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Convergence of derivative free iterative methods

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ABSTRACT. We present the local as well as the semi-local convergence of some iterative methods free of derivatives for Banach space valued operators. These methods contain the secant and the Kurchatov method as special cases. The convergence is based on weak hypotheses specializing to Lipschitz continuous or Hölder continuous hypotheses. The results are of theoretical and practical interest. In particular the method is compared favorably to other methods using concrete numerical examples to solve systems of equations containing a nondifferentiable term.

1. INTRODUCTION

Numerous problems in mathematics, computational sciences and also in engineering using mathematical modeling can be reduced to solving a nonlinear equation like

$$F(x) := H(x) + H_1(x) = 0, \tag{1.1}$$

where $H : D \subset X \longrightarrow Y, H_1 : D \subset X \longrightarrow Y, H, H_1$ are continuous operators, X and Y are Banach spaces and D is a nonempty open set. We would like to find a locally unique solution x^* of equation (1.1) preferably in closed form. However, this task is achieved only in special cases. Hence, we utilize iterative methods to generate a sequence converging to x^* under certain hypotheses.

Zabrejko and Nguen [23] have used a variant of Newton's method defined for each $n = 0, 1, 2, \ldots$ by

$$v_{n+1} = v_n - H'(v_n)^{-1}(H(v_n) + H_1(v_n)) \tag{1.2}$$

where v_n is an initial point. Numerous others [1–3, 6–12, 16–20] have improved the semi-local convergence of method (1.2). Method (1.2) uses the computationaly expensive inverse of Fréchet-derivative of H making it unsuitable for solving equation (1.1), when F is a non-differentiable operator. We propose the iterative method defined for each $n = 0, 1, 2, \ldots,$ by

$$x_{n+1} = x_n - A_n^{-1}F(x_n), \tag{1.3}$$

where $x_{-1}, x_0 \in D, 2x_0 - x_{-1} \in D, [.,.;H] : D \times D \longrightarrow L(X,Y), [.,.;H_1] : D \times D \longrightarrow L(X,Y)$ are divided differences of order one [1] and $A_n = [2x_n - x_{n-1}, x_{n-1}; H] + [x_{n-1}, x_n; H_1]$. Notice that if $H = 0$, we obtain the secant method [1], whereas if $H_1 = 0$ method (1.3) reduces to the Kurchatov method [18].

The convergence analysis uses weaker conditions than before [1]- [23] leading to the extension of the applicability of these methods. The technique used in this paper can be used to show convergence of other iterative methods. Section 2 and Section 3 contain the semi-local and local convergence of method (1.3). The numerical examples appear in Section 4.

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2. Semi-local convergence

Let $y \in X$ and $\mu > 0$. We shall denote by $B(y, \mu)$ the open ball of center y with radius μ and by $\bar{B}(y, \mu)$ its closure. The senilocal convergence of method (1.3) is based on the hypotheses (C):

(c_1) $H : D \subset X \longrightarrow Y, H_1 : D \subset X \longrightarrow Y$ are continuous operators and there exist divided differences of order one $[.,.;H] : D \times D \longrightarrow L(X,Y), [.,.;H_1] : D \times D \longrightarrow L(X,Y)$.

(c_2) For some $x_{-1}, x_0 \in D, A_0^{-1} \in L(Y, X)$ and there exist parameters $\beta > 0, \eta_0 \geq 0, \eta > 0$ such that $\|A_0^{-1}\| \leq \beta$, $\|x_{-1} - x_0\| \leq \eta_0$ and $\|A_0^{-1}F(x_0)\| \leq \eta$

(c_3) For each $x, y \in D \Longrightarrow 2y - x \in D$.

