Abel extensions of some classical Tauberian theorems

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ABSTRACT. The well-known classical Tauberian theorems given for $A_\lambda$ (the discrete Abel mean) by Armitage and Maddox in [Armitage, H. D and Maddox, J. I., Discrete Abel means, Analysis, 10 (1990), 177–186] is generalized. Similarly the “one-sided” Tauberian theorems of Landau and Schmidt for the Abel method are extended by replacing $\lim As$ with $Abel-lim A\sigma_n(s)$. Slowly oscillating of $\{s_n\}$ is a Tauberian condition of the Hardy-Littlewood Tauberian theorem for Borel summability which is also given by replacing $\lim_B t(s)$ with $\lim_B A\sigma_n(s)$, where $t$ is a continuous parameter, with $\lim_B(Bs)_n = \ell$, and further replacing it by $Abel-lim(B\sigma_n(s))_n = \ell$, where $B$ is the Borel matrix method.

1. INTRODUCTION

Let $u = \{u_n\}$ be a sequence in $\mathbb{R}$ (or $\mathbb{C}$).

Definition 1.1. ([2], [3], [5]) A series $\sum_{k=0}^{\infty} u_k$ of real (or complex) numbers is called Abel summable to $\ell$ if the series $\sum_{k=0}^{\infty} s_k x^k$ is convergent for $0 \leq x < 1$ and

$$\lim_{x \to 1^-} (1-x) \sum_{k=0}^{\infty} s_k x^k = \ell, \text{ where } s_n = \sum_{k=0}^{n} u_k.$$ 

In this case, we write $Abel-lim s_n = \ell$.

Definition 1.2. ([1]) A series $\sum_{k=0}^{\infty} u_k$ of real (or complex) numbers is called $A_\lambda$ (the discrete Abel mean) convergent to $\ell$ if the series $\sum_{k=0}^{\infty} s_k x^k$ is convergent for all $n$ and

$$\lim_{x_n \to 1^-} (1-x_n) \sum_{k=0}^{\infty} s_k x_n^k = \ell,$$

where $\lambda = \{\lambda_n\}$ is a given sequence such that $1 \leq \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$ and the sequence $\{x_n\}$ is defined by $x_n = 1 - \frac{1}{\lambda_n}$. Clearly $0 \leq x_0 < x_1 < \ldots < x_n \to 1$. In this case, we write $A_\lambda-lim s_n = \ell$.

From definition 1.2, we say that the $\{s_n\}$ is in the domain of the method $A_\lambda$ if the sequence $(A_\lambda s)_n := (1-x_n) \sum_{k=0}^{n} s_k x_n^k$ is convergent for all $n$.

For $i \in \mathbb{N}$ and $n \in \mathbb{N}^*$, define

$$\sigma_n^i(s) = \begin{cases} \frac{1}{n+1} \sum_{k=0}^{n} \sigma_k^{i-1}(s) & \text{if } i \geq 1 \\ s_n & \text{if } i = 0. \end{cases}$$
We prove this by using mathematical induction. We show that our claims true for
Abel’s well-known Limit Theorem says that the Abel summability method is regular
if \( \lim s_n = \ell \) implies \( \text{Abel} - \lim s_n = \ell \). As we know the converse is false in general, e.g.
\[
\sum_{n=0}^{\infty} (-1)^n = \frac{1}{2} \quad \text{(Abel)} \text{ but } \lim_{n \to \infty} \sum_{k=0}^{n} (-1)^k \neq \frac{1}{2}.
\]
Also, it is obvious that \( \text{Abel} - \lim s_n = \ell \) implies \( A_\lambda - \lim s_n = \ell \). Hence \( A_\lambda \) also defines a regular method. \( A_\lambda \) summability method is regular; that is, if, \( A_\lambda - \lim s_n = \ell \) then \( A_\lambda - \lim s_n^{(1)}(s) = \ell \).

