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Fixed point theorems on a γ-generalized quasi-metric spaces

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ABSTRACT. The concept of γ-generalized quasi-metric spaces is newly introduced in this paper with the symmetry assumption removed. The existence of fixed points of our newly introduced $(\gamma - \phi)$-contraction mappings, defined on γ-generalized quasi-metric spaces, is proved. Our results generalize many known related results in literature.

1. INTRODUCTION

Life is full of instances where symmetry with respect to distance is not necessarily important. In quasi-metric spaces, such distances are put into consideration. For instance, given a set X containing vehicles in a city, the driving distance between elements of X form a quasi-metric because of the one way roads. A quasi-metric space is obviously a generalization of metric space.
There are abundant literature devoted to generalization of distance spaces. 2-metric spaces were introduced by Gahler [8, 9]. Gahler claimed that a 2-metric was a generalization of the usual notion of a metric, but different authors proved that there was no relation between these two functions. For instance Ha et al. in [10] showed that a 2-metric need not be a continuous function of its variables, whereas an ordinary metric was and that there was no easy relationship between results obtained in the two settings. In particular, the contraction mapping theorem in metric spaces and in 2-metric spaces were unrelated.
Dhage [4] proposed the notion of a D-metric space in an attempt to obtain analogous results to those for metric spaces, but in a more general setting. In a subsequent series of papers (including: [3, 4, 5, 6]), Dhage presented topological structures in such spaces together with several fixed point results. These works have been the basis for a substantial number of results by other authors. Unfortunately, in 2006, Mustafa and Sims proved that these attempts were invalid. They later introduced a new structure of generalized metric spaces, an extension of usual metric space (see[13-16]). The notion of this space is as shown below:

Definition 1.1. ([13]) Let X be a non-empty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:

(i) $G(x, y, z) = 0$ if and only if $x = y = z$
(ii) $G(x, x, y) > 0, \quad \forall x, y \in X$, with $x \neq y$
(iii) $G(x, x, y) \leq G(x, y, z), \quad \forall x, y, z \in X$, with $z \neq y$
(iv) $G(x, y, z) = G(x, z, y) = G(y, x, z) = ...$ (symmetry).
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z) \quad \forall a, x, y, z \in X$ (rectangle inequality)

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The function G is called a G-metric. Then a non-empty set X with the G defined on it is called a G-metric space denoted by (X, G). They also introduced new fixed point theories for various mappings in this space.

Bakhtin [2] introduced b - metric space as a generalization of metric space and proved analogue of Banach contraction principle in it. In order to weaken the continuity in Banach contraction principle, Samet et al. introduced the following concepts:

Definition 1.2. ([19]) Let X be any nonempty set and $\alpha : X \times X \to R^+$. A mapping $T : X \to X$ is said to be:
(i) α-admissible if $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.
(ii) triangle α-admissible if it is α-admissible and $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ implies that $\alpha(x, y) \geq 1$ for all $x, y, z \in X$.

The following definition and Lemma are needed in this work:

Definition 1.3. ([19]) Suppose Ψ denote the set of all functions $\psi : R^+ \cup \{\infty\} \to R^+$ which have the following properties:
(i) ψ is continuous and non-decreasing.
(ii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ where $\psi^n(t) = \underbrace{\psi\psi...\psi(t)}_{ntimes}$ is n^{th} iterate of ψ.

The auxiliary function $\psi \in \Psi$ is known as comparison function. The following Lemma is its immediate consequence.

Lemma 1.1. ([19]) If $\psi \in \Psi$, then
(i) $\psi^n(t)$ converges to 0 as $n \to \infty$ for all $t \in [0, \infty)$ and $n \in \mathbb{N}$.
(ii) $\psi(t) < t$ for all $t > 0$.
(iii) $\psi(t) = 0$ if and only if $t = 0$.

In 2012, Samet et al. presented some new results by introducing the notion of $(\alpha - \psi)$-contractive mappings as a generalization of the Banach contraction map as shown below:

Theorem 1.1. ([19]) *Let (X, d) be a complete metric space, T, an α-admissible mapping as shown in Definition 1.2 and Ψ, the set of all functions $\psi : R^+ \cup \{\infty\} \to R^+$ which satisfy the properties in Definition 1.4. Suppose*
$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$$
for all $x, y \in X$ with $\psi \in \Psi$. Then T has a fixed point in X provided the following assertions holds:
(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.
(ii) either T is continuous or for any sequence $\{x_n\}$ in X, with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in N$.