(c_4) There exist continuous and increasing functions $\bar{\omega} : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}$, $\bar{\omega}_1 : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}$ such that for each $x, y \in D$

$$\|A_0^{-1}([2y - x, x; H] - [2x_0 - x_{-1}, x_{-1}; H])\| \leq \bar{\omega}(\|2y - x_{-1} - 3x\|, \|x - x_{-1}\|)$$

and

$$\|A_0^{-1}([2y - x, x; H_1] - [2x_0 - x_{-1}, x_{-1}, ; H_1])\| \leq \bar{\omega}_1(\|x - x_{-1} - 3x\|, \|y - x_0\|).$$

(c_5) Equation $\bar{\omega}(3t + \eta_0, t + \eta_0) + \bar{\omega}_1(t + \eta_0, t) = 1$ has positive solutions. Denote by ρ the smallest such solution. Set $D_0 = D \cap B(x_0, \rho)$.

(c_6) There exist continuous and nondecreasing functions $\omega : [0, \rho) \times [0, \rho) \longrightarrow \mathbb{R}$, $\omega_1 : [0, \rho) \longrightarrow \mathbb{R}$ such that for each $x, y, u, v \in D_0$

$$\|[x, y; H] - [u, v; H]\| \leq \omega(\|x - u\|, \|y - v\|)$$

and

$$\|[x, y; H_1] - [u, y; H_1]\| \leq \omega_1(\|x - u\|).$$

(c_7) Set $\alpha = \beta \max\{\omega(\eta + \eta_0, \eta_0) + \omega_1(\eta + \eta_0), \omega(2\eta, \eta) + \omega_1(2\eta)\}$. Equation

$$t(1 - \frac{\alpha}{\beta[\bar{\omega}(3t + \eta_0, t + \eta_0) + \bar{\omega}_1(\eta_0 + t, t)]}) - \eta = 0$$

has positive solutions. Denote by t^* the smallest such solution.

(c_9)

$$\beta[\bar{\omega}(3t^* + \eta_0, t^* + \eta_0) + \bar{\omega}_1(\eta_0 + t^*, t^*)] < 1$$

$$\delta = \frac{\alpha}{1 - \beta[\bar{\omega}(3t^* + \eta_0, t^* + \eta_0) + \bar{\omega}_1(\eta_0 + t^*, t^*)]} < 1$$

and $\bar{B}(x_0, t^*) \subseteq D$.

(c_8) There exists $\bar{t}^* \geq t^*$ such that

$$\beta[\bar{\omega}(3\bar{t}^* + \eta_0, \bar{t}^* + \eta_0) + \bar{\omega}_1(\eta_0 + \bar{t}^*, \bar{t}^*)] < 1.$$

Set $D_1 = D \cap \bar{B}(x_0, \bar{t}^*)$.

Theorem 2.1. *Suppose that the conditions (C) hold. Then, the sequence $\{x_n\}$ starting at x_{-1}, x_0 and generated by method (1.3) is well defined in $B(x_0, t^*)$, remains in $B(x_0, t^*)$, for each $n = -1, 0, 1, \ldots$ and converges to a unique solution $x^* \in D_1$ of equation $F(x) = 0$.*

Proof. We prove, by induction, that the sequence $\{x_n\}$ given by (1.3) is well defined and $x_n \in B(x_0, t^*)$ for each $n = 0, 1, 2, \ldots$. By the definition of t^*, we have $t^* = \frac{\eta}{1-\delta} > \eta$. By the condition (c_2), x_1 is well defined, and $x_1 \in B(x_0,t^*)$. Using (c_4), we obtain

$$\begin{aligned}
\|I - A_0^{-1}A_1\| &= \|A_0^{-1}([2x_1 - x_0, x_0; H] - [2x_0 - x_1, x_{-1}; H]) \\
&\quad + A_0^{-1}([x_0, x_{-1}; H_1] - [x_{-1}, x_0; H_1])\| \\
&\leq \beta[\bar\omega(\|2x_1 + x_{-1} - 3x_0\|, \|x_0 - x_{-1}\|) \\
&\quad + \bar\omega_1(\|x_0 - x_{-1}\|, \|x_1 - x_0\|)\|] \\
&\leq \beta[\bar\omega(2\eta + \eta_0, \eta_0) + \bar\omega_1(\eta_0, \eta)] \\
&\leq \beta[\bar\omega(3t^* + \eta_0, t^* + \eta_0) + \bar\omega(\eta_0 + t^*, t^*)] < 1.
\end{aligned} \tag{2.4}$$

