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Hankel determinant for m-fold symmetric bi-univalent functions

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ABSTRACT. In this paper, we consider a general subclass $H_{\Sigma_m}(\beta)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disc $\mathcal{U}$. An estimate for the second Hankel determinant for m-fold symmetric bi-univalent functions are determined.

1. INTRODUCTION

Let $\mathcal{A}$ represent the class of functions f which are analytic in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the functions f of the form (1.1) which are also univalent in $\mathcal{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z \ (z \in \mathcal{U})$ and $f\left(f^{-1}(w)\right) = w \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disc $\mathcal{U}$. For a brief history and interesting examples of functions in the class Σ, see the pioneering work on this subject by Srivastava et al. [16], which has apparently revived the study of bi-univalent functions in recent years (see also [2], [3], [4], [9], [10], [15] and [17]).

For each function $f \in \mathcal{S}$, the function

$$h(z) = \sqrt[m]{f(z^m)} \qquad (z \in \mathcal{U}, \ m \in \mathbb{N}) \tag{1.3}$$

is univalent and maps the unit disc $\mathcal{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [8], [14]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathcal{U}, \ m \in \mathbb{N}). \tag{1.4}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathcal{U}$, which are normalized by the series expansion (1.4). In fact, the functions in the class $\mathcal{S}$ are *one*-fold symmetric. Analogous to the concept of *m*-fold symmetric univalent functions, we here introduced the concept of *m*-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an *m*-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The

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normalized form of f is given as in (1.4) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [18], is given as follows:

$$\begin{aligned} g(w) &= w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1})\right] w^{2m+1} \\ &- \left[\tfrac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} \\ &+ \cdots , \end{aligned} \tag{1.5}$$

where $f^{-1} = g$. We denote by Σ_m the class of *m*-fold symmetric bi-univalent functions in $\mathcal{U}$. For $m = 1$, the formula (1.5) coincides with the formula (1.2) of the class Σ. Some examples of *m*-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad [-\log(1-z^m)]^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}}.$$

The q^{th} Hankel determinant for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas [11] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

This determinant has also been considered by several authors. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles ([1], [12], [13], [19], [20]) different subclasses of univalent and bi-univalent functions.

Note that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is well-known as Fekete-Szegö functional (see [6]).

Definition 1.1. (See [18]) A function $f \in \Sigma_m$ is said to be in the class $H_{\Sigma_m}(\beta)$, if the following conditions are satisfied:

$$\Re\left(f'(z)\right) > \beta \quad (0 \leq \beta < 1, \ z \in \mathcal{U})$$

and

$$\Re\left(g'(w)\right) > \beta \quad (0 \leq \beta < 1, w \in \mathcal{U}),$$

where $g = f^{-1}$.

2. Preliminary results

Let $\mathcal{P}$ be the class of functions $p(z)$ with positive real part consisting of all analytic functions $p : \mathcal{U} \to \mathbb{C}$ satisfying the following conditions:

$$p(0) = 1, \qquad \Re(p(z)) > 0.$$

Lemma 2.1. (See [14]) *If the function $p \in \mathcal{P}$ is defined by*

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots ,$$

then

$$|p_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}).$$

Lemma 2.2. (See [7]) *If the function $p \in \mathcal{P}$, then*

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2) \\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \end{aligned}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

3. Main results

Theorem 3.1. *Let f given by (1.4) be in the class $H_{\Sigma_m}(\beta)$, $0 \le \beta < 1$. Then*

$$\left|a_{m+1}a_{3m+1} - a_{2m+1}^2\right| \le$$

$$\begin{cases} \frac{(1-\beta)^2}{m+1}\left[\frac{4(1-\beta)^2}{m+1} + \frac{4}{3m+1}\right], \quad \beta \in \left[0, 1 - \frac{m(3m+1)+\sqrt{m^2(3m+1)^2+8(m+1)(2m+1)^2(3m+1)}}{4(2m+1)(3m+1)}\right] \\ (1-\beta)^2\left\{\frac{4}{(2m+1)^2} + \left(\frac{-\left[m(2m+1)(3m+1)(1-\beta)+(m+1)(6m^2+4m+1)\right]^2}{[(2m+1)^2(3m+1)(1-\beta)^2 - m(2m+1)(3m+1)(1-\beta)-(m+1)(5m^2+4m+1)]}\right)\right. \\ \left.\times\frac{1}{(m+1)^2(2m+1)^2(3m+1)}\right\}, \\ \beta \in \left[1 - \frac{m(3m+1)+\sqrt{m^2(3m+1)^2+8(m+1)(2m+1)^2(3m+1)}}{4(2m+1)(3m+1)}, 1\right) \end{cases}.$$

