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On the stability of two functional equations arising in mathematical biology and theory of learning

AYNUR ŞAHIN, HAKAN ARISOY and ZEYNEP KALKAN

ABSTRACT. In this paper, we prove the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of the following two functional equations

$$\varphi(x) = x\varphi((1-\alpha)x+\alpha) + (1-x)\varphi((1-\beta)x),\ x \in [0,1],\ 0 < \alpha \leq \beta < 1,$$

and

$$\varphi(x) = x\varphi(f(x)) + (1-x)\varphi(g(x)),\ x \in [0,1]$$

which is an open problem raised by Berinde and Khan [Berinde, V. and Khan, A. R., *On a functional equation arising in mathematical biology and theory of learning*, Creat. Math. Inform., **24** (2015), No. 1, 9–16].

1. INTRODUCTION AND PRELIMINARIES

The stability problems of functional equations originated from the following question of Ulam [13] concerning the stability of group homomorphisms:

Let $(G_1, \star)$ be a group and $(G_2, ., d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(x \star y), f(x).f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

Hyers [7] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Indeed, he proved that each solution of the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon,$$

for all x and y, can be approximated by an exact solution, say an additive function. Hyers's theorem was generalized by Aoki [1] for additive mappings and by Rassias [11] for linear mappings by considering an unbounded Cauchy difference. Rassias [11] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

and derived Hyers's theorem. The work of Rassias has influenced a number of mathematicians to develop the notion what is now a days referred to as Hyers-Ulam-Rassias stability of linear mappings. Since then, stability of other functional equations, differential equations, and of various integral equations has been extensively investigated by many mathematicians (cf. [2, 4, 5, 6, 8, 10, 12]).

Let $E = \{\varphi \in C[0,1] : \varphi(0) = 0, \varphi(1) = 1\}$. Then the mapping

$$\|\varphi\| = \sup_{t \neq s} \frac{|\varphi(t) - \varphi(s)|}{|t-s|}, \quad \varphi \in E,$$

is a norm on E and $(E, \|.\|)$ is a Banach space (see [9]).

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Corresponding author: Aynur Şahin; ayuce@sakarya.edu.tr

Lyubich and Shapiro [9] studied the existence and uniqueness of a continuous solution $\varphi : [0,1] \to [0,1]$ of the following functional equation

$$\varphi(x) = x\varphi((1-\alpha)x + \alpha) + (1-x)\varphi((1-\beta)x), \quad x \in [0,1],\ 0 < \alpha \leq \beta < 1 \tag{1.1}$$

in the Banach space E.

In 2015, Berinde and Khan [3] proved the following theorem for the more general functional equation by using Banach contraction mapping principle.

Theorem 1.1. (see [3, Theorem 2.2]) *If f and g are contraction mappings on $[0,1]$ (endowed with usual norm) such that $f(1) = 1$ and $g(0) = 0$, with the contraction coefficients α and β, respectively, satisfying $\alpha, \beta \in (0,1), \alpha \leq \beta$ and $2\alpha < 1$, then*

1) The functional equation

$$\varphi(x) = x\varphi(f(x)) + (1-x)\varphi(g(x)), \quad x \in [0,1] \tag{1.2}$$

has a unique solution $\overline{\varphi}$ in E.

2) The sequence of successive approximations $\{\varphi_n\}$, defined by

$$\varphi_{n+1}(x) = x\varphi_n(f(x)) + (1-x)\varphi_n(g(x)), \quad x \in [0,1],\ n \geq 0$$

converges strongly to $\overline{\varphi}$, as $n \to \infty$, for any $\varphi_0 \in E$.

3) The error estimate of $\{\varphi_n\}$ is given by

$$\|\varphi_{n+i-1} - \overline{\varphi}\| \leq \frac{(2\alpha)^i}{1-2\alpha} \|\varphi_n - \varphi_{n-1}\|, \quad n = 1,2,...;\ i = 1,2,...$$

4) The rate of convergence of the iterative method $\{\varphi_n\}$ is linear, i.e.,

$$\|\varphi_n - \overline{\varphi}\| \leq 2\alpha \|\varphi_{n-1} - \overline{\varphi}\|, \quad n = 1,2,...$$

In [3], the authors left the stability problem of the two functional equations (1.1) and (1.2) as an open problem. The purpose of the paper is to solve it.

