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Relative (p,q)-φ order and relative (p,q)-φ type based on some growth properties of composite p-adic entire functions

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ABSTRACT. Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. For any p adic entire functions $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. In this paper we study some growth properties of composite p-adic entire functions on the basis of their relative (p,q)-φ order, relative (p,q)-φ type and relative (p,q)-φ weak type where p, q are any two positive integers and $\varphi(r) : [0,+\infty) \to (0,+\infty)$ is a non-decreasing unbounded function of r.

1. INTRODUCTION AND DEFINITIONS

Let us consider an algebraically closed field $\mathbb{K}$ of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0,+\infty[$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \le R\}$ and the open disk $\{x \in \mathbb{K} : |x-\alpha| < R\}$ are denoted by $d(\alpha,R)$ and $d(\alpha,R^-)$ respectively. Also $C(\alpha,r)$ denotes the circle $\{x \in \mathbb{K} : |x-a| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represents the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$, i.e., the set of power series with an infinite radius of convergence. Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to +∞ with r therefore there exists its inverse function $\widehat{|f|} : (|f(0)|,\infty) \to (0,\infty)$ with $\lim\limits_{s\to\infty} \widehat{|f|}(s) = \infty$. For $x \in [0,\infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, $\log$ denotes the Neperian logarithm. Further we assume that throughout the present paper p, q, m and n always denote positive integers. Also throughout the paper occasionally $\varphi_1(r)$ will stand for r. Taking this into account the (p,q)-φ order $\rho^{(p,q)}(f,\varphi)$ and (p,q)-φ lower order $\lambda^{(p,q)}(f,\varphi)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ are defined as follows:

Definition 1.1. [1] Let $f \in \mathcal{A}(\mathbb{K})$. Also let $\varphi(r) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function of r. The (p,q)-φ order $\rho^{(p,q)}(f,\varphi)$ and (p,q)-φ lower order $\lambda^{(p,q)}(f,\varphi)$ of f are respectively defined as:

$$\begin{matrix} \rho^{(p,q)}(f,\varphi) \\ \lambda^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} |f|(r)}{\log^{[q]} \varphi(r)}.$$

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If $\varphi(r) = r$, then Definition 1.2 of [2] is a special case of Definition 1.1. Also for any non-decreasing unbounded function $\varphi(r) : [0, +\infty) \to (0, +\infty)$, if $\varphi(r)$ satisfies the condition $\lim\limits_{r\to+\infty} \frac{\log^{[q]} r}{\log^{[q]} \varphi(r)} = \alpha$ where $\alpha > 0$, then for any $f \in \mathcal{A}(\mathbb{K})$, one can easily verify that $\rho_g^{(p,q)}(f, \varphi) = \alpha\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f, \varphi) = \alpha\lambda_g^{(p,q)}(f)$. However, In this connection we just introduce the following definition:

Definition 1.2. An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p, q)-φ if $b < \rho^{(p,q)}(f, \varphi) < \infty$ and $\rho^{(p-1,q-1)}(f, \varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f, \varphi) < \infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f, \varphi) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f, \varphi) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f, \varphi) = 1 & \text{for } n = 1, 2, \cdots . \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f, \varphi) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f, \varphi) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f, \varphi) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f, \varphi) = 1 & \text{for } n = 1, 2, \cdots . \end{cases}$$

Next, to compare the growth of entire functions on $\mathbb{K}$ having the same (p, q)-φ order, we give the definitions of (p, q)-φ type and (p, q)-φ lower type in the following manner:

Definition 1.3. Let $f \in \mathcal{A}(\mathbb{K})$. Also let $\varphi(r) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of r. The (p, q)-φ type and (p, q)-φ lower type of f having finite positive (p, q)-φ order $\rho^{(p,q)}(f, \varphi)$ $(0 < \rho^{(p,q)}(f, \varphi) < \infty)$ are defined as :

$$\begin{matrix} \sigma^{(p,q)}(f, \varphi) \\ \overline{\sigma}^{(p,q)}(f, \varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} |f|(r)}{\left(\log^{[q-1]} \varphi(r)\right)^{\rho^{(p,q)}(f,\varphi)}}.$$

Likewise, to compare the growth of $f \in \mathbb{K}$ having the same (p, q)-φ lower order, one can also introduce the definition of (p, q)-φ weak type $\overline{\tau}^{(p,q)}(f, \varphi)$ and the growth indicator $\tau^{(p,q)}(f, \varphi)$ of f in the following manner:

