Asymptotically ideal invariant equivalence

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ABSTRACT. In this paper, the concepts of asymptotically $I_\sigma$-equivalence, $\sigma$-asymptotically equivalence, strongly $\sigma$-asymptotically equivalence and strongly $\sigma$-asymptotically $p$-equivalence for real number sequences are defined. Also, we give relationships among these new type equivalence concepts and the concept of $S_\sigma$-asymptotically equivalence which is studied in [Savaş, E. and Patterson, R. F., $\sigma$-asymptotically lacunary statistically equivalent sequences, Cent. Eur. J. Math., 4 (2006), No. 4, 648–655]

1. INTRODUCTION AND BACKGROUND

Let $\sigma$ be a mapping of the positive integers into themselves. A continuous linear functional $\phi$ on $\ell_\infty$, the space of real bounded sequences, is said to be an invariant mean or a $\sigma$-mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all $n$,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings $\sigma$ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers $n$ and $m$, where $\sigma^m(n)$ denotes the $m$ th iterate of the mapping $\sigma$ at $n$. Thus, $\phi$ extends the limit functional on $\ell$, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in \ell$.

In the case $\sigma$ is translation mappings $\sigma(n) = n + 1$, the $\sigma$-mean is often called a Banach limit and the space $V_\sigma$, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$.

It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } n \right\}.$$

Several authors have studied invariant convergent sequences (see, [5–9, 12–14, 16, 18]). The concept of strongly $\sigma$-convergence was defined by Mursaleen in [6] as follows: A bounded sequence $x = (x_k)$ is said to be strongly $\sigma$-convergent to $L$ if

$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0,$$

uniformly in $n$. It is denoted by $x_k \to L[V_\sigma]$.

By $[V_\sigma]$, we denote the set of all strongly $\sigma$-convergent sequences. In the case $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $\hat{c}$.
The concept of strongly $\sigma$-convergence was generalized by Savas \[13\] as below:

$$[V_{\sigma}]_p = \left\{ x = (x_k) : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0, \text{ uniformly in } n \right\},$$

where $0 < p < \infty$.

If $p = 1$, then $[V_{\sigma}]_p = [V_{\sigma}]$. It is known that $[V_{\sigma}]_p \subset \ell_\infty$.

The idea of statistical convergence was introduced by Fast \[1\] and studied by many authors.

A sequence $x = (x_k)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - L | \geq \varepsilon \right\} \right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

The concept of $\sigma$-statistically convergent sequence was introduced by Savas and Nuray in \[16\] as follows:

A sequence $x = (x_k)$ is $\sigma$-statistically convergent to $L$ if for every $\varepsilon > 0$

$$\lim_{m \to \infty} \frac{1}{m} \left| \left\{ k \leq m : |x_{\sigma^k(n)} - L | \geq \varepsilon \right\} \right| = 0,$$

uniformly in $n$. It is denoted by $S_{\sigma} - \lim x = L$ or $x_k \to L(S_{\sigma})$.

The idea of $I$-convergence was introduced by Kostyrko et al. \[3\] as a generalization of statistical convergence which is based on the structure of the ideal $I$ of subset of the set of natural numbers $\mathbb{N}$. Similar concepts can be seen in \[2, 9\].

A family of sets $I \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in I$, (ii) For each $A, B \in I$ we have $A \cup B \in I$, (iii) For each $A \in I$ and each $B \subseteq A$ we have $B \in I$.

An ideal is called non-trivial if $\mathbb{N} \notin I$ and non-trivial ideal is called admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

All ideals in this paper are assumed to be admissible.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if (i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

For any ideal there is a filter $\mathcal{F}(I)$ corresponding with $I$, given by

$$\mathcal{F}(I) = \left\{ M \subseteq \mathbb{N} : (\exists A \in I)(M = \mathbb{N} \setminus A) \right\}.$$ 

A sequence $x = (x_k)$ is said to be $I$-convergent to $L$ if for every $\varepsilon > 0$, the set

$$A(\varepsilon) = \left\{ k \in \mathbb{N} : |x_k - L | \geq \varepsilon \right\},$$

belongs to $I$. If $x = (x_k)$ is $I$-convergent to $L$, then we write $I - \lim x = L$.

Recently, the concepts of $\sigma$-uniform density of subset $A$ of the set $\mathbb{N}$ of positive integers and corresponding $I_{\sigma}$-convergence for real number sequences was introduced by Nuray et al. \[9\].
Let \( A \subseteq \mathbb{N} \) and 
\[
\begin{align*}
s_m &= \min_n \left| A \cap \{\sigma(n), \sigma^2(n), \ldots, \sigma^m(n)\} \right|, \\
S_m &= \max_n \left| A \cap \{\sigma(n), \sigma^2(n), \ldots, \sigma^m(n)\} \right|.
\end{align*}
\]
If the following limits exist
\[
\begin{align*}
\underline{V}(A) &= \lim_{m \to \infty} \frac{s_m}{m}, \\
\overline{V}(A) &= \lim_{m \to \infty} \frac{S_m}{m},
\end{align*}
\]
then they are called a lower \( \sigma \)-uniform density and an upper \( \sigma \)-uniform density of the set \( A \), respectively.