It follows from (2.5) and the Banach lemma on invertible operators [13] that A_1^{-1} exists and

$$\|A_1^{-1}\| \leq \frac{\beta}{1 - \beta[\bar\omega(3t^* + \eta_0, t^* + \eta_0) + \bar\omega(\eta_0 + t^*, t^*)]}. \tag{2.5}$$

Consequently, the iterate x_2 is well defined. We can also obtain from (1.3) and (c_6) that

$$\begin{aligned}
\|F(x_1)\| &= \|F(x_1) - F(x_0) + F(x_0)\| \\
&= \|H(x_1) - H(x_0) + H_1(x_1) - H_1(x_0) - A_0(x_1 - x_0)\| \\
&= \|([x_1, x_0; H] - [2x_0 - x_{-1}; x_{-1}; H] \\
&\quad + [x_1, x_0; H_1] - [x_{-1}, x_0; H_1](x_1 - x_0)\| \\
&\leq [\omega(\|x_1 + x_{-1} - 2x_0\|, \|x_0 - x_{-1}\|) + \omega_1(\|x_1 - x_{-1}\|)]\|x_1 - x_0\| \\
&\leq [\omega(\eta + \eta_0, \eta_0) + \omega_1(\eta + \eta_0)]\|x_1 - x_0\|.
\end{aligned} \tag{2.6}$$

In view of (1.3), (2.5), (c_8) and (2.7), we have

$$\begin{aligned}
\|x_2 - x_1\| &\leq \|A_1^{-1}\|\|F(x_1)\| \\
&\leq \frac{\alpha}{1 - \beta[\bar\omega(3t^* + \eta_0, t^* + \eta_0) + \bar\omega(\eta_0 + t^*, t^*)]}\|x_1 - x_0\| \\
&\leq \delta\|x_1 - x_0\|.
\end{aligned} \tag{2.7}$$

We also get

$$\|x_2 - x_0\| \leq (\delta + 1)\|x_1 - x_0\| \leq (\delta + 1)\eta < \frac{\eta}{1 - \delta} = t^*, \tag{2.8}$$

so $x_2 \in B(x_0, t^*)$.

Next we will prove by induction the following for $i = 1, 2, \dots$

(I_1) A_i^{-1} exists and $\|A_i^{-1}\| \leq \frac{\alpha}{1-\beta[\bar\omega(3t^*+\eta_0,t^*+\eta_0)+\bar\omega(\eta_0+t^*,t^*)]}$,

(II) $\|x_{i+1} - x_i\| \leq \delta\|x_i - x_{i-1}\| \leq \delta^i\|x_1 - x_0\| < \eta$.

Suppose that the operator A_i are invertible and $x_{i+1} \in B(x_0, t^*)$ for $1 \leq i \leq k-1$, and $k \geq 2$ is a fixed integer, we obtain

$$\begin{aligned}
\|I - A_0^{-1}A_k\| &= \|A_0^{-1}([2x_k - x_{k-1}, x_{k-1}; H] - [2x_0 - x_{-1}, x_{-1}; H] \\
&\quad + A_0^{-1}([x_{k-1}, x_k; H_1] - [x_{-1}, x_0; H_1])\| \\
&\leq \beta[\bar\omega(\|2(x_k - x_0) - (x_{k-1} - x_{-1})\|, \|x_{k-1} - x_{-1}\|) \\
&\quad \bar\omega_1(\|x_{k-1} - x_{-1}\|, \|x_k - x_0\|)] \\
&\leq \beta[\bar\omega(3t^* + \eta, t^* + \eta) + \bar\omega_1(\eta_0 + t^*, t^*)] < 1.
\end{aligned} \tag{2.9}$$