Definition 1.4. Let $f \in \mathcal{A}(\mathbb{K})$. Also let $\varphi(r) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of r. The (p, q)-φ weak type $\overline{\tau}^{(p,q)}(f, \varphi)$ and the growth indicator $\tau^{(p,q)}(f, \varphi)$ of f having finite positive (p, q)-φ lower order $\lambda^{(p,q)}(f, \varphi)$ $(0 < \lambda^{(p,q)}(f, \varphi) < \infty)$ are defined as :

$$\begin{matrix} \tau^{(p,q)}(f, \varphi) \\ \overline{\tau}^{(p,q)}(f, \varphi) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} |f|(r)}{\left(\log^{[q-1]} \varphi(r)\right)^{\lambda^{(p,q)}(f,\varphi)}}.$$

In order to make some progress in the study of *p*-adic analysis, recently Biswas [1] introduced the definitions of relative (p, q)-φ order $\rho_g^{(p,q)}(f, \varphi)$ and relative (p, q)-φ lower order $\lambda_g^{(p,q)}(f, \varphi)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:

Definition 1.5. Let $f, g \in \mathcal{A}(\mathbb{K})$ and $\varphi(r) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function of r. Also let the index-pairs of f and g be (m, q)-φ and (m, p) , respectively. The relative (p, q)-φ order denoted as $\rho_g^{(p,q)}(f, \varphi)$ and relative (p, q)-φ lower order denoted by $\lambda_g^{(p,q)}(f, \varphi)$ of f with respect to g are defined as

$$\begin{matrix} \rho_g^{(p,q)}(f, \varphi) \\ \lambda_g^{(p,q)}(f, \varphi) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} \varphi(r)}.$$

In this connection we also introduce the following definition which will be needed in the sequel:

Definition 1.6. An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have relative index-pair (p,q)-φ with respect to an entire function $g \in \mathcal{A}(\mathbb{K})$ if $b < \rho_g^{(p,q)}(f,\varphi) < \infty$ and $\rho_g^{(p-1,q-1)}(f,\varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho_g^{(p,q)}(f,\varphi) < \infty$, then

$$\begin{cases} \rho_g^{(p-n,q)}(f,\varphi) = \infty & \text{for } n < p, \\ \rho_g^{(p,q-n)}(f,\varphi) = 0 & \text{for } n < q, \\ \rho_g^{(p+n,q+n)}(f,\varphi) = 1 & \text{for } n = 1,2,\cdots . \end{cases}$$

Similarly for $0 < \lambda_g^{(p,q)}(f,\varphi) < \infty$, one can easily verify that

$$\begin{cases} \lambda_g^{(p-n,q)}(f,\varphi) = \infty & \text{for } n < p, \\ \lambda_g^{(p,q-n)}(f,\varphi) = 0 & \text{for } n < q, \\ \lambda_g^{(p+n,q+n)}(f,\varphi) = 1 & \text{for } n = 1,2,\cdots . \end{cases}$$

Throughout the paper, whenever we deal with any entire function $f \in \mathcal{A}(\mathbb{K})$ having relative index-pair (p,q)-φ with respect to an entire function $g \in \mathcal{A}(\mathbb{K})$, we mean that f has positive relative (p,q)-φ lower order and finite relative (p,q)-φ order with respect to g.

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as relative (p,q)-φ type and relative (p,q)-φ lower type of entire functions with respect to another entire function which are as follows:

Definition 1.7. Let f, $g \in \mathcal{A}(\mathbb{K})$ and $\varphi(r) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ type $\sigma_g^{(p,q)}(f,\varphi)$ and the relative (p,q)-φ lower type $\overline{\sigma}_g^{(p,q)}(f,\varphi)$ of f with respect to g having non-zero finite relative (p,q)-φ order $\rho_g^{(p,q)}(f,\varphi)$ are defined as :

$$\begin{matrix} \sigma_g^{(p,q)}(f,\varphi) \\ \overline{\sigma}_g^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{\left[\log^{[q-1]} \varphi(r)\right]^{\rho_g^{(p,q)}(f,\varphi)}}.$$