By [7], the series \( \sum_{n=0}^{\infty} u_n \) is Borel summable to \( \ell \) provided that
\[
\lim_{t \to \infty} e^{-t} \sum_{k=0}^{\infty} \frac{s_k t^k}{k!} = \ell.
\]
Consider as in [4] the summability matrix \( B = (b_{nk}) \) is given by
\[
b_{nk} = \frac{e^{-n} n^k}{k!}
\]
By [11], it is known that \( \{s_n\} \) is slowly oscillating if for any given \( \varepsilon > 0 \), there exists
\( \delta = \delta(\varepsilon) > 0 \) and \( N = N(\varepsilon) \) such that
\[
|s_m - s_n| < \varepsilon \text{ if } n \geq N(\varepsilon) \text{ and } n \leq m \leq (1 + \delta)n,
\]
and \( \{s_n\} \) of real numbers is slowly decreasing if
\[
\lim \inf(s_m - s_n) \geq 0 \text{ whenever } n \to \infty, \ m > n \text{ with } m \to 1.
\]
Thus, in particular, \( \{s_n\} \) is slowly oscillating when \( n \Delta s_n \) is bounded and \( \{s_n\} \) is slowly
decreasing when \( n \Delta s_n \) is bounded below.

Also, we say that \( \{s_n\} \) is strongly slowly oscillating if
\[
(s_m - s_n) \to 0 \text{ whenever } n \to \infty, \ m > n \text{ with } m \to 1;
\]
and \( \{s_n\} \) is strongly slowly decreasing if \( \{s_n\} \) is real and
\[
\lim \inf(s_m - s_n) \geq 0 \text{ whenever } n \to \infty, \ m > n \text{ with } m \to 1.
\]
Define \( t_n = \sum_{k=1}^{n} k u_k = \sum_{k=1}^{n} k \Delta s_k \) and \( \Delta s_k = s_k - s_{k-1}, s_{-1} = 0 \). We will prove that
\[
t_n = \sum_{k=1}^{n} k u_k = (n+1) s_n - \sum_{k=0}^{n} s_k \quad \text{(1.1)}
\]
We prove this by using mathematical induction. We show that our claims true for \( n = 1 : \)
\[
t_1 = 1 u_1 = 2s_1 - (s_0 + s_1) = 2s_1 - s_1 - s_0 = s_1 - s_0 = u_1.
\]
For \( n = 2, \ t_2 = u_1 + 2u_2 = 3s_2 - (s_0 + s_1 + s_2) = 2s_2 - s_0 - s_1 = 2u_2 + u_1. \)
Assume that it is true for \( n = m; \)
\[
t_m = \sum_{k=1}^{m} k u_k = (m+1) s_m - \sum_{k=0}^{m} s_k \quad \text{(1.2)}
\]
and we prove that it is true for \( n = m + 1 \): we add both sides \( (m+1) u_{m+1} \) of the equality (1.2)
\[
t_m + (m+1) u_{m+1} = \sum_{k=1}^{m} k u_k + (m+1) u_{m+1} = (m+1) s_m + (m+1) u_{m+1} - \sum_{k=0}^{m} s_k
\]
Thus proof is done. We obtain from (1.1)
\[ z_n := \frac{t_n}{n+1} = s_n - \sigma_n^1(s) = \frac{1}{n+1} \sum_{k=1}^{n} k \Delta s_k = n \Delta \sigma_n^1(s) \]
and
\[ \sigma_n^i(z) = \sigma_n^i(s) - \sigma_n^{i+1}(s) = n \Delta \sigma_n^{i+1}(s). \]

Here, \( \{z_n\} \) is known as the Kronecker identity. The classical control modulo of the oscillatory behaviour of a sequence \( \{s_n\} \) is denoted by \( w_n^m(s) = n \Delta s_n \). The general control modulo of the oscillatory behaviour of nonnegative integer order \( m \geq 1 \) of a sequence \( \{s_n\} \) is defined inductively in [3] by \( w_n^m(s) = w_n^{m-1}(s) - \sigma_n^1(w_n^{m-1}(s)) \). General control modulo is developed by Çanak in [2].