2. Main results

In this section, we consider the complete metric space (E,d) where

$$d(\varphi,\psi) = \|\varphi - \psi\| = \sup_{t \neq s} \frac{|(\varphi-\psi)(t) - (\varphi-\psi)(s)|}{|t-s|} \quad \text{for all } \varphi, \psi \in E.$$

We first prove that the functional equation (1.2) has the Hyers-Ulam stability.

Theorem 2.2. *Under the assumptions of Theorem 1.1, the equation $T\varphi = \varphi$, where T is defined by*

$$T : E \to C[0,1],\ \ (T\varphi)(x) = x\varphi(f(x)) + (1-x)\varphi(g(x))$$

for $x \in [0,1]$, has the Hyers-Ulam stability; that is, for every $\varphi \in E$ and $\epsilon > 0$ with $d(T\varphi,\varphi) \leq \epsilon$, there exists a unique $\overline{\varphi} \in E$ such that

$$T\overline{\varphi} = \overline{\varphi} \quad \textit{and} \quad d(\varphi, \overline{\varphi}) \leq K\epsilon,$$

for some $K > 0$.

Proof. Let $\varphi \in E, \epsilon > 0$ and $d(T\varphi,\varphi) \leq \epsilon$. In the proof of Theorem 1.1, the authors showed that

$$\overline{\varphi}(x) = \lim_{n\to\infty} T^n \varphi(x)$$

is a exact solution of the equation $T\varphi = \varphi$. Since $T^n\varphi$ is uniformly convergent to $\overline{\varphi}$ as $n \to \infty$, then there is a natural number N such that $d(T^n\varphi, \overline{\varphi}) \leq \epsilon$. Thus, we have

$$\begin{aligned}
& d(\varphi, \overline{\varphi}) \\
\leq\ & d(\varphi, T^n\varphi) + d(T^n\varphi, \overline{\varphi}) \\
\leq\ & d(\varphi, T\varphi) + d(T\varphi, T^2\varphi) + d(T^2\varphi, T^3\varphi) + ... + d(T^{n-1}\varphi, T^n\varphi) + d(T^n\varphi, \overline{\varphi}) \\
\leq\ & d(\varphi, T\varphi) + 2\alpha d(\varphi, T\varphi) + (2\alpha)^2 d(\varphi, T\varphi) + ... + (2\alpha)^{n-1} d(\varphi, T\varphi) + d(T^n\varphi, \overline{\varphi}) \\
\leq\ & d(\varphi, T\varphi)(1 + 2\alpha + (2\alpha)^2 + ... + (2\alpha)^{n-1}) + \epsilon \\
\leq\ & \epsilon.\frac{1}{1-2\alpha} + \epsilon = \left(\frac{2-2\alpha}{1-2\alpha}\right)\epsilon.
\end{aligned}$$

This completes the proof. □

The following example shows validity of Theorem 2.2.

Example 2.1. Let f and g be defined by

$$f(x) = \frac{x^2+5}{6}; \ \ g(x) = \frac{x^2}{5}, \quad x \in [0,1].$$

Then we get

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{x^2+5}{6} - \frac{y^2+5}{6}\right| \\
&= \frac{1}{6}\left|x^2 - y^2\right| = \frac{1}{6}|x-y|\,|x+y| \\
&\leq \frac{2}{6}|x-y| = \frac{1}{3}|x-y|
\end{aligned}$$

and

$$\begin{aligned}
|g(x) - g(y)| &= \left|\frac{x^2}{5} - \frac{y^2}{5}\right| \\
&= \frac{1}{5}\left|x^2 - y^2\right| = \frac{1}{5}|x-y|\,|x+y| \\
&\leq \frac{2}{5}|x-y|
\end{aligned}$$

for all $x, y \in [0,1]$. Hence $f, g : [0,1] \to [0,1]$ are contraction mappings such that $f(1) = 1$ and $g(0) = 0$, with the contraction coefficients $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{5}$, respectively. Also, the conditions $\alpha, \beta \in (0,1), \alpha \leq \beta$ and $2\alpha < 1$ are satisfied. On the other hand, we obtain

$$K = \frac{2 - \frac{2}{3}}{1 - \frac{2}{3}} = 4.$$

If a function $\varphi \in E$ satisfies the inequality

$$d(T\varphi, \varphi) \leq \epsilon \quad \text{for some } \epsilon > 0,$$

then Theorem 2.2 implies that there exists a unique $\overline{\varphi} \in E$ such that

$$T\overline{\varphi} = \overline{\varphi} \quad \text{and} \quad d(\varphi, \overline{\varphi}) \leq 4\epsilon.$$

We now prove the Hyers-Ulam-Rassias stability of the functional equation (1.2).