Analogously, to determine the relative growth of f having same non zero finite relative (p,q)-φ lower order with respect to g, one can introduce the definition of relative (p,q)-φ weak type $\overline{\tau}_g^{(p,q)}(f,\varphi)$ and the growth indicator $\tau_g^{(p,q)}(f,\varphi)$ of f with respect to g of finite positive relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f,\varphi)$ in the following way:

Definition 1.8. Let f, $g \in \mathcal{A}(\mathbb{K})$ and $\varphi(r) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ weak type $\overline{\tau}_g^{(p,q)}(f,\varphi)$ and the growth indicator $\tau_g^{(p,q)}(f,\varphi)$ of f with respect to g having non-zero finite relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f,\varphi)$ are defined as :

$$\begin{matrix} \tau_g^{(p,q)}(f,\varphi) \\ \overline{\tau}_g^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{\left[\log^{[q-1]} \varphi(r)\right]^{\lambda_g^{(p,q)}(f,\varphi)}}.$$

Since $\rho_g^{(p,q)}(f,r)$ $(\lambda_g^{(p,q)}(f,r))$, $\sigma_g^{(p,q)}(f,r)$ $(\overline{\sigma}_g^{(p,q)}(f,r))$ and $\overline{\tau}_g^{(p,q)}(f,r)$ are respectively known as relative (p,q)-th order (relative (p,q)-th lower order), relative (p,q)-th type (relative (p,q)-th lower type) and relative (p,q)-th weak type of f with respect to g, so for $\varphi(r) = r$, we simplify to denote $\rho_g^{(p,q)}(f,r)$ $(\lambda_g^{(p,q)}(f,r))$, $\sigma_g^{(p,q)}(f,r)$ $(\overline{\sigma}_g^{(p,q)}(f,r))$ and

$\tau_g^{(p,q)}(f,r)$ $(\overline{\tau}_g^{(p,q)}(f,r))$ by $\rho_g^{(p,q)}(f)$ $(\lambda_g^{(p,q)}(f))$, $\sigma_g^{(p,q)}(f)$ $(\overline{\sigma}_g^{(p,q)}(f))$ and $\tau_g^{(p,q)}(f)$ $(\overline{\tau}_g^{(p,q)}(f))$ respectively. For detail about relative (p,q)-th order (relative (p,q)-th lower order), relative (p,q)-th type (relative (p,q)-th lower type) and relative (p,q)-th weak type of p-adic entire functions, one may see [2].

The purpose of this paper is to deal with some growth properties of composite p-adic entire functions in the light of their relative (p,q)-φ order, relative (p,q)-φ type and relative (p,q)-φ weak type where $\varphi(r):[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function of r.

2. Main results

First of all, we recall one related known property which can be found in [3] or [4] and will be needed in order to prove our results, as we see in the following lemma:

Lemma 2.1. *Let $f,g\in\mathcal{A}(\mathbb{K})$. Then for all sufficiently large positive numbers of r the following equality holds*

$$|f\circ g|(r)=|f|(|g|(r)).$$

Theorem 2.1. *Let $f,g,h\in\mathcal{A}(\mathbb{K})$. Also let the relative index pair of f with respect to h and the index pair of g be (p,q)-φ_1 and (m,n)-φ respectively. Then*
(i) *the relative index-pair of $f\circ g$ is (p,n)-φ when $q=m$ and either $\lambda_h^{(p,q)}(f,\varphi_1)>0$ or $\lambda^{(m,n)}(g,\varphi)>0$. Also*

$$\begin{aligned}(a)\ \lambda_h^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi) &\le \rho_h^{(p,n)}(f\circ g,\varphi)\\ &\leqslant \rho_h^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)\ \text{if}\ \lambda_h^{(p,q)}(f,\varphi_1)>0\end{aligned}$$

and

$$\begin{aligned}(b)\ \rho_h^{(p,q)}(f,\varphi_1)\,\lambda^{(m,n)}(g,\varphi) &\le \rho_h^{(p,n)}(f\circ g,\varphi)\\ &\leqslant \rho_h^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)\ \text{if}\ \lambda^{(m,n)}(g,\varphi)>0;\end{aligned}$$

