Some properties of the analytic functions with bounded radius rotation

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ABSTRACT. In the present paper, we introduce a new subclass of normalized analytic starlike functions by using bounded radius rotation associated with $q-$ analogues in the open unit disc $D$. We investigate growth theorem, radius of starlikeness and coefficient estimate for the new subclass of starlike functions by using bounded radius rotation associated with $q-$ analogues denoted by $R_k(q)$, where $k \geq 2$, $q \in (0, 1)$.

1. INTRODUCTION

Let $A$ be the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $D = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = 0$, $f'(0) = 1$ for every $z \in D$. We say that $f_1$ is subordinate to $f_2$, written as $f_1 \prec f_2$, if there exists a Schwarz function $\phi$ which is analytic in $D$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, such that $f_1(z) = f_2(\phi(z))$. In particular, when $f_2$ is univalent, then the above subordination is equivalent to $f_1(0) = f_2(0)$ and $f_1(D) \subset f_2(D)$ (Subordination principle [3]).

In 1971, Pinchuk [4] introduced and studied the classes $P_k$ and $R_k$, where $R_k$ generalizes the class of starlike functions. Here $P_k$ denotes the class of functions $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, analytic in $D$ with $p(0) = 1$ and having the representation

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

where $\mu$ is real-valued function of bounded variation for which

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k.$$

The class $R_k$, defined by Pinchuk in [4], consists of those functions $f$ which satisfy the condition

$$\int_{-\pi}^{\pi} \left| \text{Re} \left( \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right) \right| d\theta \leq k \pi, 0 < r < 1, z = re^{i\theta}.$$

Geometrically, the condition (1.2) is the total variation of the angle between radius vector $f(re^{i\theta})$ makes with the positive real axis is bounded by $k \pi$. Thus $R_k$ is the class of bounded radius rotation bounded by $k \pi$.
Denote by $\mathcal{P}_q$ the family of functions $p$ of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, analytic $\mathbb{D}$ and satisfy the condition
\[
\left| p(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q},
\]
where $q \in (0, 1)$ is a fixed real number.

The following lemma is first introduced in [6], later given in [2]:

**Lemma 1.1.** $p$ is an element of $\mathcal{P}_q$ if and only if $p(z) \prec \frac{1+z}{1-qz}$. This result is sharp for the functions $p(z) = \frac{1+\phi(z)}{1-q\phi(z)}$, where $\phi$ is a Schwarz function.

Using the definitions $\mathcal{P}_k$ and $\mathcal{P}_q$, Noor and Noor introduced the class $\mathcal{P}_k(q)$ in [5] as below:

**Definition 1.1.** A function $p$ of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, analytic in $\mathbb{D}$ with $p(0) = 1$ is said to be in the class $\mathcal{P}_k(q)$, $k \geq 2$, $q \in (0, 1)$ if and only if there exists $p_1^{(1)}, p_2^{(2)} \in \mathcal{P}_q$ such that
\[
p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(2)}(z).
\]

For $q \to 1^-$, $\mathcal{P}_k(q)$ reduces to $\mathcal{P}_k$, (see [4]); for $k = 2$, $\mathcal{P}_k(q)$ reduces to $\mathcal{P}_q$; for $k = 2$, $q \to 1^-$, $\mathcal{P}_k(q)$ reduces to $\mathcal{P}$ which is the well known class of functions with positive real part.

In the present paper, we give a new subclass of starlike functions with bounded radius rotation associated with $q$– analogues denoted by $R_k(q)$.

**Definition 1.2.** Let $f$ of the form (1.1) be an element of $\mathcal{A}$. If $f$ satisfies the condition
\[
z \frac{f'(z)}{f(z)} = p(z), \quad p \in \mathcal{P}_k(q),
\]
with $k \geq 2$, $q \in (0, 1)$, then $f$ is called starlike function with bounded radius rotation with $q$– analogues denoted by $R_k(q)$.

Motivated by Definition 1.2, we investigate growth theorem, radius of starlikeness and coefficient inequality for the class $R_k(q)$.

2. Main Results

We first give growth theorem for the class $R_k(q)$.

**Theorem 2.1.** If $f \in R_k(q)$, then
\[
r F(q, k, -r) \leq |f(z)| \leq r F(q, k, r),
\]
where
\[
F(q, k, r) = \left(\frac{1+qr}{1-qr}\right)^{\frac{k+1+q}{2}}.
\]

$k \geq 2$, $q \in (0, 1)$. 

