoted by $x \overset{S_\sigma}{\sim} y$) and simply S_σ-asymptotically statistical equivalent, if $L = 1$.

2. Asymptotically $\mathcal{I}_\sigma$-Equivalence

In this section, the concepts of asymptotically $\mathcal{I}_\sigma$-equivalence, σ-asymptotically equivalence, strongly σ-asymptotically equivalence and strongly σ-asymptotically p-equivalence for real number sequences are defined. Also, we examine relationships among these new type equivalence concepts and the concept of S_σ-asymptotically equivalence which is studied in this area before.

Definition 2.1. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathcal{I}_\sigma$-equivalent of multiple L if for every $\varepsilon > 0$

$$A_\varepsilon := \left\{ k \in \mathbb{N} : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_\sigma;$$

i.e., $V(A_\varepsilon) = 0$. In this case, we write $x \overset{\mathcal{I}_\sigma^L}{\sim} y$ and simply asymptotically $\mathcal{I}_\sigma$-equivalent, if $L = 1$.

The set of all asymptotically $\mathcal{I}_\sigma$-equivalent of multiple L sequences will be denoted by $\mathfrak{I}_\sigma^L$.

Definition 2.2. Two nonnegative sequence $x = (x_k)$ and $y = (y_k)$ are σ-asymptotically equivalent of multiple L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} = L,$$

uniformly in m. In this case, we write $x \overset{V_\sigma^L}{\sim} y$ and simply σ-asymptotically equivalent, if $L = 1$.

Theorem 2.1. *Suppose that $x = (x_k)$ and $y = (y_k)$ are bounded sequences. If x and y are asymptotically $\mathcal{I}_\sigma$-equivalent of multiple L, then these sequences are σ-asymptotically equivalent of multiple L.*

Proof. Let $m, n \in \mathbb{N}$ be an arbitrary and $\varepsilon > 0$. Now, we calculate

$$t(m,n) := \left| \frac{1}{n} \sum_{k=1}^{n} \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|.$$

We have

$$t(m,n) \leq t^{(1)}(m,n) + t^{(2)}(m,n),$$

where

$$t^{(1)}(m,n) := \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon}}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \quad \text{and} \quad t^{(2)}(m,n) := \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| < \varepsilon}}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|.$$

We get $t^{(2)}(m,n) < \varepsilon$, for every $m = 1, 2, \ldots$. The boundedness of $x = (x_k)$ and $y = (y_k)$ implies that there exists a $M > 0$ such that

$$\left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \leq M,$$

for $k = 1, 2, \ldots;\ m = 1, 2, \ldots$. Then, this implies that

$$\begin{aligned} t^{(1)}(m,n) &\leq \frac{M}{n} \left| \left\{ 1 \leq k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| \\ &\leq M \frac{\max_m \left| \left\{ 1 \leq k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right|}{n} = M \frac{S_n}{n}, \end{aligned}$$

hence x and y are σ-asymptotically equivalent to multiple L. □

The converse of Theorem 2.1 does not hold. For example, $x = (x_k)$ and $y = (y_k)$ are the sequences defined by following;

$$x_k := \begin{cases} 2 \ , & \text{if } k \text{ is an even integer} \\ 0 \ , & \text{if } k \text{ is an odd integer} \end{cases} \quad ; \qquad y_k := 1$$

When $\sigma(m) = m + 1$, this sequence is σ-asymptotically equivalent but it is not asymptotically $\mathcal{I}_\sigma$-equivalent.

Definition 2.3. Two nonnegative sequence $x = (x_k)$ and $y = (y_k)$ are strongly σ-asymptotically equivalent of multiple L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| = 0,$$

uniformly in m. In this case, we write $x \overset{[V_\sigma^L]}{\sim} y$ and simply strongly σ-asymptotically equivalent, if $L = 1$.

Definition 2.4. Let $0 < p < \infty$. Two nonnegative sequence $x = (x_k)$ and $y = (y_k)$ are strongly σ-asymptotically p-equivalent of multiple L if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|^p = 0,$$

uniformly in m. In this case, we write $x \overset{[V_\sigma^L]_p}{\sim} y$ and simply strongly σ-asymptotically p-equivalent, if $L = 1$.

The set of all strongly σ-asymptotically p-equivalent of multiple L sequences will be denoted by $[\mathfrak{V}_\sigma^L]_p$.

