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Some properties of the analytic functions with bounded radius rotation

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ABSTRACT. In the present paper, we introduce a new subclass of normalized analytic starlike functions by using bounded radius rotation associated with $q-$ analogues in the open unit disc $\mathbb{D}$. We investigate growth theorem, radius of starlikeness and coefficient estimate for the new subclass of starlike functions by using bounded radius rotation associated with $q-$ analogues denoted by $\mathcal{R}_k(q)$, where $k \geq 2, q \in (0,1)$.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = 0, f'(0) = 1$ for every $z \in \mathbb{D}$. We say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$, if there exists a Schwarz function ϕ which is analytic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, such that $f_1(z) = f_2(\phi(z))$. In particular, when f_2 is univalent, then the above subordination is equivalent to $f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ (Subordination principle [3]).

In 1971, Pinchuk [4] introduced and studied the classes $\mathcal{P}_k$ and $\mathcal{R}_k$, where $\mathcal{R}_k$ generalizes the class of starlike functions. Here $\mathcal{P}_k$ denotes the class of functions $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, analytic in $\mathbb{D}$ with $p(0) = 1$ and having the representation

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

where μ is real-valued function of bounded variation for which

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k.$$

The class $\mathcal{R}_k$, defined by Pinchuk in [4], consists of those functions f which satisfy the condition

$$\int_{-\pi}^{\pi} \left| Re\left(re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right) \right| d\theta \leq k\pi, 0 < r < 1, z = re^{i\theta}. \tag{1.2}$$

Geometrically, the condition (1.2) is the total variation of the angle between radius vector $f(re^{i\theta})$ makes with the positive real axis is bounded by $k\pi$. Thus $\mathcal{R}_k$ is the class of bounded radius rotation bounded by $k\pi$.

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Denote by $\mathcal{P}_q$ the family of functions p of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, analytic $\mathbb{D}$ and satisfy the condition

$$\left| p(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q},$$

where $q \in (0,1)$ is a fixed real number.

The following lemma is first introduced in [6], later given in [2]:

Lemma 1.1. *p is an element of $\mathcal{P}_q$ if and only if $p(z) \prec \dfrac{1+z}{1-qz}$. This result is sharp for the functions $p(z) = \dfrac{1+\phi(z)}{1-q\phi(z)}$, where ϕ is a Schwarz function.*

Using the definitions $\mathcal{P}_k$ and $\mathcal{P}_q$, Noor and Noor introduced the class $\mathcal{P}_k(q)$ in [5] as below:

Definition 1.1. A function p of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, analytic in $\mathbb{D}$ with $p(0) = 1$ is said to be in the class $\mathcal{P}_k(q)$, $k \geq 2$, $q \in (0,1)$ if and only if there exists $p_1^{(1)}, p_2^{(2)} \in \mathcal{P}_q$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(2)}(z). \tag{1.3}$$

For $q \to 1^-$, $\mathcal{P}_k(q)$ reduces to $\mathcal{P}_k$, (see [4]); for $k = 2$, $\mathcal{P}_k(q)$ reduces to $\mathcal{P}_q$; for $k = 2$, $q \to 1^-$, $\mathcal{P}_k(q)$ reduces to $\mathcal{P}$ which is the well known class of functions with positive real part.

In the present paper, we give a new subclass of starlike functions with bounded radius rotation associated with $q-$ analogues denoted by $\mathcal{R}_k(q)$.

Definition 1.2. Let f of the form (1.1) be an element of $\mathcal{A}$. If f satisfies the condition

$$z\frac{f'(z)}{f(z)} = p(z), \quad p \in \mathcal{P}_k(q), \tag{1.4}$$

with $k \geq 2, q \in (0,1)$, then f is called starlike function with bounded radius rotation with $q-$ analogues denoted by $\mathcal{R}_k(q)$.

Motivated by Definition 1.2, we investigate growth theorem, radius of starlikeness and coefficient inequality for the class $\mathcal{R}_k(q)$.

2. Main results

We first give growth theorem for the class $\mathcal{R}_k(q)$.

Theorem 2.1. *If $f \in \mathcal{R}_k(q)$, then*

$$rF(q,k,-r) \leq |f(z)| \leq rF(q,k,r), \tag{2.5}$$

where

$$F(q,k,r) = \frac{\left(1+qr\right)^{\left(\frac{k}{4}-\frac{1}{2}\right)\frac{(1+q)}{q}}}{\left(1-qr\right)^{\left(\frac{k}{4}+\frac{1}{2}\right)\frac{(1+q)}{q}}},$$

$k \geq 2$, $q \in (0,1)$.