Throughout this paper, the symbols \( s_n = o(1) \) and \( s_n = O(1) \) mean that \( s_n \to 0 \) as \( n \to \infty \) and that \( \{s_n\} \) is bounded for large enough \( n \), respectively.

**Theorem 1.1.** ([1]) Let \( \{\lambda_n\} \) be a strictly increasing sequence of real numbers which tends to infinity such that
\[ \lim_n \frac{\lambda_{n+1}}{\lambda_n} = 1. \]
If the \( A_\lambda - \lim s_n = \ell \) and \( \{s_n\} \) is slowly decreasing, then \( \lim s_n = \ell \).

**Lemma 1.1.** ([1]) If \( \{s_n\} \) is slowly decreasing, then \( \left\{ \frac{t_n}{n} \right\} \) is bounded below.

Now, we will prove that the hypothesis \( A_\lambda - \lim s_n = \ell \) and slowly decreasing of \( \{s_n\} \) can be replaced by \( A_\lambda - \lim \sigma_n^1(s) = \ell \) and slowly decreasing of \( \{z_n\} \). So, we generalize some classical types of Tauberian theorems for given \( A_\lambda \). Moreover, we extend the “one-sided” Tauberian theorems of Landau and Schmidt’s Tauberian theorems for the Abel method by replacing \( \lim A \) with \( A \ell_n \sigma_n^1(s) \).

Before proving our statements, we recall more results that we will need in the sequel.

**Theorem 1.2.** ([9], [11])
1. If Abel-lim \( s_n = \ell \) and \( n \Delta s_n \geq -c \) for a positive number \( c \) then \( \lim s_n = \ell \).
2. Let a sequence \( \{s_n\} \) of real numbers be slowly decreasing. Then
   \( A_\lambda - \lim s_n = \ell \) implies \( \lim s_n = \ell \).
3. If Borel - lim \( s_n = \ell \) and \( \Delta s_n = o(1) \) then \( \lim \sigma_n^1(s) = \ell \).

2. **Main results**

**Lemma 2.2.** If the \( \{s_n\} \) is in the domain of method \( A_\lambda \) for which \( \lambda_n = n^\beta \), for some \( \beta \geq 1 \) and \( n \Delta s_n \geq -c \) for some positive \( c \), then the transformed sequence \( n \Delta(A_\lambda \sigma_k^1(s)) \) is also of one-sided, that is \( n \Delta(A_\lambda \sigma_k^1(s))_{n \geq c_1} \) for some positive \( c_1 \).

**Proof.** By the proof of Theorem 2.5 in [6], if \( n \Delta s_n \geq -c \) for a positive number \( c \) then \( n \Delta \sigma_n^1(s) \geq -c \). In [1] Armitage and Maddox showed that if we let \( v_k(x) = \frac{x^k}{k} \) then
\[ (1-x) \sum_{k=0}^{\infty} \sigma_k^1(s)x^k = \sum_{k=1}^{\infty} \Delta \sigma_k^1(s)x^k = \sum_{k=1}^{\infty} y_k(v_k(x) - v_{k+1}(x)), \quad 0 < x < 1, \]
where \( y_k = \sum_{j=1}^{k} j \Delta \sigma_j^1(s) \). It follows that from \( n \Delta \sigma_n^1(s) \geq -c, y_k = \sum_{j=1}^{k} j \Delta \sigma_j^1(s) \geq -kc. \)