Proof. Let \( p \) be an element of \( \mathcal{P}_q \) and \(|z| = r < 1\), then by Lemma 1.1 we have

\[
\frac{1-r}{1+qr} \leq \text{Rep}(z) \leq |p(z)| \leq \frac{1+r}{1-qr}.
\]

After simple calculations in (2.6), we get

\[
\frac{1 - \frac{k}{2}(1+q)r + qr^2}{(1-qr)(1+qr)} \leq \text{Rep}(z) \leq \frac{1 + \frac{k}{2}(1+q)r + qr^2}{(1-qr)(1+qr)}.
\]

Inequality in (2.7) shows that the set of variability of \( p \in \mathcal{P}_k(q) \) is the closed disc

\[
\left|p(z) - \frac{1 + qr^2}{1 - q^2r^2}\right| \leq \frac{k}{2}(1+q)r
\]

On the other hand from definition of \( \mathcal{R}_k(q) \), we can write

\[
\left|zf'(z)f(z) - \frac{1 + qr^2}{1 - q^2r^2}\right| \leq \frac{k}{2}(1+q)r,
\]

which gives

\[
\frac{1 - \frac{k}{2}(1+q)r + qr^2}{r(1-qr)(1+qr)} \leq \text{Re} z f'(z)f(z) \leq \frac{1 + \frac{k}{2}(1+q)r + qr^2}{r(1-qr)(1+qr)}.
\]

Taking integration on both sides of (2.11), we obtain

\[
r F(q, k, -r) \leq |f(z)| \leq r F(q, k, r),
\]

where

\[
F(q, k, r) = \frac{(1+q)\left(\frac{q}{q - 1}\right)^{\frac{1+q}{q}}}{(1-qr)\left(\frac{q}{q + 1}\right)^{\frac{1+q}{q}}},
\]

This estimate is sharp because extremal function is

\[
f(z) = \frac{z(1+qz)^{\left(\frac{q}{q - 1}\right)^{\frac{1+q}{q}}}}{(1-qr)\left(\frac{q}{q + 1}\right)^{\frac{1+q}{q}}}.
\]

\[\square\]

Corollary 2.1. If we take \( q = 0 \) in (2.9), we obtain

\[
\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{kr}{2},
\]

which gives

\[
\frac{1}{r} - \frac{k}{2} \leq \frac{\partial}{\partial r} \log|f(z)| \leq \frac{1}{r} + \frac{k}{2}.
\]

Integrating both sides of (2.12), we obtain

\[
r F(k, -r) \leq |f(z)| \leq r F(k, r),
\]
where $F(k, r) = e^{kr}$. The inequality in (2.13) is sharp because extremal function is $f(z) = ze^{kr}$.

**Theorem 2.2.** For $k \geq 2$ and $q \in (0, 1)$, starlikeness of the class $R_k(q)$ is

$$r^*(f) = \frac{k(1 + q) - \sqrt{k^2(1 + q)^2 - 16q}}{4q}. \quad (2.14)$$

**Proof.** Let $f \in A$, then the real number

$$r^*(f) = \sup \left\{ r > 0 \mid \Re \left( z \frac{f'(z)}{f(z)} \right) > 0 \text{ for all } z \in \mathbb{D} \right\}$$

is called the starlikeness of the class $A$. Then the inequality in (2.10) gives the starlikeness of the class $R_k(q)$, that is

$$\Re \left( z \frac{f'(z)}{f(z)} \right) \geq 1 - \frac{k}{2} (1 + q)r + qr^2.$$ 

Hence for $r < r^*$ the right side of the preceding inequality is positive if

$$r^*(f) = \frac{k(1 + q) - \sqrt{k^2(1 + q)^2 - 16q}}{4q}.$$ 

$\square$

**Remark 2.1.** If $q \to 1^-$, then radius in (2.14) reduces to $r^*(f) = \frac{k - \sqrt{k^2 - 4}}{2}$. This is the radius of starlikeness of the class $R_k$ which was obtained by Pinchuk [4].

We now prove coefficient inequality for the class $R_k(q)$. For our main theorem, we need the following two lemmas.

**Lemma 2.2.** [1] If $p$ is an element of $P_q$, then $|p_n| \leq 1 + q$ for all $n \geq 1$. This result is sharp.

**Lemma 2.3.** Let $p(z) = 1 + p_1z + p_2z^2 + \ldots$ be an element of $P_k(q)$, then

$$|p_n| \leq \frac{k}{2} (1 + q)$$

for all $n \geq 1$, $k \geq 2$ and $q \in (0, 1).$ This result is sharp for the functions

$$p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z),$$

where $p_1^{(1)}, p_2^{(2)} \in P_q$.