(ii) *the relative index-pair of $f\circ g$ is $(p,q+n-m)$-φ when $q>m$ and either $\lambda_h^{(p,q)}(f,\varphi_1)>0$ or $\lambda^{(m,n)}(g,\varphi)>0$. Also*

$$(a)\ \lambda_h^{(p,q)}(f,\varphi_1)\le\rho_h^{(p,q+n-m)}(f\circ g,\varphi)\leqslant\rho_h^{(p,q)}(f,\varphi_1)\ \text{if}\ \lambda_h^{(p,q)}(f,\varphi_1)>0\ \text{and}$$

$$(b)\ \rho_h^{(p,q+n-m)}(f\circ g,\varphi)=\rho_h^{(p,q)}(f,\varphi_1)\ \text{if}\ \lambda^{(m,n)}(g,\varphi)>0;$$

(iii) *the relative index-pair of $f\circ g$ is $(p+m-q,n)$-φ when $q<m$ and either $\lambda_h^{(p,q)}(f,\varphi_1)>0$ or $\lambda^{(m,n)}(g,\varphi)>0$. Also*

$$(a)\ \rho_h^{(p+m-q,n)}(f\circ g,\varphi)=\rho^{(m,n)}(g,\varphi)\ \text{if}\ \lambda_h^{(p,q)}(f,\varphi_1)>0\ \text{and}$$

$$(b)\ \lambda^{(m,n)}(g,\varphi)\le\rho_h^{(p+m-q,n)}(f\circ g,\varphi)\leqslant\rho^{(m,n)}(g,\varphi)\ \text{if}\ \lambda^{(m,n)}(g,\varphi)>0\,.$$

Proof. In view of Lemma 2.1, it follows for all sufficiently large values of r that

$$\log^{[p]}\widehat{|h|}(|f\circ g|(r))\ge\left(\lambda_h^{(p,q)}(f,\varphi_1)-\varepsilon\right)\log^{[q]}|g|(r) \tag{2.1}$$

and also for a sequence of values of r tending to infinity we get that

$$\log^{[p]}\widehat{|h|}(|f\circ g|(r))\ge\left(\rho_h^{(p,q)}(f,\varphi_1)-\varepsilon\right)\log^{[q]}|g|(r)\,. \tag{2.2}$$

Likewise, we have for all sufficiently large values of r that

$$\log^{[p]}\widehat{|h|}(|f\circ g|(r))\leqslant\left(\rho_h^{(p,q)}(f,\varphi_1)+\varepsilon\right)\log^{[q]}|g|(r)\,. \tag{2.3}$$

Now the following three cases may arise:
Case I. Let $q = m$. In this case we have from (2.3) for all sufficiently large values of r that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[n]}\varphi\left(r\right)}\leqslant\rho_h^{(p,q)}\left(f,\varphi_1\right)\rho^{(m,n)}\left(g,\varphi\right). \tag{2.4}$$

Also from (2.1) we obtain for a sequence of values of r tending to infinity that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[n]}\varphi\left(r\right)}\geq\lambda_h^{(p,q)}\left(f,\varphi_1\right)\rho^{(m,n)}\left(g,\varphi\right). \tag{2.5}$$

Similarly we get from (2.2) that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[n]}\varphi\left(r\right)}\geq\rho_h^{(p,q)}\left(f,\varphi_1\right)\lambda^{(m,n)}\left(g,\varphi\right). \tag{2.6}$$

Therefore from (2.4) and (2.5) , we get for $\lambda_h^{(p,q)}\left(f,\varphi_1\right)>0$ that

$$\lambda_h^{(p,q)}\left(f,\varphi_1\right)\rho^{(m,n)}\left(g,\varphi\right)\leq\rho_h^{(p,n)}\left(f\circ g,\varphi\right)\leqslant\rho_h^{(p,q)}\left(f,\varphi_1\right)\rho^{(m,n)}\left(g,\varphi\right). \tag{2.7}$$

Similarly, from (2.4) and (2.6) we obtain for $\lambda^{(m,n)}\left(g,\varphi\right)>0$ that

$$\rho_h^{(p,q)}\left(f,\varphi_1\right)\lambda^{(m,n)}\left(g,\varphi\right)\leq\rho_h^{(p,n)}\left(f\circ g,\varphi\right)\leqslant\rho_h^{(p,q)}\left(f,\varphi_1\right)\rho^{(m,n)}\left(g,\varphi\right). \tag{2.8}$$