Hence we see that since \( \{\sigma_k^1(s)\} \) verifies the one-sided Tauberian condition, \( \{y_k\} \) is bounded below by \(-kM\) for some positive number \( M \). If the Abel transform of \( \{\sigma_n^1(s)\} \) is denoted by \( A(x) = (A\sigma_k^1(s))_x \) then, for such a positive constant \( M \), we have

\[
n \Delta A(x_n) = n \left(A(x_n) - A(x_{n-1})\right) = n \left(\sum_{k=1}^{\infty} y_k \int_{x_{n-1}}^{x_n} y^{k-1}(1-y)dy\right)
\]

\[
\geq -n \left(M \sum_{k=1}^{\infty} k \int_{x_{n-1}}^{x_n} y^{k-1}(1-y)dy\right) = -n \left(M \int_{x_{n-1}}^{x_n} ky^{k-1}(1-y)dy\right)
\]

\[
= -n \left(M \int_{x_{n-1}}^{x_n} (1-y)^{-1}dy\right) \geq n \left(- M \log \frac{n^\beta}{(n-1)^\beta}\right) \geq -M \log \left(\frac{n^\beta}{(n-1)^\beta}\right)^n
\]

\[
= -M \log \left(\frac{n^\beta}{(n-1)^\beta} + 1 \right) + \log(1 + \frac{1}{n^\beta})^n
\]

\[
= -M \left[\beta \log \left(1 + \frac{1}{n^{-1}}\right) + \log \left[\left[1 + \left(\frac{1}{n}\right)^\beta\right]^{n^{1-\beta}}\right]\right]
\]

\[
= -M \left[\beta \log \left(1 + \frac{1}{n^{-1}}\right)^{n-1} \left(1 + \frac{1}{n^{-1}}\right) + \log \left[\left[1 + \left(\frac{1}{n}\right)^\beta\right]^{n^{1-\beta}}\right]\right]
\]

Hence, we obtain

\[
\liminf_n n \Delta A_\lambda(x_n) \geq -M \left[\beta \log \left(1 + \frac{1}{n^{-1}}\right)^{n-1} \left(1 + \frac{1}{n^{-1}}\right) + \log \left[\left[1 + \left(\frac{1}{n}\right)^\beta\right]^{n^{1-\beta}}\right]\right]
\]

\[
= -M \limsup_n \left[\beta \log \left(1 + \frac{1}{n^{-1}}\right)^{n-1} \left(1 + \frac{1}{n^{-1}}\right) + \log \left[\left[1 + \left(\frac{1}{n}\right)^\beta\right]^{n^{1-\beta}}\right]\right]
\]

\[
= -M (\beta + \log c^0) = -M \beta.
\]

Consequently, we see that the sequence \( (A_\lambda \sigma_k^1(s))_n \) obeys the one-sided Tauberian condition.

**Lemma 2.3.** If the \( \{s_n\} \) is in the domain of the method \( A_\lambda \) and is of slowly decreasing then the transformed sequence \( (A_\lambda \sigma_k^1(s))_n \) is also of slowly decreasing.

**Proof.** Proof is similar to one of Lemma 2.2. \( \square \)

**Lemma 2.4.** If \( \{s_n\} \) is slowly decreasing then \( \{\sigma_n^i(s)\} \) for all \( i \geq 1 \) is slowly decreasing.

**Proof.** We claim that \( \{\sigma_n^i(s)\} \) for all \( i \geq 1 \) is slowly decreasing. We will prove this by using mathematical induction. We show that our claims true for \( i = 1 \). Let \( \{s_n\} \) be slowly decreasing. By Lemma 1.1, \( z_n = \frac{t_n}{n} = s_n - \sigma_n^1(s) \) is bounded below. Hence, \( z_n = n \Delta \sigma_n^1(s) \) is bounded below. Consequently, \( \{\sigma_n^1(s)\} \) is slowly decreasing. Assume that it is true for
i = t−1, and we will prove that it is true for i = t. By assumption, since \( \{\sigma^{t-1}_n(s)\} \) is slowly decreasing and applying Lemma 1.1, we obtain \( \sigma^{t-1}_n(z) = n\Delta\sigma^t_n(s) \) is bounded below. Hence, there exists a positive constant \( M \) such that \( n\Delta\sigma^t_n(s) \geq -M \) for all \( n \). For \( n \) large enough, \( n > N_1, \sigma^{t}_m(s) - \sigma^{t}_n(s) = \sum_{k=n+1}^{m} \Delta\sigma^t_k(s) \geq -\sum_{k=n+1}^{m} \frac{M}{k} = -M(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{m}) \geq -M(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{m+n+1}) \geq -M(1 - \frac{m}{m+n}) \geq -M(1 - 1) = 0, \) by \( \frac{m}{n} \to 1. \)