It follows from (2.10) and the Banach lemma on invertible operators that A_k^{-1} exists, so that

$$\|A_k^{-1}\| \leq \frac{\beta}{1 - \beta[\bar\omega(3t^* + \eta, t^* + \eta) + \bar\omega_1(\eta_0 + t^*, t^*)]}, \tag{2.10}$$

and the iterate x_{k+1} is well defined. Similarly to (2.7), we can get

$$\begin{aligned}
(2.11)\quad \|F(x_k)\| &= \|F(x_k) - F(x_{k-1}) + F(x_{k-1})\| \\
&= \|H(x_k) - H(x_{k-1}) + H_1(x_k) - H_1(x_{k-1}) - A_{k-1}(x_k - x_{k-1})\| \\
&= \|([x_k, x_{k-1}; H] - [2x_{k-1} - x_{k-2}; x_{k-2}; H] \\
&\quad +[x_k, x_{k-1}; H_1] - [x_{k-2}, x_{k-1}; H_1])(x_k - x_{k-1})\| \\
&\leq [\omega(\|x_k + x_{k-2} - 2x_{k-1}\|, \|x_{k-1} - x_{k-2}\|) \\
&\quad +\omega_1(\|x_k - x_{k-2}\|)]\|x_k - x_{k-1}\| \\
&\leq [\omega((\delta^{k-1} + \delta^{k-2})\eta, \delta^{k-2}\eta) + \omega_1((\delta^{k-1} + \delta^{k-2})\eta)]\|x_k - x_{k-1}\| \\
(2.12)\qquad &\leq [\omega(2\eta, \eta) + \omega_1(2\eta)]\|x_k - x_{k-1}\| \leq \frac{\alpha}{\beta}\|x_k - x_{k-1}\|.
\end{aligned}$$

Hence, we get

$$(2.13)\qquad \|x_{k+1} - x_k\| \leq \|A_k^{-1}\|\|F(x_k)\| \leq \delta\|x_k - x_{k-1}\| \leq \delta^k\|x_1 - x_0\| < \eta.$$

Then, from (c_2) and (II_i), it follows that

$$\begin{aligned}
(2.14)\qquad \|x_{k+1} - x_k\| &\leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \ldots_{\|} x_1 - x_0\| \\
&\leq (\delta^k + \delta^{k-1} + \ldots + 1)\|x_1 - x_0\| \\
&\leq \frac{1 - \delta^{k+1}}{1 - \delta}\eta < \frac{\eta}{1 - \delta} = t^*.
\end{aligned}$$

That is, $x_{k+1} \in B(x_0, t^*)$ and the induction is complete.

Moreover, we prove that $\{x_n\}$ is a complete sequence. For $k \geq 1$, we obtain

$$\begin{aligned}
\|x_{n+k} - x_n\| &\leq \|x_{n+k} - x_{n+k-1}\| + \|x_{n+k-1} - x_{n+k-2}\| + \ldots + \|x_{n+1} - x_n\| \\
(2.15)\qquad &\leq (\delta^{k-1} + \delta^{k-2} + \ldots + 1)\|x_{n+1} - x_n\| \\
&\leq \frac{1 - \delta^k}{1 - \delta}\delta^n\|x_1 - x_0\| < \frac{\delta^n}{1 - \delta}\|x_1 - x_0\|.
\end{aligned}$$

Hence $\{x_n\}$ is a complete sequence on a Banach space X and as such it converges to $x^*\bar{B}(x_0, t^*)$. Setting $n \longrightarrow \infty$ in (2.12), we obtain $F(x^*) = 0$.