Proof. Let $f \in H_{\Sigma_m}(\beta)$. Then

$$f'(z) = \beta + (1-\beta)p(z), \tag{3.6}$$

$$g'(w) = \beta + (1-\beta)q(w), \tag{3.7}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \cdots$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \cdots.$$

It follows from (3.6) and (3.7) that

$$(m+1)a_{m+1} = (1-\beta)p_m, \tag{3.8}$$

$$(2m+1)a_{2m+1} = (1-\beta)p_{2m}, \tag{3.9}$$

$$(3m+1)a_{3m+1} = (1-\beta)p_{3m}, \tag{3.10}$$

$$-(m+1)a_{m+1} = (1-\beta)q_m, \tag{3.11}$$

$$(2m+1)\left[(m+1)a_{m+1}^2 - a_{2m+1}\right] = (1-\beta)q_{2m}, \tag{3.12}$$

$$-(3m+1)\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] = (1-\beta)q_{3m}. \tag{3.13}$$

From (3.8) and (3.11), we obtain

$$p_m = -q_m \tag{3.14}$$

and

$$a_{m+1} = \frac{(1-\beta)}{m+1}p_m. \tag{3.15}$$

Subtracting (3.9) from (3.12), we have

$$a_{2m+1}=\frac{(1-\beta)^{2}}{2\left(m+1\right)}p_{m}^{2}+\frac{(1-\beta)}{2\left(2m+1\right)}\left(p_{2m}-q_{2m}\right). \tag{3.16}$$

Also, subtracting (3.10) from (3.13), we have

$$a_{3m+1}=\frac{\left(3m+2\right)\left(1-\beta\right)^{2}}{4\left(m+1\right)\left(2m+1\right)}p_{m}\left(p_{2m}-q_{2m}\right)+\frac{(1-\beta)}{2\left(3m+1\right)}\left(p_{3m}-q_{3m}\right). \tag{3.17}$$

Then, we can establish that

$$\begin{aligned}\left|a_{m+1}a_{3m+1}-a_{2m+1}^{2}\right| &= \left|-\tfrac{(1-\beta)^{4}}{4(m+1)^{2}}p_{m}^{4}+\tfrac{m(1-\beta)^{3}}{4(m+1)^{2}(2m+1)}p_{m}^{2}\left(p_{2m}-q_{2m}\right)\right.\\ &\quad \left.+\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}p_{m}\left(p_{3m}-q_{3m}\right)-\tfrac{(1-\beta)^{2}}{4(2m+1)^{2}}\left(p_{2m}-q_{2m}\right)^{2}\right|\end{aligned} \tag{3.18}$$

According to Lemma 2.2 and (3.14), we write

$$\left.\begin{array}{l} 2p_{2m}=p_{m}^{2}+x(4-p_{m}^{2}) \\ 2q_{2m}=q_{m}^{2}+y(4-q_{m}^{2}) \end{array}\right\} \Rightarrow p_{2m}-q_{2m}=\frac{4-p_{m}^{2}}{2}(x-y) \tag{3.19}$$

and

$$4p_{3m} = p_{m}^{3}+2(4-p_{m}^{2})p_{m}x-p_{m}(4-p_{m}^{2})x^{2}+2(4-p_{m}^{2})(1-|x|^{2})z \tag{3.20}$$

$$4q_{3m} = q_{m}^{3}+2(4-q_{m}^{2})q_{m}y-q_{m}(4-q_{m}^{2})y^{2}+2(4-q_{m}^{2})(1-|y|^{2})w \tag{3.21}$$

$$\begin{aligned}p_{3m}-q_{3m} &= \tfrac{p_{m}^{3}}{2}+\tfrac{p_{m}(4-p_{m}^{2})}{2}(x+y)-\tfrac{p_{m}(4-p_{m}^{2})}{4}(x^{2}+y^{2})\\ &+\tfrac{4-p_{m}^{2}}{2}\left[(1-|x|^{2})z-(1-|y|^{2})w\right].\end{aligned}$$