**Lemma 2.5.** If \( \{s_n\} \) is slowly oscillating then \( \{\sigma^t_n(u)\} \) for all \( i \geq 1 \) is slowly oscillating.

**Proof.** Proof is similar to one of Lemma 2.4. □

**Lemma 2.6.** If \( \{s_k\} \) is slowly oscillating then \( \{(B\sigma^i_k(s))_n\} \) is slowly oscillating.

**Proof.** By Lemma 2.5, slow oscillating of \( \{s_k\} \) implies both \( \{\sigma^i_k(s)\} \) for all \( i \geq 1 \) is slowly oscillating and \( \Delta\sigma^i_k(s) = o(1) \). It follows that \( \sum_{k=n+1}^{m} \Delta\sigma^i_k(s) \leq \frac{\varepsilon}{2} \) for \( n \) large enough. Thus we have

\[
(B\sigma^i_k(s))_m - (B\sigma^i_k(s))_n = \left| (B\sigma^i_k(s))_{n+r} - (B\sigma^i_k(s))_n \right|
\]

\[
= \left| \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} b_{r,j} b_{n,p-j} \sigma^i_p(s) - \sum_{k=0}^{\infty} b_{n,k} \sigma^i_k(s) \right|
\]

\[
= \left| \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} \sigma^i_{k+j}(s) - \sum_{k=0}^{\infty} b_{n,k} \sigma^i_k(s) \right|
\]

\[
= \left| \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} (\sigma^i_{k+j}(s) - \sigma^i_k(s)) \right|
\]

\[
= \left| \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} \left| \Delta\sigma^i_{k+p}(s) \right| + \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} \left| \sum_{p=j}^{j_0+1} \Delta\sigma^i_{k+p}(s) \right| \right|
\]

\[
\leq \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} \sum_{p=1}^{j_0} \left| \Delta\sigma^i_{k+p}(s) \right| + \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} \left| \sum_{p=j}^{j_0+1} \Delta\sigma^i_{k+p}(s) \right| \right|
\]

\[
\leq \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} \sum_{p=1}^{j_0} \left| \Delta\sigma^i_{k+p}(s) \right| + \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} \frac{\varepsilon}{2}
\]

\[
\leq \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k} \frac{1}{k+1} + \frac{\varepsilon}{2} \sum_{j=0}^{\infty} b_{r,j} \sum_{k=0}^{\infty} b_{n,k}
\]

\[
= \sum_{n=0}^{\infty} \frac{n}{n(k+1)} + \frac{\varepsilon}{2} = \frac{1}{n} \sum_{k=0}^{\infty} b_{n,k} \frac{n}{k+1} + \frac{\varepsilon}{2}
\]

\[
= \frac{1}{n} \sum_{k=0}^{\infty} b_{n,k+1} + \frac{\varepsilon}{2} - \frac{1}{n} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

for \( n \geq n_0 \) large enough. □
Theorem 2.3. Let \( \lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1 \). If \( A_\lambda - \lim \sigma_n^i(s) = \ell \) and \( \{z_n\} \) is slowly decreasing then \( \lim s_n = \ell \).