Furthermore, to show the uniqueness, we assume that there exists a solution $y^* \in D_1$. Consider the operator $A = [y^*, x^*; H] + [y^*, x^*; H_1]$, if the operator A is invertible, then since $A(y^* - x^*) = F(y^*) - F(x^*)$, we have $x^* = y^*$. In particular we obtain in turn

$$\begin{aligned}
(2.16)\qquad \|I - A_0^{-1}A\| &= \|A_0^{-1}(A - A_0)\| \\
&\leq \|A_0^{-1}\|(\|[y^*, x^*, H] - [2x_0 - x_{-1}, x_{-1}; H]\| \\
&\quad +\|[y^*, x^*; H_1] - [x_{-1}, x_0; H_1]\|) \\
&\leq \beta[\bar{\omega}(\|y^* + x_{-1} - 2x_0\|, \|x^* - x_{-1}\|) \\
&\quad +\bar{\omega}_1(\|y^* - x_{-1}\|, \|x^* - x_0\|)] \\
&\leq \beta[\bar{\omega}(t^* + \eta_0, t^* + \eta_0) + \bar{\omega}_1(t^* + \eta_0, t^*)] < 1,
\end{aligned}$$

so operator A^{-1} exists. □

Remark 2.1. Condition (c_3) is satisfied if e.g.,$D = X$. It can also be dropped if in (c_8), $\bar{B}(x_0, t^*) \subseteq D$ is replaced by $\bar{B}(x_0, 3t^*) \subseteq D$. Indeed, we shall have in this case that for $x, y \in \bar{B}(x_0, t^*)$

$$\|2y - x - x_0\| \leq 2\|y - x_0\| + \|x - x_0\| \leq 3t^*,$$

so $2y - x \in \bar{B}(x_0, 3t^*)$. Similar replacements of (a_3) can be made for condition (c_3) in the local convergence analysis that follows.

3. Local convergence

The local convergence analysis of method (1.3) is based on the hypotheses (A):

(a_1) $H : D \subset X \longrightarrow Y$ is a Fréchet-differentiable operator, $H_1 : D \subset X \longrightarrow Y$ is a continuous operator and there exist divided differences of order one $[.,.;H] : D \times D \longrightarrow L(X,Y), [.,.;H_1] : D \times D \longrightarrow L(X,Y)$.

(a_2) There exist $x^* \in D$ and $\tilde{x} \in D$ be such that $F(x^*) = 0, \|\tilde{x} - x^*\| = \xi$, so that operator $A_*^{-1} = (H'(x^*) + [\tilde{x}, x^*; H_1])^{-1}$ exists and $\|A_*^{-1}\| \leq \lambda$.

(1) (a_3)=(c_3)

(a_4) There exist continuous and increasing functions $\bar{w} : [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$, $\bar{w}_1 : [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$ such that for each $x, y \in D$

$$\|[x,y;H] - H'(x^*)\| \leq \bar{w}(\|x - x^*\|, \|y - x^*\|)$$

and

$$\|[2y - x, x; H_1] - [\tilde{x}, x^*; H_1])\| \leq \bar{w}_1(\|2y - x - \tilde{x}\|, \|x - x^*\|).$$

(a_5) Define function $\varphi : [0,+\infty) \longrightarrow \mathbb{R}$ by $\varphi(t) = \lambda(\bar{w}(t,t) + \bar{w}_1(3t + \xi, t))$. Suppose equation $\varphi(t) = 1$ has positive solutions. Denote by r the smallest such solution. Set $D_2 = D \cap B(x^*, r)$.

(a_6) There exist continuous and increasing functions $w : [0,r) \times [0,r) \longrightarrow \mathbb{R}$, $w_1 : [0,r) \longrightarrow \mathbb{R}$ such that for each $x, y, u, v \in D_2$

$$\|[x,y;H_1] - [y, x^*;; H_1]\| \leq w_1(\|x - y\|, \|y - x^*\|)$$

and

$$\|[2y - x, x; H] - [y, x^*; H]\| \leq w(\|y - x\|, \|x - x^*\|).$$

(a_7) Equation $\psi(t) = 1$, has positive solutions, where

$$\psi(t) = \frac{\lambda(w(2t,t) + w_1(2t,t))}{1 - \varphi(t)}.$$

Denote by r^* the smallest such solution.