If \( \underline{V}(A) = \overline{V}(A) \), then \( \overline{V}(A) = \underline{V}(A) = V(A) \) is called the \( \sigma \)-uniform density of \( A \).

Denote by \( I_{\sigma} \) the class of all \( A \subseteq \mathbb{N} \) with \( \overline{V}(A) = 0 \).

Throughout the paper we take \( I_{\sigma} \) as an admissible ideal in \( \mathbb{N} \).

A sequence \( x = (x_k) \) is said to be \( I_{\sigma} \)-convergent to \( L \) if for every \( \varepsilon > 0 \), the set
\[
A_\varepsilon = \left\{ k : |x_k - L| \geq \varepsilon \right\},
\]
belongs to \( I_{\sigma} \); i.e., \( V(A_\varepsilon) = 0 \). It is denoted by \( I_{\sigma} \lim x_k = L \).

Marouf [4] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the concept of asymptotically equivalence has been developed by many other researchers (see, [10, 11, 15, 17]).

Two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be asymptotically equivalent if
\[
\lim_{k \to \infty} \frac{x_k}{y_k} = 1.
\]
It is denoted by \( x \sim y \).

Two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are \( S_{\sigma} \)-asymptotically equivalent of multiple \( L \) provided that for every \( \varepsilon > 0 \)
\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| = 0,
\]
uniformly in \( m = 1, 2, \ldots \), (denoted by \( x \overset{S_{\sigma}}{\sim} y \)) and simply \( S_{\sigma} \)-asymptotically statistical equivalent, if \( L = 1 \).

2. ASYMPTOTICALLY \( I_{\sigma} \)-EQUIVALENCE

In this section, the concepts of asymptotically \( I_{\sigma} \)-equivalence, \( \sigma \)-asymptotically equivalence, strongly \( \sigma \)-asymptotically equivalence and strongly \( \sigma \)-asymptotically \( p \)-equivalence for real number sequences are defined. Also, we examine relationships among these new type equivalence concepts and the concept of \( S_{\sigma} \)-asymptotically equivalence which is studied in this area before.

Definition 2.1. Two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be asymptotically \( I_{\sigma} \)-equivalent of multiple \( L \) if for every \( \varepsilon > 0 \)
\[
A_\varepsilon := \left\{ k \in \mathbb{N} : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I_{\sigma};
\]
i.e., \( V(A_\varepsilon) = 0 \). In this case, we write \( x \overset{I_{\sigma}}{\sim} y \) and simply asymptotically \( I_{\sigma} \)-equivalent, if \( L = 1 \).

The set of all asymptotically \( I_{\sigma} \)-equivalent of multiple \( L \) sequences will be denoted by \( \mathcal{G}_{\sigma}^L \).
**Definition 2.2.** Two nonnegative sequence \( x = (x_k) \) and \( y = (y_k) \) are \( \sigma \)-asymptotically equivalent of multiple \( L \) if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} = L,
\]

uniformly in \( m \). In this case, we write \( x \overset{\sigma}{\sim}^L y \) and simply \( \sigma \)-asymptotically equivalent, if \( L = 1 \).

**Theorem 2.1.** Suppose that \( x = (x_k) \) and \( y = (y_k) \) are bounded sequences. If \( x \) and \( y \) are asymptotically \( I_\sigma \)-equivalent of multiple \( L \), then these sequences are \( \sigma \)-asymptotically equivalent of multiple \( L \).

**Proof.** Let \( m, n \in \mathbb{N} \) be an arbitrary and \( \varepsilon > 0 \). Now, we calculate

\[
t(m, n) := \left| \frac{1}{n} \sum_{k=1}^{n} \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right|.
\]

We have

\[
t(m, n) \leq t^{(1)}(m, n) + t^{(2)}(m, n),
\]

where

\[
t^{(1)}(m, n) := \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon
\]

\[
t^{(2)}(m, n) := \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| < \varepsilon
\]

We get \( t^{(2)}(m, n) < \varepsilon \), for every \( m = 1, 2, \ldots \). The boundedness of \( x = (x_k) \) and \( y = (y_k) \) implies that there exists a \( M > 0 \) such that

\[
\left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \leq M,
\]

for \( k = 1, 2, \ldots; m = 1, 2, \ldots \). Then, this implies that

\[
t^{(1)}(m, n) \leq \frac{M}{n} \left\{ \left| 1 \leq k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right| \right\}
\]

\[
\leq M \max_m \left\{ \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} = M \frac{S_n}{n},
\]

hence \( x \) and \( y \) are \( \sigma \)-asymptotically equivalent to multiple \( L \). \( \Box \)