Then, using (3.19) and (3.20), in (3.18), we obtain

$$\begin{aligned}\left|a_{m+1}a_{3m+1}-a_{2m+1}^{2}\right| &= \left|-\tfrac{(1-\beta)^{4}}{4(m+1)^{2}}p_{m}^{4}+\tfrac{m(1-\beta)^{3}}{4(m+1)^{2}(2m+1)}p_{m}^{2}\tfrac{4-p_{m}^{2}}{2}(x-y)\right.\\ &+\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}\tfrac{p_{m}^{4}}{2}+\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}p_{m}^{2}\tfrac{4-p_{m}^{2}}{2}(x+y)-\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}p_{m}^{2}\tfrac{(4-p_{m}^{2})}{4}(x^{2}+y^{2})\\ &\left.+\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}p_{m}\tfrac{(4-p_{m}^{2})}{2}\left[\left(1-|x|^{2}\right)z-\left(1-|y|^{2}\right)w\right]-\tfrac{(1-\beta)^{2}}{4(2m+1)^{2}}\tfrac{(4-p_{m}^{2})^{2}}{4}(x-y)^{2}\right|\end{aligned}$$

and

$$\begin{aligned}\left|a_{m+1}a_{3m+1}-a_{2m+1}^{2}\right| &\leq \tfrac{(1-\beta)^{4}}{4(m+1)^{2}}p_{m}^{4}+\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}\tfrac{p_{m}^{4}}{2}+\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}p_{m}(4-p_{m}^{2})\\ &+\left[\tfrac{m(1-\beta)^{3}}{4(m+01)^{2}(2m+1)}p_{m}^{2}\tfrac{\left(4-p_{m}^{2}\right)}{2}+\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}p_{m}^{2}\tfrac{\left(4-p_{m}^{2}\right)}{2}\right](|x|+|y|)\\ &+\left[\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}p_{m}^{2}\tfrac{(4-p_{m}^{2})}{4}-\tfrac{(1-\beta)^{2}}{2(m+1)(3m+1)}p_{m}\tfrac{(4-p_{m}^{2})}{2}\right](|x|^{2}+|y|^{2})\\ &+\tfrac{(1-\beta)^{2}}{4(2m+1)^{2}}\tfrac{(4-p_{m}^{2})^{2}}{4}(|x|+|y|)^{2}.\end{aligned} \tag{3.22}$$

Since $p\in\mathcal{P}$, so $|p_{m}|\leq 2$. Letting $|p_{m}|=p$, we may assume without restriction that $p\in[0,2]$. For $\eta=|x|\leq 1$ and $\mu=|y|\leq 1$, we get

$$\left|a_{m+1}a_{3m+1}-a_{2m+1}^{2}\right|\leq T_{1}+(\eta+\mu)\,T_{2}+\left(\eta^{2}+\mu^{2}\right)T_{3}+(\eta+\mu)^{2}\,T_{4}=G(\eta,\mu)$$

where

$$\begin{aligned}
T_1 &= T_1(p) = \frac{(1-\beta)^2}{2(m+1)}\left[\left(\frac{(1-\beta)^2}{2(m+1)}+\frac{1}{2(3m+1)}\right)p^4-\frac{1}{3m+1}p^3+\frac{4}{3m+1}p\right] \geq 0\\
T_2 &= T_2(p) = \frac{(1-\beta)^2}{4(m+1)}p^2(4-p^2)\left[\frac{m(1-\beta)}{2(m+1)(2m+1)}+\frac{1}{3m+1}\right] \geq 0\\
T_3 &= T_3(p) = \frac{(1-\beta)^2}{8(m+1)(3m+1)}p(4-p^2)(p-2) \leq 0\\
T_4 &= T_4(p) = \frac{(1-\beta)^2}{4(2m+1)^2}\frac{(4-p^2)^2}{4} \geq 0.
\end{aligned}$$

We now need to maximize the function $G(\eta,\mu)$ on the closed square $[0,1]\times[0,1]$. We must investigate the maximum of $G(\eta,\mu)$ according to $p\in(0,2)$, $p=0$ and $p=2$ taking into account the sign of $G_{\eta\eta}.G_{\mu\mu}-(G_{\eta\mu})^2$.

Firstly, let $p\in(0,2)$. Since $T_3<0$ and $T_3+2T_4>0$ for $p\in(0,2)$, we conclude that

$$G_{\eta\eta}.G_{\mu\mu}-(G_{\eta\mu})^2<0.$$

Thus the function G cannot have a local maximum in the interior of the square. Now, we investigate the maximum of G on the boundary of the square.