Mohamed and Samet [11] also introduced a generalized metric space, a generalization of both metric and b-metric space as shown below:

Definition 1.4. ([12]) Let X be a non-empty set and $D : X^2 \to [0, \infty]$ be a given mapping. For every $x \in X$, define the set
$$C(D, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} D(x_n, x) = 0\}$$
D is said to be a generalized metric on X if it satisfies the following conditions:

(D_1) For every $(x, y) \in X \times X$ we have $D(x, y) = 0$ implies $x = y$;
(D_2) For every $(x, y) \in X \times X$ we have $D(x, y) = D(y, x)$;
(D_3) There exists a constant $C > 1$ such that if $(x, y) \in X \times X$ and $\{x_n\} \in C(D, X, x)$ then $D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y)$.

The pair (X, D) is a generalized metric space.
In this paper, the concepts and results were extended from distance type between two points to distance type among three points and weaken the axioms by removing the symmetric property. Existence of fixed point of $(\gamma - \phi)$-contraction mappings, a generalization of $(\alpha - \psi)$-contractive mappings shown above are proved. The notion of α-admissible and rectangular α-admissible, which generalize the concept of α-admissible and triangular α-admissible are also introduced in this work. Some examples are included which shows that our generalizations are genuine.

2. Main results

Motivated by the work of Mohammed and Samet in [12] and the applications in G-metric spaces by authors in [11], the following definitions are introduced. These generalize generalized metric space introduced by Mohammed et al. [12]
Let X be a nonempty set and $G : X \times X \times X \to R^+ \cup \{\infty\}$ a given mapping. For every $x \in X$, let us define the set

$$(G, X, x) = \{\{x_n\} \subseteq X : \lim_{n\to\infty} G(x_n, x, x) = 0\}.$$

Definition 2.5. Let X be a nonempty set and $\gamma : R^+ \cup \{\infty\} \to R^+$ be a continuous and nondecreasing for all $t > 0$ and $\gamma(0) = 0$. A function $G : X \times X \times X \to R^+ \cup \{\infty\}$ is called a γ-generalized quasi-metric on X if the following two conditions are satisfied:
$(G)_1$: for all $x, y, z \in X, G(x, y, z) = 0 \Rightarrow x = y = z$.
$(G)_2$: if $(x, y, z) \in X \times X \times X$ and $\{x_n\} \in (G, X, x)$, then

$$G(x, y, z) \leq \gamma(\limsup_{n\to\infty} G(x_n, y, z)).$$

Then, the pair (X, G) is called a γ-generalized quasi metric space.

Remark 2.1. (i) The notion of a γ- generalized quasi metric space is a generalization of a generalized quasi-metric space introduced by Samet et al. [19].
(ii) For a γ-generalized quasi-metric G, the conjugate γ-generalized quasi metric G^{-1} on X of G is defined by $G^{-1}(x, y, z) = \dfrac{1}{2}\big\{G(x, x, y) + G(x, x, z)\big\}$ i.e $G(x, y, y) = G(x, x, y)$.
(iii) If G is a γ-generalized quasi-metric on X, then the function G^u defined by $G \bigvee G^{-1}$, that is $G^u(x, y, z) = max\{G(x, y, z), G^{-1}(x, y, z)\}$ defines a γ- generalized quasi-metric on X.

Symmetry, not present in the definition of quasi-metric spaces may cause a lot of difficulties in completeness, compactness and total boundedness. However, there are lot of completeness notions in these spaces, all upholding the usual notion of completeness in the case of metric spaces. Now, some of these notions along with some of their properties are as follows:

Definition 2.6. A sequence $\{x_n\}$ in a γ-generalized quasi-metric space (X, G_γ) is said to be:
(i) G-convergent or left convergent to x if $G(x, x, x_n) \to 0$ as $n \to \infty$.
(ii) G^{-1}-convergent or right convergent to x if $G(x_n, x, x) \to 0$ as $n \to \infty$.
(iii) G^u-convergent if and only if it is both left convergent and right convergent.

Definition 2.7. A sequence $\{x_n\}$ in a γ-generalized quasi-metric space $(X, G_)$ is called:
(i) left K-Cauchy if for every $\epsilon > 0$ there exists $n_\epsilon \in N$ such that $G(x_l, x_m, x_n) < \epsilon$ for all $l, m, n \in N$ with $n_\epsilon \leq l \leq m \leq n$
(ii) right K-Cauchy if for every $\epsilon > 0$ there exists $n_\epsilon \in N$ such that $G(x_l, x_m, x_n) < \epsilon$ for all $l, m, n \in N$ with $n_\epsilon \leq n \leq m \leq l$

(iii) G^u-Cauchy if for every $\epsilon > 0$, there exists $n_\epsilon \in N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $l, m, n \in N$ with $l, m, n \geq n_\epsilon$.

