On the stability of two functional equations arising in mathematical biology and theory of learning

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ABSTRACT. In this paper, we prove the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the following two functional equations

\[ \varphi(x) = x\varphi((1-\alpha)x + \alpha) + (1-x)\varphi((1-\beta)x), \quad x \in [0,1], \quad 0 < \alpha \leq \beta < 1, \]

and

\[ \varphi(x) = x\varphi(f(x)) + (1-x)\varphi(g(x)), \quad x \in [0,1] \]


1. INTRODUCTION AND PRELIMINARIES

The stability problems of functional equations originated from the following question of Ulam [13] concerning the stability of group homomorphisms:

Let \((G_1, \ast)\) be a group and \((G_2, \cdot, d)\) be a metric group with the metric \(d(., .)\). Given \(\epsilon > 0\), does there exist a \(\delta > 0\) such that if a function \(f : G_1 \rightarrow G_2\) satisfies the inequality \(d(f(x \ast y), f(x) \cdot f(y)) < \delta\) for all \(x, y \in G_1\), then there exists a homomorphism \(h : G_1 \rightarrow G_2\) with \(d(f(x), h(x)) < \epsilon\) for all \(x \in G_1\)?

Hyers [7] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Indeed, he proved that each solution of the inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \epsilon, \]

for all \(x\) and \(y\), can be approximated by an exact solution, say an additive function. Hyers’s theorem was generalized by Aoki [1] for additive mappings and by Rassias [11] for linear mappings by considering an unbounded Cauchy difference. Rassias [11] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

\[ \|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \]

and derived Hyers’s theorem. The work of Rassias has influenced a number of mathematicians to develop the notion what is now a days referred to as Hyers-Ulam-Rassias stability of linear mappings. Since then, stability of other functional equations, differential equations, and of various integral equations has been extensively investigated by many mathematicians (cf. [2, 4, 5, 6, 8, 10, 12]).

Let \(E = \{ \varphi \in C[0,1] : \varphi(0) = 0, \varphi(1) = 1 \}\). Then the mapping

\[ \|\varphi\| = \sup_{t \neq s} \left| \frac{\varphi(t) - \varphi(s)}{|t - s|} \right|, \quad \varphi \in E, \]

is a norm on \(E\) and \((E, \|\|\)) is a Banach space (see [9]).
Lyubich and Shapiro [9] studied the existence and uniqueness of a continuous solution 
\( \varphi : [0,1] \to [0,1] \) of the following functional equation

\[
\varphi(x) = x\varphi((1 - \alpha)x + \alpha) + (1 - x)\varphi((1 - \beta)x), \quad x \in [0,1], \; 0 < \alpha \leq \beta < 1
\]
in the Banach space \( E \).

In 2015, Berinde and Khan [3] proved the following theorem for the more general functional equation by using Banach contraction mapping principle.

**Theorem 1.1.** (see [3, Theorem 2.2]) If \( f \) and \( g \) are contraction mappings on \([0,1]\) (endowed with usual norm) such that \( f(1) = 1 \) and \( g(0) = 0 \), with the contraction coefficients \( \alpha \) and \( \beta \), respectively, satisfying \( \alpha, \beta \in (0,1), \alpha \leq \beta \) and \( 2\alpha < 1 \), then

1) The functional equation

\[
\varphi(x) = x\varphi(f(x)) + (1 - x)\varphi(g(x)), \quad x \in [0,1]
\]

has a unique solution \( \varphi \) in \( E \).

2) The sequence of successive approximations \( \{\varphi_n\} \), defined by

\[
\varphi_{n+1}(x) = x\varphi_n(f(x)) + (1 - x)\varphi_n(g(x)), \quad x \in [0,1], \; n \geq 0
\]

converges strongly to \( \varphi \), as \( n \to \infty \), for any \( \varphi_0 \in E \).

3) The error estimate of \( \{\varphi_n\} \) is given by

\[
\|\varphi_{n+i-1} - \varphi\| \leq \frac{(2\alpha)^i}{1 - 2\alpha} \|\varphi_n - \varphi_{n-1}\|, \quad n = 1, 2, \ldots; \; i = 1, 2, \ldots
\]

4) The rate of convergence of the iterative method \( \{\varphi_n\} \) is linear, i.e.,

\[
\|\varphi_n - \varphi\| \leq 2\alpha \|\varphi_{n-1} - \varphi\|, \quad n = 1, 2, \ldots
\]

In [3], the authors left the stability problem of the two functional equations (1.1) and (1.2) as an open problem. The purpose of the paper is to solve it.

2. **Main Results**

In this section, we consider the complete metric space \((E, d)\) where

\[
d(\varphi, \psi) = \|\varphi - \psi\| = \sup_{t \neq s} \frac{|(\varphi - \psi)(t) - (\varphi - \psi)(s)|}{|t - s|} \quad \text{for all } \varphi, \psi \in E.
\]

We first prove that the functional equation (1.2) has the Hyers-Ulam stability.