For $\eta=0$ and $0\leq\mu\leq 1$ (similarly $\mu=0$ and $0\leq\eta\leq 1$), we obtain

$$G(0,\mu)=H(\mu)=(T_3+T_4)\mu^2+T_2\mu+T_1.$$

i. The case $T_3 + T_4 \geq 0$: In this case for $0<\mu<1$ and any fixed p with $0<p<2$, it is clear that $H'(\mu)=2(T_3+T_4)\mu+T_2>0$, that is, $H(\mu)$ is an increasing function. Hence, for fixed $p\in(0,2)$, the maximum of $H(\mu)$ occurs at $\mu=1$, and

$$\max H(\mu)=H(1)=T_1+T_2+T_3+T_4.$$

ii. The case $T_3+T_4<0$: Since $T_2+2(T_3+T_4)\geq 0$ for $0<\mu<1$ and any fixed p with $0<p<2$, it is clear that $T_2+2(T_3+T_4)<2(T_3+T_4)\mu+T_2<T_2$ and so $H'(\mu)>0$. Hence for fixed $p\in(0,2)$, the maximum of $H(\mu)$ occurs at $\mu=1$.

Also for $p=2$ we obtain

$$G(\eta,\mu)=\frac{(1-\beta)^2}{m+1}\left[\frac{4(1-\beta)^2}{m+1}+\frac{4}{3m+1}\right] \tag{3.23}$$

Taking into account the value (3.23), and the cases i and ii, for $0\leq\mu\leq 1$ and any fixed p with $0\leq p\leq 2$,

$$\max H(\mu)=H(1)=T_1+T_2+T_3+T_4.$$

For $\eta=1$ and $0\leq\mu\leq 1$ (similarly $\mu=1$ and $0\leq\eta\leq 1$), we obtain

$$G(1,\mu)=F(\mu)=(T_3+T_4)\mu^2+(T_2+2T_4)\mu+T_1+T_2+T_3+T_4.$$

Similarly to the above cases of T_3+T_4, we get that

$$\max F(\mu)=F(1)=T_1+2T_2+2T_3+4T_4.$$

Since $H(1)\leq F(1)$ for $p\in[0,2]$, $\max G(\eta,\mu)=G(1,1)$ on the boundary of the square. Thus the maximum of G occurs at $\eta=1$ and $\mu=1$ in the closed square.

Let $K:[0,2]\to\mathbb{R}$

$$K(p)=\max G(\eta,\mu)=G(1,1)=T_1+2T_2+2T_3+4T_4. \tag{3.24}$$

Substituting the values of T_1, T_2, T_3 and T_4 in the function K defined by (3.24), yield

$$
\begin{aligned}
K(p) &= \frac{(1-\beta)^2}{2}\left\{\left(\frac{(1-\beta)^2}{2(m+1)^2} - \frac{m(1-\beta)}{2(m+1)^2(2m+1)} - \frac{1}{(m+1)(3m+1)} + \frac{1}{2(2m+1)^2}\right)p^4 \right.\\
&\left. + \left(\frac{2m(1-\beta)}{(m+1)^2(2m+1)} + \frac{6}{(m+1)(3m+1)} - \frac{4}{(2m+1)^2}\right)p^2 + \frac{8}{(2m+1)^2}\right\}.
\end{aligned}
$$

Assume that $K(p)$ has a maximum value in an interior of $p \in [0,2]$, by elementary calculation

$$
\begin{aligned}
K'(p) &= (1-\beta)^2\left\{\left(\frac{(1-\beta)^2}{(m+1)^2} - \frac{m(1-\beta)}{(m+1)^2(2m+1)} - \frac{2}{(m+1)(3m+1)} + \frac{1}{(2m+1)^2}\right)p^3 \right.\\
&\left. + \left(\frac{2m(1-\beta)}{(m+1)^2(2m+1)} + \frac{6}{(m+1)(3m+1)} - \frac{4}{(2m+1)^2}\right)p\right\}.
\end{aligned}
$$