Proof. Let p be an element of $\mathcal{P}_q$ and $|z| = r < 1$, then by Lemma 1.1 we have

$$\frac{1-r}{1+qr} \leq Rep(z) \leq |p(z)| \leq \frac{1+r}{1-qr}. \tag{2.6}$$

After simple calculations in (2.6), we get

$$\frac{1-\frac{k}{2}(1+q)r+qr^2}{(1-qr)(1+qr)} \leq Rep(z) \leq \frac{1+\frac{k}{2}(1+q)r+qr^2}{(1-qr)(1+qr)}. \tag{2.7}$$

Inequality in (2.7) shows that the set of variability of $p \in \mathcal{P}_k(q)$ is the closed disc

$$\left|p(z) - \frac{1+qr^2}{1-q^2r^2}\right| \leq \frac{\frac{k}{2}(1+q)r}{1-q^2r^2}. \tag{2.8}$$

On the other hand from defintion of $\mathcal{R}_k(q)$, we can write

$$\left|z\frac{f'(z)}{f(z)} - \frac{1+qr^2}{1-q^2r^2}\right| \leq \frac{\frac{k}{2}(1+q)r}{1-q^2r^2}, \tag{2.9}$$

which gives

$$\frac{1-\frac{k}{2}(1+q)r+qr^2}{(1-qr)(1+qr)} \leq Rez\frac{f'(z)}{f(z)} \leq \frac{1+\frac{k}{2}(1+q)r+qr^2}{(1-qr)(1+qr)}. \tag{2.10}$$

Since

$$Re\left(z\frac{f'(z)}{f(z)}\right) = r\frac{\partial}{\partial r}log|f(z)|,$$

then the equality (2.10) can be written

$$\frac{1-\frac{k}{2}(1+q)r+qr^2}{r(1-qr)(1+qr)} \leq \frac{\partial}{\partial r}log|f(z)| \leq \frac{1+\frac{k}{2}(1+q)r+qr^2}{r(1-qr)(1+qr)}. \tag{2.11}$$

Taking integration on both sides of (2.11), we obtain

$$rF(q,k,-r) \leq |f(z)| \leq rF(q,k,r),$$

where

$$F(q,k,r) = \frac{\left(1+qr\right)^{\left(\frac{k}{4}-\frac{1}{2}\right)\frac{(1+q)}{q}}}{\left(1-qr\right)^{\left(\frac{k}{4}+\frac{1}{2}\right)\frac{(1+q)}{q}}}.$$

This estimate is sharp because extremal function is

$$f(z) = \frac{z\left(1+qz\right)^{\left(\frac{k}{4}-\frac{1}{2}\right)\frac{(1+q)}{q}}}{\left(1-qz\right)^{\left(\frac{k}{4}+\frac{1}{2}\right)\frac{(1+q)}{q}}}.$$

□

Corollary 2.1. *If we take $q = 0$ in (2.9), we obtain*

$$\left|z\frac{f'(z)}{f(z)} - 1\right| \leq \frac{kr}{2},$$

which gives

$$\frac{1}{r} - \frac{k}{2} \leq \frac{\partial}{\partial r}log|f(z)| \leq \frac{1}{r} + \frac{k}{2}. \tag{2.12}$$

Integrating both sides of (2.12), we obtain

$$rF(k,-r) \leq |f(z)| \leq rF(k,r), \tag{2.13}$$

where $F(k,r) = e^{\frac{kr}{2}}$. The inequality in (2.13) is sharp because extremal function is

$$f(z) = ze^{\frac{kz}{2}}.$$

Theorem 2.2. *For $k \geq 2$ and $q \in (0,1)$, starlikeness of the class $\mathcal{R}_k(q)$ is*

$$r^*(f) = \frac{k(1+q) - \sqrt{k^2(1+q)^2 - 16q}}{4q}. \tag{2.14}$$

Proof. Let $f \in \mathcal{A}$, then the real number

$$r^*(f) = \sup\left\{r > 0, Re(z\frac{f'(z)}{f(z)}) > 0 \quad \text{for all} \quad z \in \mathbb{D}\right\}$$

is called the starlikeness of the class $\mathcal{A}$. Then the inequality in (2.10) gives the starlikeness of the class $\mathcal{R}_k(q)$, that is

$$Re\left(z\frac{f'(z)}{f(z)}\right) \geq \frac{1 - \frac{k}{2}(1+q)r + qr^2}{(1-qr)(1+qr)}.$$

Hence for $r < r^*$ the right side of the preceding inequality is positive if

$$r^*(f) = \frac{k(1+q) - \sqrt{k^2(1+q)^2 - 16q}}{4q}.$$

□

Remark 2.1. If $q \to 1^-$, then radius in (2.14) reduces to $r^*(f) = \frac{k-\sqrt{k^2-4}}{2}$. This is the radius of starlikeness of the class $\mathcal{R}_k$ which was obtained by Pinchuk [4].