**Theorem 2.2.** Under the assumptions of Theorem 1.1, the equation \( T\varphi = \varphi \), where \( T \) is defined by

\[
T : E \to C[0,1], \quad (T\varphi)(x) = x\varphi(f(x)) + (1 - x)\varphi(g(x))
\]

for \( x \in [0,1] \), has the Hyers-Ulam stability; that is, for every \( \varphi \in E \) and \( \epsilon > 0 \) with \( d(T\varphi, \varphi) \leq \epsilon \), there exists a unique \( \overline{\varphi} \in E \) such that

\[
T\overline{\varphi} = \overline{\varphi} \quad \text{and} \quad d(\varphi, \overline{\varphi}) \leq K\epsilon,
\]

for some \( K > 0 \).

**Proof.** Let \( \varphi \in E, \; \epsilon > 0 \) and \( d(T\varphi, \varphi) \leq \epsilon \). In the proof of Theorem 1.1, the authors showed that

\[
\overline{\varphi}(x) = \lim_{n \to \infty} T^n\varphi(x)
\]
is a exact solution of the equation $T\varphi = \varphi$. Since $T^n\varphi$ is uniformly convergent to $\varphi$ as $n \to \infty$, then there is a natural number $N$ such that $d(T^n\varphi, \varphi) \leq \epsilon$. Thus, we have

$$
\begin{align*}
d(\varphi, \varphi) &\leq d(T\varphi, \varphi) + d(T^n\varphi, \varphi) \\
&\leq d(\varphi, T\varphi) + d(T^2\varphi, T^3\varphi) + \ldots + d(T^n\varphi, \varphi) \\
&\leq d(\varphi, T\varphi) + 2ad(\varphi, T\varphi) + (2\alpha)^2d(\varphi, T\varphi) + \ldots + (2\alpha)^n-1d(\varphi, T\varphi) + d(T^n\varphi, \varphi) \\
&\leq d(\varphi, T\varphi)(1 + 2\alpha + (2\alpha)^2 + \ldots + (2\alpha)^n-1) + d(T^n\varphi, \varphi) \\
&\leq \epsilon.
\end{align*}
$$

This completes the proof. \hfill \Box

The following example shows validity of Theorem 2.2.

**Example 2.1.** Let $f$ and $g$ be defined by

$$
f(x) = \frac{x^2 + 5}{6}; \quad g(x) = \frac{x^2}{5}, \quad x \in [0, 1].
$$

Then we get

$$
|f(x) - f(y)| = \left| \frac{x^2 + 5}{6} - \frac{y^2 + 5}{6} \right| \\
= \frac{1}{6} |x^2 - y^2| = \frac{1}{6} |x - y| |x + y| \\
\leq \frac{2}{6} |x - y| = \frac{1}{3} |x - y|
$$

and

$$
|g(x) - g(y)| = \left| \frac{x^2}{5} - \frac{y^2}{5} \right| \\
= \frac{1}{5} |x^2 - y^2| = \frac{1}{5} |x - y| |x + y| \\
\leq \frac{2}{5} |x - y|
$$

for all $x, y \in [0, 1]$. Hence $f, g : [0, 1] \to [0, 1]$ are contraction mappings such that $f(1) = 1$ and $g(0) = 0$, with the contraction coefficients $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{5}$, respectively. Also, the conditions $\alpha, \beta \in (0, 1), \alpha \leq \beta$ and $2\alpha < 1$ are satisfied. On the other hand, we obtain

$$
K = \frac{2 - \frac{2}{3}}{1 - \frac{2}{3}} = 4.
$$

If a function $\varphi \in E$ satisfies the inequality

$$
d(T\varphi, \varphi) \leq \epsilon \quad \text{for some } \epsilon > 0,
$$

then Theorem 2.2 implies that there exists a unique $\varphi \in E$ such that

$$
T\varphi = \varphi \quad \text{and} \quad d(\varphi, \varphi) \leq 4\epsilon.
$$

We now prove the Hyers-Ulam-Rassias stability of the functional equation (1.2).
Theorem 2.3. Under the assumptions of Theorem 1.1, the equation \( T\varphi = \varphi \), where \( T \) is defined by

\[
T : E \to C[0,1], \quad (T\varphi)(x) = x\varphi(f(x)) + (1-x)\varphi(g(x))
\]

for \( x \in [0,1] \), has the Hyers-Ulam-Rassias stability; that is, for every \( \varphi \in E \) and \( \sigma(x) > 0 \) for all \( x \in [0,1] \) with \( d(T\varphi, \varphi) \leq \sigma(x) \), there exists a unique \( \varphi \in E \) such that

\[
T\varphi = \varphi \quad \text{and} \quad d(\varphi, \varphi) \leq K_1\sigma(x)
\]

for some \( K_1 > 0 \).