**Proof.** Let $p_1^{(1)} = 1 + a_1z + a_2z^2 + \ldots$ and $p_2^{(2)} = 1 + b_1z + b_2z^2 + \ldots$. Since $p \in P_k(q)$, then we have

$$p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1^{(1)}(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2^{(2)}(z)$$

$$= \left( \frac{k}{4} + \frac{1}{2} \right) (1 + a_1z + a_2z^2 + \ldots) - \left( \frac{k}{4} - \frac{1}{2} \right) (1 + b_1z + b_2z^2 + \ldots).$$

Then, for $n$th term, we have

$$p_n = \left( \frac{k}{4} + \frac{1}{2} \right) a_n - \left( \frac{k}{4} - \frac{1}{2} \right) b_n.$$
Taking into account Lemma 2.2, \(|a_n| \leq 1 + q\) and \(|b_n| \leq 1 + q\) for all \(n \geq 1\). Therefore

\[
|p_n| = \left| \left( \frac{k}{4} + \frac{1}{2} \right) a_n - \left( \frac{k}{4} - \frac{1}{2} \right) b_n \right|
\leq \left( \frac{k}{4} + \frac{1}{2} \right) |a_n| + \left( \frac{k}{4} - \frac{1}{2} \right) |b_n|
\leq \left( \frac{k}{4} + \frac{1}{2} \right) (1 + q) + \left( \frac{k}{4} - \frac{1}{2} \right) (1 + q).
\]

This shows that,

\[
|p_n| \leq \frac{k}{2} (1 + q)
\]

for all \(n \geq 1, k \geq 2\) and \(q \in (0, 1)\).

**Theorem 2.3.** If \(f \in \mathcal{R}_k(q)\), then

\[
|a_n| \leq \frac{1}{(n-1)!} \prod_{\nu=0}^{n-2} \left( \nu + \frac{k}{2} (1 + q) \right).
\]

This inequality is sharp for every \(n \geq 2, k \geq 2\) and \(q \in (0, 1)\).

**Proof.** In view of definition of the class \(\mathcal{R}_k(q)\) and subordination principle, we can write

\[
z f'(z) = p(z),
\]

where \(p \in \mathcal{P}_k(q)\) with \(p(0) = 1\). Since \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\) and \(p(z) = 1 + p_1 z + p_2 z^2 + \ldots\), then we have

\[
z f'(z) = f(z) p(z).
\]

Therefore,

\[
z + 2 a_2 z^2 + 3 a_3 z^3 + \ldots = z + (a_2 + p_1) z^2 + (a_3 + p_1 a_2 + p_2) z^3 + (a_4 + p_1 a_3 + p_2 a_2 + p_3) z^4 + \ldots
\]

Comparing the coefficients of \(z^n\) on both sides, we obtain

\[
n a_n = a_n + p_1 a_{n-1} + p_2 a_{n-2} + \ldots + p_{n-2} a_2 + p_{n-1}
\]

for all integer \(n \geq 2\). In view of Lemma 2.3, we get

\[
(n-1) |a_n| \leq \frac{k}{2} (1 + q) (|a_{n-1}| + \ldots + |a_2| + 1),
\]

or equivalently

\[
|a_n| \leq \frac{1}{(n-1)!} \frac{k}{2} (1 + q) \sum_{\nu=1}^{n-1} |a_{\nu}|, \quad |a_1| = 1.
\]

Induction shows that we have

\[
|a_n| \leq \frac{1}{(n-1)!} \prod_{\nu=0}^{n-2} \left( \nu + \frac{k}{2} (1 + q) \right).
\]

This estimate is sharp because extremal function is

\[
z f'(z) = \left( \frac{k}{4} + \frac{1}{2} \right) \frac{1 + z}{1 - qz} - \left( \frac{k}{4} - \frac{1}{2} \right) \frac{1 - z}{1 + qz}
\]

which gives

\[
f(z) = \frac{z(1 + q z)^{(\frac{k}{4} - \frac{1}{2}) (1 + q)}}{(1 - q z)^{(\frac{k}{4} + \frac{1}{2}) (1 + q)}}.
\]
Remark 2.2. Taking $q \to 1^-$ and choosing $k = 2$ in (2.15), we get $|a_n| \leq n$ for every $n \geq 2$. This result is the well known coefficient inequality for starlike functions.

REFERENCES