As a result of some calculations we can do the following examine: □

Case 1. Let $\left(\frac{(1-\beta)^2}{2(m+1)^2} - \frac{m(1-\beta)}{2(m+1)^2(2m+1)} - \frac{1}{(m+1)(3m+1)} + \frac{1}{2(2m+1)^2}\right) \geq 0$. Therefore $\beta \in \left[0, 1 - \frac{m(3m+1)+\sqrt{m^2(3m+1)^2+4(m+1)(3m+1)(5m^2+4m+1)}}{2(2m+1)(3m+1)}\right]$ and $K'(p) > 0$ for $p \in (0,2)$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $p \in [0,2]$, that is, $p = 2$. Thus, we have

$$
\max K(p) = K(2) = \frac{(1-\beta)^2}{m+1}\left[\frac{4(1-\beta)^2}{m+1} + \frac{4}{3m+1}\right].
$$

Case 2. Let $\left(\frac{(1-\beta)^2}{2(m+1)^2} - \frac{m(1-\beta)}{2(m+1)^2(2m+1)} - \frac{1}{(m+1)(3m+1)} + \frac{1}{2(2m+1)^2}\right) < 0$. that is, $\beta \in \left(1 - \frac{m(3m+1)+\sqrt{m^2(3m+1)^2+4(m+1)(3m+1)(5m^2+4m+1)}}{2(2m+1)(3m+1)}, 1\right)$. Then $K'(p) = 0$ implies the real critical points $p_{01} = 0$ or

$$
p_{02} = \sqrt{\frac{-2\left[m(2m+1)(3m+1)(1-\beta) + (m+1)(6m^2+4m+1)\right]}{(2m+1)^2(3m+1)(1-\beta)^2 - m(2m+1)(3m+1)(1-\beta) - (m+1)(5m^2+4m+1)}}.
$$

When

$$
\begin{aligned}
\beta \in & \left(1 - \frac{m(3m+1) + \sqrt{m^2(3m+1)^2 + 4(m+1)(3m+1)(5m^2+4m+1)}}{2(2m+1)(3m+1)},\right.\\
& \left. 1 - \frac{m(3m+1) + \sqrt{m^2(3m+1)^2 + 8(m+1)(2m+1)^2(3m+1)}}{4(2m+1)(3m+1)}\right]
\end{aligned}
$$

we observe that $p_{02} \geq 2$, that is, p_{02} is out of the interval $(0,2)$. Therefore the maximum value of $K(p)$ occurs at $p_{01} = 0$ or $p = p_{02}$ which contradicts our assumption of having the maximum value at the interior point of $p \in [0,2]$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $p \in [0,2]$, that is, $p = 2$. Thus, we have

$$
\max K(p) = K(2) = \frac{(1-\beta)^2}{m+1}\left[\frac{4(1-\beta)^2}{m+1} + \frac{4}{3m+1}\right].
$$

When $\beta \in \left(1 - \frac{m(3m+1)+\sqrt{m^2(3m+1)^2+8(m+1)(2m+1)^2(3m+1)}}{4(2m+1)(3m+1)}, 1\right)$ we observe that $p_{02} < 2$, that is, p_{02} is interior of the interval $[0,2]$. Since $K''(p_{02}) < 0$, the maximum value of $K(p)$

occurs at $p = p_{02}$. Thus, we have

$$K(p_{02}) = (1-\beta)^2 \left\{ \frac{4}{(2m+1)^2} \right.$$

$$\left. - \frac{\left[m(2m+1)(3m+1)(1-\beta)+(m+1)(6m^2+4m+1)\right]^2}{(m+1)^2(2m+1)^2(3m+1)[(2m+1)^2(3m+1)(1-\beta)^2-m(2m+1)(3m+1)(1-\beta)-(m+1)(5m^2+4m+1)]} \right\}.$$

This completes the proof.

Remark 3.1. (See [5]) Putting $m = 1$ in Theorem 3.1 we have the second Hankel determinant for the well-known class $H_{\Sigma_m}(\beta) = H_{\Sigma}(\beta)$.

Remark 3.2. Let f given by (1.4) be in the class $S^*_{\Sigma}(\beta)$ and $0 \leq \beta < 1$. Then

$$\left|a_2 a_4 - a_3^2\right| \leq \begin{cases} \frac{(1-\beta)^2}{2}\left(2\beta^2 - 4\beta + 3\right) & \beta \in \left[0, \frac{11-\sqrt{37}}{12}\right) \\ \frac{(1-\beta)^2}{9}\left(4 - \frac{(17-6\beta)^2}{16(9\beta^2-15\beta+1)}\right) & \beta \in \left(\frac{11-\sqrt{37}}{12}, 1\right) \end{cases}.$$

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