Theorem 2.3. *Under the assumptions of Theorem 1.1, the equation $T\varphi = \varphi$, where T is defined by*

$$T : E \to C[0,1], \quad (T\varphi)(x) = x\varphi(f(x)) + (1-x)\varphi(g(x))$$

for $x \in [0,1]$, has the Hyers-Ulam-Rassias stability; that is, for every $\varphi \in E$ and $\sigma(x) > 0$ for all $x \in [0,1]$ with $d(T\varphi, \varphi) \le \sigma(x)$, there exists a unique $\overline{\varphi} \in E$ such that

$$T\overline{\varphi} = \overline{\varphi} \quad \textit{and} \quad d(\varphi, \overline{\varphi}) \le K_1\sigma(x),$$

for some $K_1 > 0$.

Proof. Let $\varphi \in E$, σ be a non-negative function on $[0,1]$ such that $d(T\varphi, \varphi) \le \sigma(x)$, and let $\overline{\varphi} \in E$ be the unique solution of the functional equation (1.2) on E. Then, we have

$$\begin{aligned} d(\varphi, \overline{\varphi}) &\le d(\varphi, T\varphi) + d(T\varphi, \overline{\varphi}) \\ &\le \sigma(x) + d(T\varphi, \overline{\varphi}). \end{aligned} \tag{2.3}$$

Also, we obtain

$$d(T\varphi, \overline{\varphi}) = d(T\varphi, T\overline{\varphi}) \le 2\alpha d(\varphi, \overline{\varphi}). \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$d(\varphi, \overline{\varphi}) \le \sigma(x) + 2\alpha d(\varphi, \overline{\varphi})$$

which implies that

$$d(\varphi, \overline{\varphi}) \le K_1\sigma(x)$$

with $K_1 = \dfrac{1}{1-2\alpha}$. Hence, the functional equation (1.2) has the Hyers-Ulam-Rassias stability. □

Next, we give an example to support Theorem 2.3.

Example 2.2. Let $f : [0,1] \to [0,1]$ be given by

$$f(x) = -\frac{1}{5}x + 1, \;\; x \in \left[0, \frac{1}{2}\right) \;\text{ and } f(x) = \frac{1}{5}x + \frac{4}{5}, \;\; x \in \left[\frac{1}{2}, 1\right].$$

To verify that f is contraction mapping with the contraction coefficient $\alpha = \dfrac{1}{5}$, consider the following cases:

Case I: Let $x, y \in \left[0, \frac{1}{2}\right)$, then

$$|f(x) - f(y)| = \left| -\frac{1}{5}x + 1 - \left(-\frac{1}{5}y + 1 \right) \right| = \frac{1}{5}|x-y|.$$

Case II: Let $x, y \in \left[\frac{1}{2}, 1\right]$, then

$$|f(x) - f(y)| = \left| \frac{1}{5}x + \frac{4}{5} - \left(\frac{1}{5}y + \frac{4}{5} \right) \right| = \frac{1}{5}|x-y|.$$

Case III: Let $x \in \left[0, \frac{1}{2}\right)$ and $y \in \left[\frac{1}{2}, 1\right]$, then

$$|f(x) - f(y)| = \left| -\frac{1}{5}x + 1 - \left(\frac{1}{5}y + \frac{4}{5} \right) \right| = \frac{1}{5}|x+y-1|.$$

For $|f(x) - f(y)| \le \frac{1}{5}|x-y|$, we must have $|x+y-1| \le y - x$, i.e, $2x \le 1 \le 2y$, which is obviously true.