**Proof.** Since \( A_\lambda \) method is regular, \( A_\lambda - \lim \sigma_n^{i+1}(s) = \ell \). Hence, we have \( A_\lambda - \lim \sigma_n^i(z) = 0 \). By Lemma 2.4, as \( z_n = n\Delta \sigma_n^i(s) \) is slowly decreasing, \( (\sigma_n^i(z)) = n\Delta \sigma_n^{i+1}(s) \) for all \( i \geq 1 \) is slowly decreasing. Since \( \{\sigma_n^i(z)\} \) is \( A_\lambda \) summability to 0, \( \lim \sigma_n^i(z) = \lim n\Delta \sigma_n^{i+1}(s) = 0 \). \( \lim n\Delta \sigma_n^{i+1}(s) = 0 \) implies \( n\Delta \sigma_n^{i+1}(s) \) is bounded below, that is, \( n\Delta \sigma_n^{i+1}(s) \geq -c \) for some positive c. It follows that \( \{\sigma_n^{i+1}(s)\} \) is slowly decreasing. If \( \sigma_n^i(z) = \sigma_n^i(s) - \sigma_n^{i+1}(s) \) is slowly decreasing, then \( \{\sigma_n^i(s)\} \) is slowly decreasing. From \( A_\lambda - \lim \sigma_n^i(s) = \ell \), we have \( \lim \sigma_n^i(s) = \ell \). By the fact that every sequence \( C, 1 \) limitable is Abel limitable, we have \( Abel - \lim \sigma_n^{-1}(s) = \ell \). Abel - \lim \sigma_n^{-1}(s) = \ell \) implies \( A_\lambda - \lim \sigma_n^{-1}(s) = \ell \). If we continue in that way, we obtain, \( A_\lambda - \lim s_n = \ell \). By Theorem 1.1, \( \lim s_n = \ell \).

Theorem 2.3 generalises Theorem 1.1. For example, if we consider the case \( i = 1 \) then the sequence \( \{s_n\} \) which is the Taylor coefficients of the function \( f \) defined by \( f(t) = \sin(1-t)^{-1} \) on \( 0 < t < 1 \) is not \( A_\lambda \) convergent however, Cesaro of the sequence \( \{s_n\} \) is \( A_\lambda \) convergent.

An immediate consequence of Theorem 2.3 is that the boundedness below of \( n\Delta z_n \) is a Tauberian condition for \( A_\lambda \).

**Corollary 2.1.** Let \( \lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1 \). If \( A_\lambda - \lim \sigma_n^i(s) = \ell \) and \( n\Delta z_n \geq -c \) for some positive c, then \( \lim s_n = \ell \).

Also, by considering \( \{s_n\} \) as a complex sequence we deduce the following result.

**Corollary 2.2.** Let \( \lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1 \). If \( A_\lambda - \lim \sigma_n^i(s) = \ell \) and slowly oscillating of \( \{z_n\} \), then \( \lim s_n = \ell \).

The proof of Lemma 1.1 in [1] can be modified to show that if \( \{z_n\} \) is strongly slowly decreasing, then \( \lim \inf \left( \frac{t_n}{n} \right) \geq 0 \). In view of this, the proof of Theorem 2.3 can be adapted to yield the following result:

**Theorem 2.4.** Let \( \frac{\lambda_{n+1}}{\lambda_n} = O(1) \). If \( A_\lambda - \lim \sigma_n^i(s) = \ell \) and \( \{z_n\} \) is the strongly slowly decreasing, then \( \lim s_n = \ell \).

It follows that for a complex sequence \( \{\sigma_n^i(s)\} \) the strongly slowly oscillating of \( \{z_n\} \) is a Tauberian condition for \( A_\lambda \) when \( \frac{\lambda_{n+1}}{\lambda_n} = O(1) \).