We now prove coefficient inequality for the class $\mathcal{R}_k(q)$. For our main theorem, we need the following two lemmas.

Lemma 2.2. [1] *If p is an element of $\mathcal{P}_q$, then $|p_n| \leq 1+q$ for all $n \geq 1$. This result is sharp.*

Lemma 2.3. *Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be an element of $\mathcal{P}_k(q)$, then*

$$|p_n| \leq \frac{k}{2}(1+q)$$

for all $n \geq 1$, $k \geq 2$ and $q \in (0,1)$.This result is sharp for the functions

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(2)}(z),$$

where $p_1^{(1)}, p_2^{(2)} \in \mathcal{P}_q$.

Proof. Let $p_1^{(1)} = 1 + a_1 z + a_2 z^2 + \ldots$ and $p_2^{(2)} = 1 + b_1 z + b_2 z^2 + \ldots$. Since $p \in \mathcal{P}_k(q)$, then we have

$$\begin{aligned} p(z) &= \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(2)}(z) \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)(1 + a_1 z + a_2 z^2 + \ldots) - \left(\frac{k}{4} - \frac{1}{2}\right)(1 + b_1 z + b_2 z^2 + \ldots). \end{aligned}$$

Then, for n th term, we have

$$p_n = \left(\frac{k}{4} + \frac{1}{2}\right) a_n - \left(\frac{k}{4} - \frac{1}{2}\right) b_n.$$

Taking into account Lemma 2.2, $|a_n| \leq 1+q$ and $|b_n| \leq 1+q$ for all $n \geq 1$. Therefore

$$\begin{aligned} |p_n| &= \left| \left(\frac{k}{4}+\frac{1}{2}\right) a_n - \left(\frac{k}{4}-\frac{1}{2}\right) b_n \right| \\ &\leq \left(\frac{k}{4}+\frac{1}{2}\right) |a_n| + \left(\frac{k}{4}-\frac{1}{2}\right) |b_n| \\ &\leq \left(\frac{k}{4}+\frac{1}{2}\right) (1+q) + \left(\frac{k}{4}-\frac{1}{2}\right) (1+q). \end{aligned}$$

This shows that,

$$|p_n| \leq \frac{k}{2}(1+q)$$

for all $n \geq 1$, $k \geq 2$ and $q \in (0,1)$. □

Theorem 2.3. *If $f \in \mathcal{R}_k(q)$, then*

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{\nu=0}^{n-2} \left(\nu + \frac{k}{2}(1+q)\right). \tag{2.15}$$

This inequality is sharp for every $n \geq 2$, $k \geq 2$ and $q \in (0,1)$.

Proof. In view of definition of the class $\mathcal{R}_k(q)$ and subordination principle, we can write

$$z\frac{f'(z)}{f(z)} = p(z),$$

where $p \in \mathcal{P}_k(q)$ with $p(0)=1$. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, then we have

$$zf'(z) = f(z)p(z).$$

Therefore,

$$z + 2a_2z^2 + 3a_3z^3 + \ldots = z + (a_2+p_1)z^2 + (a_3+p_1a_2+p_2)z^3 + (a_4+p_1a_3+p_2a_2+p_3)z^4 + \ldots \quad .$$

Comparing the coefficients of z^n on both sides, we obtain

$$na_n = a_n + p_1a_{n-1} + p_2a_{n-2} + \ldots + p_{n-2}a_2 + p_{n-1}$$

for all integer $n \geq 2$. In view of Lemma 2.3, we get

$$(n-1)|a_n| \leq \frac{k}{2}(1+q)(|a_{n-1}| + \ldots + |a_2| + 1),$$

or equivalently

$$|a_n| \leq \frac{1}{(n-1)}\frac{k}{2}(1+q)\sum_{\nu=1}^{n-1}|a_\nu|, \quad |a_1| = 1. \tag{2.16}$$

Induction shows that we have

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{\nu=0}^{n-2} \left(\nu + \frac{k}{2}(1+q)\right).$$

This estimate is sharp because extremal function is

$$z\frac{f'(z)}{f(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+z}{1-qz} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-z}{1+qz}$$

which gives

$$f(z) = \frac{z\big(1+qz\big)^{\left(\frac{k}{4}-\frac{1}{2}\right)\frac{(1+q)}{q}}}{\big(1-qz\big)^{\left(\frac{k}{4}+\frac{1}{2}\right)\frac{(1+q)}{q}}}.$$

□

Remark 2.2. Taking $q \to 1^-$ and choosing $k = 2$ in (2.15), we get $|a_n| \leq n$ for every $n \geq 2$. This result is the well known coefficient inequality for starlike functions.

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