Example 2.1. Let $X = (0, 1)$ and define G on X by:

$$G(x,y,z) \quad = \quad \begin{cases} |z-x|, & \text{if} \quad x \leq y \leq z; \\ 1, & \text{if} \quad otherwise. \end{cases}$$

and $\gamma(t) = 2t$ for all $t \geq 0$. Then (X, G) is a γ-generalized quasi-metric space.
This can be shown as seen below.
Clearly, from the definition $G(x, y, z) = 0 \Rightarrow x = y = z$. So the property $(G)_1$ is satisfied.
For $(G)_2$, we assume $\{x_n\}_{n\in\mathbb{N}} \in (G, X, x)$ and consider the following cases.
Case 1: Let $x, y, z \in X$ and $\{x_n\} \in (G, X, x)$.

$$\lim_{n\to\infty} G(x_n, x, x) = 0.$$

There exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $G(x_n, x, x) = |x_n - x|$ and $x_n \leq x$. Hence, $x_n \to x$ and $x_n \leq x \; \forall n \geq n_0$
Let $y, z \in X$. If $y \in [x_n, z] \forall n \in \mathbb{N}$, then it follows that $y \in [x, z]$ and

$$\begin{aligned} \gamma(\limsup_{n\to\infty} G(x_n, y, z)) &= 2(\limsup_{n\to\infty} G(x_n, y, z)) = 2(\limsup_{n\to\infty} |x_n - z|) \\ &= 2|x - z| \geq |x - z| = G(x, y, z) \end{aligned}$$

Case 2: If $y < x_n \; \forall n \in \mathbb{N}$, then y is not in $[x, z]$. Also if $y > z$, y is not in $[x, z]$ and

$$\gamma(\limsup_{n\to\infty} G(x_n, y, z)) = 2(\limsup_{n\to\infty} G(x_n, y, z)) = 2(1) > G(x, y, z)$$

Therefore, (X, G) is a γ-generalized quasi metric space.

The definitions below generalize the α-admissible and triangular α-admissible mappings introduced by Samet et al. [19].

Definition 2.8. Let X be any nonempty set and $\alpha : X \times X \times X \to R^+$. A mapping $T : X \to X$ is said to be:
(i) α-admissible if $\alpha(x, y, z) \geq 1$ implies that $\alpha(Tx, Ty, Tz) \geq 1$ for all $x, y, z \in X$.
(ii) rectangular α-admissible if it is α-admissible and $\alpha(x, v, v) \geq 1$ and $\alpha(v, y, z) \geq 1$ implies that $\alpha(x, y, z) \geq 1$ for all $v, x, y, z \in X$.

Example 2.2. Let $X = R$, $Tx = \sqrt{x}$ and $\alpha(x, y, z) = e^{\frac{x-y}{z}}$ with $z \neq 0$, then T is a rectangular α-admissible mapping. Indeed if $\alpha(x, y, z) = e^{\frac{x-y}{z}} \geq 1$ then $\dfrac{x}{z} \geq \dfrac{y}{z}$ which implies that $\dfrac{Tx}{Tz} \geq \dfrac{Ty}{Tz}$. That means $\alpha(Tx, Ty, Tz) = e^{\frac{Tx-Ty}{Tz}} \geq 1$. Also, $\alpha(x, v, v) \geq 1$ and $\alpha(v, y, z) \geq 1$ implies that $\alpha(x, y, z) \geq 1$ for all $v, x, y, z \in X$.

The following Lemmas are newly introduced and will be needed in the main proof.

Lemma 2.2. *Let T be a rectangular α-admissible mapping on a nonempty set X. Suppose there exists $x_0 \in X$ such that $\alpha(x_0, x_0, Tx_0) \geq 1$ and $\alpha(x_0, Tx_0, Tx_0) \geq 1$. If we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n \in N$. Then we have $\alpha(x_n, x_m, x_l) \geq 1$ for all $l, m, n \in N$ with $n < m < l$.*

Proof. Since T is α admissible and $\alpha(x_0, x_0, Tx_0) \geq 1$, we deduce that

$$\alpha(x_1, x_1, x_2) = \alpha(Tx_0, Tx_0, Tx_1) \geq 1.$$

By continuing this process, we get $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all $n \geq 1$. If $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_{n+1}, x_{n+2}) \geq 1$, by rectangular α admissible, $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$. Hence, we have proved that $\alpha(x_n, x_m, x_l) \geq 1$ for all $l, m, n \in N$ with $n < m < l$. $\square$