(a_8) $\varphi(r^*) < 1$ and $\bar{B}(x^*, r^*) \subseteq D$.

(a_9) There exists continuous and increasing functions $v : [0,r) \longrightarrow \mathbb{R}$ and $v_1 : [0,r) \longrightarrow \mathbb{R}$ such that

$$\|[y, x^*; H] - H'(x^*)\| \leq v(\|y - x^*\|)$$

and

$$\|[y, x^*; H_1] - [\tilde{x}, x^*; H_1]\| \leq v_1(\|y - \tilde{x}\|).$$

Equation $\lambda(v(t) + v_1(t + \xi)) = 1$ has positive solutions. Denote by r_1 the smallest such solution. There exists $r_2 \geq r_1$. Set $D_3 = D \cap \bar{B}(x^*, r_2)$.

Theorem 3.2. *Suppose that the conditions (A) hold. Then, sequence $\{x_n\}$ starting from $x_{-1}, x_0, 2x_0 - x_{-1} \in B(x^*, r^*) - \{x^*\}$ and generated by method (1.3) is well defined in $B(x^*, r^*)$, remains in $B(x^*, r^*)$ for each $n = -1, 0, 1, \ldots$ and converges to x^*. Moreover, the following estimates hold*

$$\|x_{n+1} - x^*\| \leq \psi(r^*)\|x_n - x^*\|. \tag{3.17}$$

Furthermore, the point x^* is the only solution of equation $F(x) = 0$ in D_3.

Proof. Let $x, y, 2y - x \in B(x^*, r^*)$. Using the definition of r and (a_4), we get in turn that

$$\begin{aligned} & \|A_*^{-1}([x,y;H] + [2y-x,x;H_1] - A^*)\| \\ \leq\ & \|A_*^{-1}\|[\|[x,y;H] - H'(x^*)\| \\ & +\|[2y-x,x;H_1] - [\tilde{x},x^*;H_1]\|] \\ \leq\ & \lambda(\bar{\omega}(\|x-x^*\|, \|y-x^*\|) + \bar{\omega}_1(\|2y-x-\tilde{x}\|, \|x-\tilde{x}\|)) \\ \leq\ & \lambda(\bar{\omega}(r,r) + \bar{\omega}_1(3r+\xi, r)) = \varphi(r) < 1, \end{aligned} \tag{3.18}$$

so $A = A(x,y) = [x,y;H] + [2y-x,x;H_1]$ is invertible and

$$\|A^{-1}A_*\| \leq \frac{\lambda}{1-\varphi(r)}. \tag{3.19}$$

We can write

$$\begin{aligned} & A - (H+H_1)(y) \\ =\ & [2y-x,x;H] + [x,y;H_1] \\ & -[y,x^*;H] - [y,x^*;H_1] \\ =\ & ([2y-x,x;H] - [y,x^*;H_1]) + ([x,y;H_1] - [y,x^*;H_1]) \end{aligned} \tag{3.20}$$

so by (a_6)

$$\begin{aligned} \|A - (H+H_1)(y)\| &\leq w(\|2y-x-y\|, \|x-x^*\|) + w_1(\|x-y\|, \|y-x^*\|) \\ &\leq w(\|x-x^*\| + \|y-x^*\|, \|x-x^*\|) \\ &\quad +w_1(\|x-x^*\| + \|y-x^*\|, \|y-x^*\|) \\ &\leq w(2r,r) + w_1(2r,r). \end{aligned} \tag{3.21}$$