Theorem 2.4 is a generalization of the Theorem 7 in [1]. The strongly slowly decreasing of \( \{z_n\} \) does not imply the strongly slowly decreasing of \( \{s_n\} \). As an example, if we take \( s_n = \sum_{j=1}^{n+1} \frac{1}{j^2} + \sum_{k=1}^{1} \frac{1}{k} \sum_{j=1}^{n} \frac{1}{j^2} \), we see that \( z_n = \sum_{j=1}^{n} \frac{1}{j^2} \) is slowly decreasing but clearly, \( \{s_n\} \) is not slowly decreasing. Next theorem extends the classical Tauberian theorems of Hardy and Littlewood in [7] and [10] respectively.

**Theorem 2.5.** Let \( \lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1 \) and \( \lambda_n = n^\beta \), for some \( \beta \geq 1 \). If Abel-\( \lim (A_\lambda \sigma_n^i(s))_k = \ell \) and \( n\Delta s_n \geq -c \), then \( \lim s_n = \ell \).
Proof. By the proof of Theorem 2.5 of in [6], \( n\Delta s_n \geq -c \) for a positive number \( c \) implies \( n\Delta \sigma^i_n(s) \geq -c \). Hence, by Lemma 2.2, we see that \( (A_\lambda \sigma^i_n(s))_k \) obeys the one-sided Tauberian condition. From Abel-lim \((A_\lambda \sigma^i_n(s))_k = \ell \), we have lim \( A_\lambda \sigma^i_n(s) = \ell \), by above (1) in Theorem 1.2. Now by Theorem 1.1 implies that \( \sigma^i_n(s) \) is Abel summable to \( \ell \). Since \( n\Delta \sigma^i_n(s) \geq -c \), lim \( \sigma^i_n(s) = \ell \). By the fact that every sequence (C, 1) limitable is Abel limitable, we have \( Abel - \lim \sigma^i_{n-1}(s) = \ell \). Since \( Abel - \lim \sigma^i_{n-1}(s) = \ell \) and \( n\Delta \sigma^i_{n-1}(s) \geq -c \), we obtain that \( \lim \sigma^i_{n-1}(s) = \ell \). If we continue in that way, we obtain, \( Abel - \lim s_n = \ell \). By (1) in Theorem 1.2, \( \lim s_n = \ell \).

**Remark 2.1.** The following result, which is analogous to Theorem 2.5, may be proved for the slow decrease condition by using the Tauberian theorems results provided in [11] and Lemma 2.4. This then extends the classical Tauberian theorem of Schmidt [11].

**Theorem 2.6.** Let \( \lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1 \). If Abel-lim \((A_\lambda \sigma^i_n(s))_k = \ell \) and the \( \{s_n\} \) is slowly decreasing then \( \lim s_n = \ell \).

Final theorem is a Abel extension of Hardy and Littlewood’s Tauberian theorem in [8] for Borel summability.

**Theorem 2.7.** If Abel-lim \((B\sigma^i_n(s))_k = \ell \) and \( \{s_n\} \) is slowly oscillating then \( \lim s_n = \ell \).

**Proof.** By Lemma 2.3, slowly oscillating of \( \{s_n\} \) implies both slowly oscillating of \( \{\sigma^i_n(s)\} \) and \( \Delta \sigma^i_n(s) = o(1) \). By Lemma 2.6, we conclude that \( (B\sigma^i_n(s))_k \) is slowly oscillating. This allows us to apply (2) in Theorem 1.2 that \( \lim (B\sigma^i_n(s))_k = \ell \). Now (3) in Theorem 1.2 gives \( \lim \sigma^i_{n-1}(s) = \ell \). By the fact that every sequence (C, 1) limitable is Abel limitable, we have \( Abel - \lim \sigma^i_n(s) = \ell \). Since \( Abel - \lim \sigma^i_n(s) = \ell \) and \( \{\sigma^i_n(s)\} \) is slowly decreasing, we obtain \( \lim \sigma^i_n(s) = \ell \). If we continue in that way, we obtain \( Abel - \lim s_n = \ell \). Again by (2) in Theorem 1.2, we get \( \lim s_n = \ell \).

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**References**