Lemma 2.3. *Let T be a rectangular α-admissible mapping on a nonempty set X. Suppose there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0, x_0) \geq 1$ and $\alpha(Tx_0, Tx_0, x_0) \geq 1$. If we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for each $n \in N$. Then we have $\alpha(x_n, x_m, x_l) \geq 1$ for all $l, m, n \in N$ with $n > m > l$.*

Proof. Since T is α admissible and $\alpha(Tx_0, x_0, x_0) \geq 1$, we deduce that

$$\alpha(x_2, x_1, x_1) = \alpha(Tx_1, Tx_0, Tx_0) \geq 1.$$

By continuing this process, we get $\alpha(x_{n+1}, x_n, x_n) \geq 1$ for all $n \geq 1$. If $\alpha(x_{n+2}, x_{n+1}, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_{n+1}, x_n) \geq 1$, by rectangular α admissible, $\alpha(x_{n+2}, x_{n+1}, x_n) \geq 1$. Hence, we have proved that $\alpha(x_n, x_m, x_l) \geq 1$ for all $l, m, n \in N$ with $n > m > l$. □

Definition 2.9. A γ-generalized quasi metric space (X, G) is called:
(i) left K-complete if every left K-Cauchy sequence in X is left convergent.
(ii) right K-complete if every right K-Cauchy sequence in X is right convergent.
(iii) Smyth complete if every G^u-Cauchy sequence in X is G^u convergent.

Definition 2.10. Let (X, G) be a γ- generalized quasi-metric space and $T : X \to X$ a given mapping. T is called a $(\gamma - \phi)$-contraction mapping if there exist two functions $\gamma, \phi : X \times X \times X \to R^+$ such that:

$$\gamma(x, y, z)G(Tx, Ty, Tz) \leq \phi(G(x, y, z))$$

Remark 2.2. If $\gamma(x, y, z) = 1 \forall x, y, z \in X$, $\phi(t) = kt, k \in [0, 1), t \geq 0$ with $y = z$ and $G(x, y, y) = d(x, y)$ in Definition 2.10, mapping in Banach contraction principle is obtained.

We now prove the main theorem:

Theorem 2.2. *Let (X, G) be a Smyth complete γ-generalized quasi-metric space and $T : X \to X$, a $(\gamma - \phi)$-contraction mapping which satisfies:*
(i) T is a rectangular α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0, x_0) \geq 1$, $\alpha(x_0, x_0, Tx_0) \geq 1$ and $\delta(G, T, x_0) < \infty$ where $\delta(G, T, x) = \sup\{G(T^i x, T^j x, T^k x) : i \neq j \neq k \in N_0\}$ for every $x \in X$:
(iii) $u, v \in F(T)$ implies $G(u, v, v) < \infty$, $\alpha(u, v, v) \geq 1$ and $\alpha(x_0, v, v) \geq 1$.
Then T has a unique fixed point v and $\{T^n x_0\}$ is G-convergent to the fixed point v of T.

Proof. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By (ii), there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0, x_0) \geq 1$ and $\alpha(x_0, x_0, Tx_0) \geq 1$. Let us define a sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$, then x_{n_0} is the fixed point of T. For the rest of the proof, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}_0$. Regarding assumption (i), we derive

$$\alpha(x_1, x_0, x_0) = \alpha(Tx_0, x_0, x_0) \geq 1 \Rightarrow \alpha(Tx_1, Tx_0, Tx_0) = \alpha(x_2, x_1, x_1) \geq 1$$

recursively, we get

$$\alpha(x_{n+1}, x_n, x_n) = \alpha(T^{n+1}x_0, T^n x_0, T^n x_0) \geq 1 \forall v \in \mathbb{N}_0.$$

For $i, j, k \in \mathbb{N}_0, i \neq j \neq k$. Taking Definition 2.10 into account and applying Lemma 2.2, we obtain

$$\begin{aligned} G(x_{n+i}, x_{n+j}, x_{n+k}) &\leq \gamma(x_{n-1+i}, x_{n-1+j}, x_{n-1+k})G(x_{n+i}, x_{n+j}, x_{n+k}) \\ &\leq \phi(G(x_{n-1+i}, x_{n-1+j}, x_{n-1+k}). \end{aligned}$$

which implies

$$\delta(G, T, x_n) \leq \phi(\delta(G, T, x_{n-1}))$$

Then for every $n \in N$, we have

$$\delta(G,T,x_n) \leq \phi^n(\delta(G,T,x_0))$$

Thus for every $n,m,l \in \mathbb{N}$, we have

$$G(x_n,x_{n+m},x_{n+m+l}) \leq \delta(G,T,x_n) \leq \phi^n(\delta(G,T,x_0))$$

Using the fact that $\delta(G,T,x_0) < \infty$, we obtain

$$\lim_{l,m,n\to\infty} G(x_n,x_{n+m},x_{n+m+l}) = 0$$

Then $\{T^n x_0\}$ is a left K-Cauchy sequence in (X,G). Since the space (X,G) is generalized Smyth complete quasi-metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} G(x_n,v,v) = 0 \text{ and } \lim_{n\to\infty} G(v,v,x_n) = 0$$