Case IV: Let $x \in \left[\frac{1}{2}, 1\right]$ and $y \in \left[0, \frac{1}{2}\right)$, then

$$|f(x) - f(y)| = \left| \frac{1}{5}x + \frac{4}{5} - \left(-\frac{1}{5}y + 1 \right) \right| = \frac{1}{5}|x+y-1|.$$

For $|f(x)-f(y)|\leq \frac{1}{5}|x-y|$, we must have $|x+y-1|\leq x-y$, i.e, $2y\leq 1\leq 2x$, which is obviously true.

Let now $g:[0,1]\to[0,1]$ be given by

$$g(x)=\frac{4}{5}x,\ \ x\in\left[0,\frac{1}{2}\right)\ \text{ and } g(x)=-\frac{4}{5}x+\frac{4}{5},\ \ x\in\left[\frac{1}{2},1\right].$$

Similarly, it can be shown that g is contraction mapping with the contraction coefficient $\beta=\dfrac{4}{5}$. On the other hand, the contraction coefficients α and β satisfy the conditions of Theorem 1.1. Also, we get

$$K_1=\frac{1}{1-\frac{2}{5}}=\frac{5}{3}.$$

If a function $\varphi\in E$ satisfies the inequality

$$d(T\varphi,\varphi)\leq\sigma(x)\quad\text{for some } \sigma(x)>0,$$

then Theorem 2.3 implies that there exists a unique $\overline{\varphi}\in E$ such that

$$T\overline{\varphi}=\overline{\varphi}\quad\text{and}\quad d(\varphi,\overline{\varphi})\leq\frac{5}{3}\sigma(x).$$

If we take $f(x)=(1-\alpha)x+\alpha$ and $g(x)=(1-\beta)x$ for each $x\in[0,1]$ in Theorem 2.2 and Theorem 2.3, we get the following corollary.

Corollary 2.1. *If f and g are given by $f(x)=(1-\alpha)x+\alpha$ and $g(x)=(1-\beta)x$ in (1.2), then the functional equation (1.1) has the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability.*

REFERENCES

[1] Aoki, T., *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Jpn., **2** (1950), 64–66

[2] Bae, J. H. and Park, W. G., *A fixed point approach to the stability of a Cauchy-Jensen functional equation*, Abst. Appl. Anal., **2012** (2012), Article ID 205160, 1–10

[3] Berinde, V. and Khan, A. R., *On a functional equation arising in mathematical biology and theory of learning*, Creat. Math. Inform., **24** (2015), No. 1, 9–16

[4] Cădariu, L. and Radu, V., *Fixed points and the stability of Jensen's functional equation*, JIPAM. J. Inequal. Pure Appl. Math., **4** (2003), No. 1, Article 4, 7 pp.

[5] Castro, L. P. and Guerra, R. C., *Hyers-Ulam-Rassias stability of Volterra integral equations within weighted spaces*, Libertas Math. (new series), **33** (2013), No. 2, 21–35

[6] Gachpazan, M. and Bagdani, O., *Hyers-Ulam stability of nonlinear integral equation*, Fixed Point Theory Appl., **2010** (2010), Article ID 927640, 1-6

[7] Hyers, D. H., *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci., USA, **27** (1941), 222–224

[8] Jung, S. Mo and Min, S., *A fixed point approach to the stability of the functional equation $f(x+y)=F[f(x),f(y)]$*, Fixed Point Theory Appl., **2009** (2009), Article ID 912046, 1–8

[9] Lyubich, I. Yu. and Shapiro, A. P., *On a functional equation (Russian)*, Teor. Funkts., Funkts. Anal. Prilozh., **17** (1973), 81–84

[10] Morales, J. S. and Rojas, E. M., *Hyers-Ulam and Hyers-Ulam-Rassias stability of nonlinear integral equations with delay*, Int. J. Nonlinear Anal. Appl., **2** (2011), No. 2, 1–6

[11] Rassias, Th. M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300

[12] Rus, I. A., *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math., **26** (2010), No. 1, 103–107

[13] Ulam, S. M., *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1960

DEPARTMENT OF MATHEMATICS
SAKARYA UNIVERSITY
SAKARYA, 54050 TURKEY
Email address: ayuce@sakarya.edu.tr
Email address: hakanarisoy34@hotmail.com
Email address: zeynepyildiz28@gmail.com