Proof. Let \( \varphi \in E, \sigma \) be a non-negative function on \( [0,1] \) such that \( d(T\varphi, \varphi) \leq \sigma(x) \), and let \( \varphi \in E \) be the unique solution of the functional equation (1.2) on \( E \). Then, we have

\[
d(\varphi, \varphi) \leq d(\varphi, T\varphi) + d(T\varphi, \varphi) \\
\leq \sigma(x) + d(T\varphi, \varphi).
\]

(2.3)

Also, we obtain

\[
d(T\varphi, \varphi) = d(T\varphi, T\varphi) \leq 2\alpha d(\varphi, \varphi).
\]

(2.4)

Combining (2.3) and (2.4), we get

\[
d(\varphi, \varphi) \leq \sigma(x) + 2\alpha d(\varphi, \varphi)
\]

which implies that

\[
d(\varphi, \varphi) \leq K_1\sigma(x)
\]

with \( K_1 = \frac{1}{1-2\alpha} \). Hence, the functional equation (1.2) has the Hyers-Ulam-Rassias stability. \( \square \)

Next, we give an example to support Theorem 2.3.

Example 2.2. Let \( f : [0,1] \to [0,1] \) be given by

\[
f(x) = -\frac{1}{5}x + 1, \quad x \in \left[0, \frac{1}{2}\right) \quad \text{and} \quad f(x) = \frac{1}{5}x + \frac{4}{5}, \quad x \in \left[\frac{1}{2}, 1\right].
\]

To verify that \( f \) is contraction mapping with the contraction coefficient \( \alpha = \frac{1}{5} \), consider the following cases:

Case I: Let \( x, y \in \left[0, \frac{1}{2}\right) \), then

\[
|f(x) - f(y)| = \left| -\frac{1}{5}x + 1 - \left(-\frac{1}{5}y + 1\right) \right| = \frac{1}{5} |x - y|.
\]

Case II: Let \( x, y \in \left[\frac{1}{2}, 1\right] \), then

\[
|f(x) - f(y)| = \left| \frac{1}{5}x + \frac{4}{5} - \left(\frac{1}{5}y + \frac{4}{5}\right) \right| = \frac{1}{5} |x - y|.
\]

Case III: Let \( x \in \left[0, \frac{1}{2}\right) \) and \( y \in \left[\frac{1}{2}, 1\right] \), then

\[
|f(x) - f(y)| = \left| -\frac{1}{5}x + 1 - \left(\frac{1}{5}y + \frac{4}{5}\right) \right| = \frac{1}{5} |x + y - 1|.
\]

For \( |f(x) - f(y)| \leq \frac{1}{5} |x - y| \), we must have \( |x + y - 1| \leq y - x \), i.e. \( 2x \leq 1 \leq 2y \), which is obviously true.

Case IV: Let \( x \in \left[\frac{1}{2}, 1\right] \) and \( y \in \left[0, \frac{1}{2}\right) \), then

\[
|f(x) - f(y)| = \left| \frac{1}{5}x + \frac{4}{5} - \left(-\frac{1}{5}y + 1\right) \right| = \frac{1}{5} |x + y - 1|.
\]
For $|f(x) - f(y)| \leq \frac{1}{5} |x - y|$, we must have $|x + y - 1| \leq x - y$, i.e., $2y \leq 1 \leq 2x$, which is obviously true.

Let now $g : [0, 1] \rightarrow [0, 1]$ be given by

$$g(x) = \frac{4}{5} x, \quad x \in \left[0, \frac{1}{2}\right] \quad \text{and} \quad g(x) = -\frac{4}{5} x + \frac{4}{5}, \quad x \in \left[\frac{1}{2}, 1\right].$$

Similarly, it can be shown that $g$ is contraction mapping with the contraction coefficient $\beta = \frac{4}{5}$. On the other hand, the contraction coefficients $\alpha$ and $\beta$ satisfy the conditions of Theorem 1.1. Also, we get

$$K_1 = \frac{1}{1 - \frac{2}{5}} = \frac{5}{3}.$$

If a function $\varphi \in E$ satisfies the inequality

$$d(T\varphi, \varphi) \leq \sigma(x) \quad \text{for some } \sigma(x) > 0,$$

then Theorem 2.3 implies that there exists a unique $\overline{\varphi} \in E$ such that

$$T\overline{\varphi} = \overline{\varphi} \quad \text{and} \quad d(\varphi, \overline{\varphi}) \leq \frac{5}{3} \sigma(x).$$

If we take $f(x) = (1 - \alpha)x + \alpha$ and $g(x) = (1 - \beta)x$ for each $x \in [0, 1]$ in Theorem 2.2 and Theorem 2.3, we get the following corollary.

**Corollary 2.1.** If $f$ and $g$ are given by $f(x) = (1 - \alpha)x + \alpha$ and $g(x) = (1 - \beta)x$ in (1.2), then the functional equation (1.1) has the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability.

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