On the other hand, we have, since T is a $(\gamma-\phi)$-contractive and by $(G)_2$ for all $n \in \mathbb{N}$,

$$\begin{aligned} G(v,v,Tv) &\leq \gamma(\limsup_{n\to\infty} G(x_n,x_n,Tv)) \\ &\leq \gamma(\limsup_{n\to\infty} \alpha(x_{n-1},x_{n-1},v)G(x_n,x_n,Tv)) \\ &\leq \gamma(\phi(\limsup_{n\to\infty} G(x_{n-1},x_{n-1},v))) = 0. \end{aligned}$$

This implies $G(Tv,v,v) = 0$ and by $(G)_1$, we get $Tv = v$.
Now, suppose $u \in X$ is another fixed point of T such that $G(u,v,v) < \infty$. Since T is a $(\gamma-\phi)$-contraction, we have

$$G(u,v,v) = G(Tu,Tv,Tv) \leq \alpha(u,v,v)G(Tu,Tv,Tv) \leq \psi(G(u,v,v))$$

If $G(u,v,v) > 0$, then by Lemma 1.4, we have

$$G(u,v,v) = G(Tu,Tv,Tv) \leq \phi(G(u,v,v)) < G(u,v,v)$$

a contradiction. Hence, $G(u,v,v) = 0$ and from $(G)_1$, it follows that $u = v$. ☐

Example 2.3. Let $X = \mathbb{R}$ endowed with the γ-generalized quasi-metric $G(x,y,z) = |x-z|$ if $y \in [x,z]$ and 0 if otherwise. Define the mapping $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{3}{4x}, & \text{if } x > 2; \\ \dfrac{x}{2}, & \text{if } x \in [0,2] \\ 0, & \text{if } x < 0. \end{cases}$$

At first, it is observed that Banach contraction principle in G-metric space introduced by Mohamed and Sims (2006) cannot be applied in this case because

$$G(T2,T1,T3) = G(1,0.5,0.25) = |1-0.25| = 0.75 > kG(2,1,3) = 0, k \in [0,1)$$

Define the mapping $\alpha : X \times X \times X \to [0,\infty)$ by

$$\alpha(x,y,z) = \begin{cases} 1, & \text{if } x,y,z \in [0,2] \\ 0, & otherwise. \end{cases}$$

Obviously, T is $(\gamma-\phi)$-contraction mapping newly introduced with $\phi(t) = 2 + t$ for all $t \geq 0$. As a matter of fact, for all $x,y,z \in X$

$$\alpha(x,y,z)G(Tx,Ty,Tz) \leq 2 + G(x,y,z).$$

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0, x_0) \geq 1$. In fact, for $x_0 = 1$,

$$\alpha(1, 1, T1) = \alpha(1, 1, 0.5) = 1 \quad \text{and} \quad \alpha(T1, 1, 1) = \alpha(0.5, 1, 1) = 1.$$

Clearly, $\sup\{G(T^i x, T^j x, T^k x) : i \neq j \neq k \in \mathbb{N}\} < \infty$ for every $x \in X$ and so it remains to show that T is rectangular α-admissible. Let $x, y, z \in X$ such that $\alpha(x, y, z) \geq 1$. It implies $x, y, z \in [0, 2]$ and by the definition of T and α, $Tx, Ty, Tz \in [0, 1]$ and $\alpha(Tx, Ty, Tz) = 1$. So, T is α-admissible. Let $w, x, y, z \in X$ such that $\alpha(x, w, w) \geq 1$ and $\alpha(w, y, z) \geq 1$. It implies $w, x, y, z \in [0, 2]$ and by the definition of α, $\alpha(x, y, z) = 1$. Then T is rectangular α-admissible.
Let u and v be two fixed points of T. It implies $u = v = 0$ and then $G(u, v, v) < \infty$, $\alpha(u, v, v) = 1$ and $\alpha(x_0, v, v) = 1$.
All the conditions in Theorem 2.2 are satisfied. So, T has a unique fixed point v and $\{Tx_n\}$ is G-convergent to the fixed point v of T.

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