Theorem 2.2. *Let $0 < p < \infty$. Then, $x \overset{[V_\sigma^L]_p}{\sim} y \Rightarrow x \overset{\mathcal{I}_\sigma^L}{\sim} y$.*

Proof. Let $x \overset{[V_\sigma^L]_p}{\sim} y$ and given $\varepsilon > 0$. Then, for every $m \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|^p \geq \sum_{\substack{k=1 \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon}}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|^p &\geq \varepsilon^p \cdot \left| \left\{ 1 \leq k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| \\ &\geq \varepsilon^p \cdot \max_m \left| \left\{ 1 \leq k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right|\end{aligned}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|^p \geq \varepsilon^p \cdot \frac{\max_m \left| \left\{ 1 \leq k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right|}{n} = \varepsilon^p \cdot \frac{S_n}{n},$$

for every $m = 1, 2, \ldots$. This implies $\lim_{n\to\infty} \frac{S_n}{n} = 0$ and so $x \overset{\mathcal{I}_\sigma^L}{\sim} y$. □

Theorem 2.3. *Let $0 < p < \infty$ and $x, y \in \ell_\infty$. Then, $x \overset{\mathcal{I}_\sigma^L}{\sim} y \Rightarrow x \overset{[V_\sigma^L]_p}{\sim} y$.*

Proof. Suppose that $x, y \in \ell_\infty$ and $x \overset{\mathcal{I}_\sigma^L}{\sim} y$. Let $\varepsilon > 0$. By assumption, we have $V(A_\varepsilon) = 0$. The boundedness of x and y implies that there exists a $M > 0$ such that

$$\left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \leq M,$$

for $k = 1, 2, \ldots;\ m = 1, 2, \ldots$. Observe that, for every $m \in \mathbb{N}$ we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|^p &= \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon}}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|^p + \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| < \varepsilon}}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|^p \\ &\leq M \frac{\max_m \left| \left\{ 1 \leq k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right|}{n} + \varepsilon^p \\ &\leq M \frac{S_n}{n} + \varepsilon^p.\end{aligned}$$

Hence, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right|^p=0,$$

uniformly in m.

□

Theorem 2.4. *Let $0<p<\infty$. Then, $\mathfrak{I}_\sigma^L\cap\ell_\infty=[\mathfrak{V}_\sigma^L]_p\cap\ell_\infty$.*

Proof. This is an immediate consequence of Theorem 2.2 and Theorem 2.3. □

Now we shall state a theorem that gives a relationship between asymptotically $\mathcal{I}_\sigma$-equivalence and S_σ-asymptotically equivalence.

Theorem 2.5. *The sequences $x=(x_k)$ and $y=(y_k)$ are asymptotically $\mathcal{I}_\sigma$-equivalent to multiple L if and only if they are S_σ-asymptotically equivalent of multiple L.*

References

[1] Fast, H., *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244
[2] Kostyrko, P., Macaj, M., Šalát, T. and Sleziak, M., *$\mathcal{I}$-Convergence and Extermal $\mathcal{I}$-limits points*, Math. Slovaca, **55** (2005), 443–464
[3] Kostyrko, P., Šalát, T. and Wilczyński, W., *$\mathcal{I}$-Convergence*, Real Anal. Exchange, **26** (2000), No. 2, 669–686
[4] Marouf, M., *Asymptotic equivalence and summability*, Int. J. Math. Math. Sci., **16** (1993), No. 4, 755–762
[5] Mursaleen, M., *On finite matrices and invariant means*, Indian J. Pure Appl. Math., **10** (1979), 457–460
[6] Mursaleen, M., *Matrix transformation between some new sequence spaces*, Houston J. Math., **9** (1983), 505–509
[7] Mursaleen, M. and Edely, O. H. H., *On the invariant mean and statistical convergence*, Appl. Math. Lett., **22** (2009), No. 11, 1700–1704
[8] Nuray, F. and Savaş, E., *Invariant statistical convergence and A-invariant statistical convergence*, Indian J. Pure Appl. Math., **25** (1994), No. 3, 267–274
[9] Nuray, F., Gök, H. and Ulusu, U., *$\mathcal{I}_\sigma$-convergence*, Math. Commun., **16** (2011), 531–538
[10] Patterson, R. F., *On asymptotically statistically equivalent sequences*, Demostratio Math., **36** (2003), No. 1, 149–153
[11] Patterson, R. F. and Savaş, E., *On asymptotically lacunary statistically equivalent sequences*, Thai J. Math., **4** (2006), No. 2, 267–272
[12] Raimi, R. A., *Invariant means and invariant matrix methods of summability*, Duke Math. J., **30** (1963), No. 1, 81–94
[13] Savaş, E., *Some sequence spaces involving invariant means*, Indian J. Math., **31** (1989), 1–8
[14] Savaş, E., *Strongly σ-convergent sequences*, Bull. Calcutta Math., **81** (1989), 295–300
[15] Savaş, E., *On $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences*, Adv. Difference Equ., **111** (2013), 7 pages. doi:10.1186/1687-1847-2013-111
[16] Savaş, E. and Nuray, F., *On σ-statistically convergence and lacunary σ-statistically convergence*, Math. Slovaca, **43** (1993), No. 3, 309–315
[17] Savaş, E. and Patterson, R. F., *σ-asymptotically lacunary statistical equivalent sequences*, Cent. Eur. J. Math., **4** (2006), No. 4, 648–655
[18] Schaefer, P., *Infinite matrices and invariant means*, Proc. Amer. Math. Soc., **36** (1972), 104–110

Department of Mathematics
Afyon Kocatepe University
03200 Afyonkarahisar, Turkey
Email address: ulusu@aku.edu.tr