In particular, for $x = x_{n-1}, y = x_n$, we get by (1.3), (a_2), (a_7), (3.21) , (3.22) that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|x_n - x^* - A_n^{-1}(H+H_1)(x_n)\| \\ &\leq \|A_n^{-1}\|\|A_n - (H+H_1)(x_n)\|\|x_n - x^*\| \\ &\leq \frac{\lambda(w(2r^*,r^*) + w_1(2r^*,r^*))}{1-\varphi(r^*)}\|x_n - x^*\| \leq c\|x_n - x^*\| \\ &\leq \|x_n - x^*\| < r^*, \end{aligned} \tag{3.22}$$

so (3.17) holds and $x_{n+1} \in B(x^*, r^*)$, where $c = \psi(r^*) \in [0,1)$. Therefore $\lim_{n\longrightarrow+\infty} x_n = x^*$.

To show the uniqueness part, as in (2.17) but using (a_9), we obtain in turn for $y^* \in D_3$ with $F(y^*) = 0$ that

$$\begin{aligned} \|A_*^{-1}\|\|A_* - A\| &\leq \lambda\|([y^*,x^*;H] - H'(x^*)) \\ &\quad +([y^*,x^*;H_1] - [\tilde{x},x^*;H_1])\| \\ &\leq \lambda(v(\|y^*-x^*\|) + v_1(\|y^*-\tilde{x}\|)) \\ &\leq \lambda(v(r_2) + v_1(r_2+\xi)) < 1. \end{aligned} \tag{3.23}$$

□

4. Numerical examples

We complete this study with a numerical example.

Example 4.1. Let $X = Y = (\mathbb{R}^2, \|.\|_\infty)$. Consider the system

$$\begin{aligned} 3x^2y + y^2 - 1 + |x-1| &= 0, \\ x^4 + xy^3 - 1 + |y| &= 0. \end{aligned}$$

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
-1	5	5	
0	1	0	5.000E+00
1	0.989800874210782	0.012627489072365	1.262E-02
2	0.921814765493287	0.307939916152262	2.953E-01
3	0.900073765669214	0.325927010697792	2.174E-02
4	0.894939851625105	0.327725437396226	5.133E-03
5	0.894658420586013	0.327825363500783	2.814E-04
6	0.894655375077418	0.327826521051833	3.045E-04
7	0.094655373334698	0.3278266521746293	1.742E-09
8	0.894655373334687	0.327826521746298	1.076E-14
9	0.894655373334687	0.327826521746298	5.421E-20

TABLE 1. Iterations and error for secant method

Set

$$\|x\|_\infty = \|(x', x'')\|_\infty = \max\{|x'|, |x''|\},$$
$$H = (P_1, P_2),\ H_1 = (Q_1, Q_2),\ \text{and } F = H + H_1.$$

For $x = (x', x'') \in \mathbb{R}^2$ we take

$$P_1(x', x'') = 3(x')^2 x'' + (x'')^2 - 1,\ P_2(x', x'') = (x')^4 + x'(x'')^3 - 1,$$
$$Q_1(x', x'') = |x' - 1|,\ Q_2(x', x'') = |x''|.$$

We shall take $[x, y, H_1] \in M_{2\times 2}(\mathbb{R})$ as

$$[x, y; H]_{i,1} = \frac{P_i(y', y'') - P_i(x', y'')}{y' - x'},$$
$$[x, y; H]_{1,2} = \frac{P_i(x', y'') - P_i(x', x'')}{y'' - x''}, i = 1, 2,$$

provided that $y' \neq x'$ and $y'' \neq x''$. Otherwise define $[x, y; H]$ to be the zero matrix in $M_{2\times 2}(\mathbb{R}$. Similarly we define divided difference $[2y - x, x; H_1]$. Using the secant method with $x_{-1} = (5, 5)$, and $x_0 = (1, 0)$, we obtain the values in Table 1. It is clear that the hypotheses of Theorem 2.1 are satisfied for the methods for the starting points close to the solution

$$x^* = (0.894655373334687, 0.327826521746298).$$

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