The converse of Theorem 2.1 does not hold. For example, \( x = (x_k) \) and \( y = (y_k) \) are the sequences defined by following;

\[
x_k := \begin{cases} 2, & \text{if } k \text{ is an even integer} \\ 0, & \text{if } k \text{ is an odd integer} \end{cases} \quad y_k := 1
\]

When \( \sigma(m) = m + 1 \), this sequence is \( \sigma \)-asymptotically equivalent but it is not asymptotically \( I_\sigma \)-equivalent.

**Definition 2.3.** Two nonnegative sequence \( x = (x_k) \) and \( y = (y_k) \) are strongly \( \sigma \)-asymptotically equivalent of multiple \( L \) if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| = 0,
\]
uniformly in $m$. In this case, we write $x \overset{V^L}{\sim} y$ and simply strongly $\sigma$-asymptotically equivalent, if $L = 1$.

**Definition 2.4.** Let $0 < p < \infty$. Two nonnegative sequence $x = (x_k)$ and $y = (y_k)$ are strongly $\sigma$-asymptotically $p$-equivalent of multiple $L$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right|^p = 0,$$

uniformly in $m$. In this case, we write $x \overset{[V^L_\sigma]}{\sim} y$ and simply strongly $\sigma$-asymptotically $p$-equivalent, if $L = 1$.

The set of all strongly $\sigma$-asymptotically $p$-equivalent of multiple $L$ sequences will be denoted by $[\Omega^L_\sigma]^p$.

**Theorem 2.2.** Let $0 < p < \infty$. Then, $x \overset{[V^L_\sigma]}{\sim} y \Rightarrow x \overset{\sigma}{\sim} y$.

**Proof.** Let $x \overset{[V^L_\sigma]}{\sim} y$ and given $\varepsilon > 0$. Then, for every $m \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right|^p \geq \sum_{k=1}^{n} \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right|^p \geq \varepsilon^p \cdot \max_{m} \left\{ 1 \leq k \leq n : \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right| \geq \varepsilon \right\},$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right|^p \geq \varepsilon^p \cdot \frac{\max_{m} \left\{ 1 \leq k \leq n : \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right| \geq \varepsilon \right\}}{n} = \varepsilon^p \cdot \frac{S_n}{n},$$

for every $m = 1, 2, \ldots$. This implies $\lim_{n \to \infty} \frac{S_n}{n} = 0$ and so $x \overset{\sigma}{\sim} y$. \hfill $\square$

**Theorem 2.3.** Let $0 < p < \infty$ and $x, y \in \ell_\infty$. Then, $x \overset{\sigma}{\sim} y \Rightarrow x \overset{[V^L_\sigma]}{\sim} y$.

**Proof.** Suppose that $x, y \in \ell_\infty$ and $x \overset{\sigma}{\sim} y$. Let $\varepsilon > 0$. By assumption, we have $V(A_\varepsilon) = 0$. The boundedness of $x$ and $y$ implies that there exists a $M > 0$ such that

$$\left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right| \leq M,$$

for $k = 1, 2, \ldots; m = 1, 2, \ldots$. Observe that, for every $m \in \mathbb{N}$ we have

$$\frac{1}{n} \sum_{k=1}^{n} \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right|^p \geq \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right|^p \leq \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right|^p + \varepsilon^p \cdot \frac{\max_{m} \left\{ 1 \leq k \leq n : \left| \frac{x^{\sigma_k}(m)}{y^{\sigma_k}(m)} - L \right| \geq \varepsilon \right\}}{n} \leq M \frac{S_n}{n} + \varepsilon^p.$$
Hence, we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_{\sigma^k(m)} - y_{\sigma^k(m)}}{L} \right|^p = 0,
\]
uniformly in \(m\).

\[\Box\]

**Theorem 2.4.** Let \(0 < p < \infty\). Then, \(\mathcal{J}_p^L \cap \ell_\infty = [\mathcal{M}_\sigma^L]_p \cap \ell_\infty\).

**Proof.** This is an immediate consequence of Theorem 2.2 and Theorem 2.3.

Now we shall state a theorem that gives a relationship between asymptotically \(\mathcal{I}_\sigma\)-equivalence and \(S_\sigma\)-asymptotically equivalence.

**Theorem 2.5.** The sequences \(x = (x_k)\) and \(y = (y_k)\) are asymptotically \(\mathcal{I}_\sigma\)-equivalent to multiple \(L\) if and only if they are \(S_\sigma\)-asymptotically equivalent of multiple \(L\).

**REFERENCES**


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