Also from (2.7) and (2.8) one can easily verify that

$$\rho_h^{(p-1,n)}\left(f\circ g,\varphi\right)=\infty,$$

$$\rho_h^{(p,n-1)}\left(f\circ g,\varphi\right)=0$$

and

$$\rho_h^{(p+1,n+1)}\left(f\circ g,\varphi\right)=1,$$

and therefore we obtain that the relative index-pair of $f\circ g$ is (p,n)-φ when $q=m$ and either $\lambda_h^{(p,q)}\left(f,\varphi_1\right)>0$ or $\lambda^{(m,n)}\left(g,\varphi\right)>0$ and thus the first part of the theorem is established.
Case II. Let $q>m$. Now we obtain from (2.3) for all sufficiently large values of r that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)\leqslant\left(\rho_h^{(p,q)}\left(f,\varphi_1\right)+\varepsilon\right)\log^{[q-m]}\left[\left(\rho^{(m,n)}\left(g,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right)\right]$$

$$i.e.,\ \log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)\leqslant\left(\rho_h^{(p,q)}\left(f,\varphi_1\right)+\varepsilon\right)\log^{[q+n-m]}\varphi\left(r\right)+O(1)$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[q+n-m]}\varphi\left(r\right)}\leqslant\rho_h^{(p,q)}\left(f,\varphi_1\right). \tag{2.9}$$

Also from (2.1) we have for a sequence of values of r tending to infinity that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)\geq\left(\lambda_h^{(p,q)}\left(f,\varphi_1\right)-\varepsilon\right)\log^{[q-m]}\left[\left(\rho^{(m,n)}\left(g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)\right]$$

$$i.e.,\ \log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)\geq\left(\lambda_h^{(p,q)}\left(f,\varphi_1\right)-\varepsilon\right)\log^{[q-m+n]}\varphi\left(r\right)+O(1)$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[q+n-m]}\varphi\left(r\right)}\geq\lambda_h^{(p,q)}\left(f,\varphi_1\right). \tag{2.10}$$

Similarly we get from (2.2) that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(r\right)\right)}{\log^{[q+n-m]}\varphi\left(r\right)}\geq\rho_h^{(p,q)}\left(f,\varphi_1\right). \tag{2.11}$$

Therefore from (2.9) and (2.10) , we get for $\lambda_h^{(p,q)}(f,\varphi_1) > 0$ that

$$\lambda_h^{(p,q)}(f,\varphi_1) \leq \rho_h^{(p,q+n-m)}(f\circ g,\varphi) \leqslant \rho_h^{(p,q)}(f,\varphi_1). \tag{2.12}$$

Likewise, from (2.9) and (2.11) we get for $\lambda^{(m,n)}(g,\varphi) > 0$ that

$$\rho_h^{(p,q+n-m)}(f\circ g,\varphi) = \rho_h^{(p,q)}(f,\varphi_1). \tag{2.13}$$

Hence from (2.12) and (2.13) one can easily verify that

$$\rho_h^{(p-1,q+n-m)}(f\circ g,\varphi) = \infty,$$

$$\rho_h^{(p,q+n-m-1)}(f\circ g,\varphi) = 0$$

and

$$\rho_h^{(p+1,q+n-m+1)}(f\circ g,\varphi) = 1,$$

and therefore we get that the relative index-pair of $f\circ g$ is $(p,q+n-m)$-φ when $q>m$ and either $\lambda_h^{(p,q)}(f,\varphi_1) > 0$ or $\lambda^{(m,n)}(g,\varphi) > 0$ and thus the second part of the theorem follows.

Case III. Let $q<m$. Then we obtain from (2.3) for all sufficiently large values of r that

$$\log^{[p+m-q]}\widehat{|h|}(|f\circ g|(r)) \leqslant \left(\rho^{(m,n)}(g,\varphi)+\varepsilon\right)\log^{[n]}\varphi(r)+O(1)$$

$$i.e., \lim_{r\to+\infty}\frac{\log^{[p+m-q]}\widehat{|h|}(|f\circ g|(r))}{\log^{[n]}\varphi(r)} \leqslant \rho^{(m,n)}(g,\varphi). \tag{2.14}$$

Also from (2.1) we have for a sequence of values of r tending to infinity that

$$\log^{[p+m-q]}\widehat{|h|}(|f\circ g|(r)) \geq \left(\rho^{(m,n)}(g,\varphi)-\varepsilon\right)\log^{[n]}\varphi(r)+O(1)$$

$$\limsup_{r\to\infty}\frac{\log^{[p+m-q]}\widehat{|h|}(|f\circ g|(r))}{\log^{[n]}\varphi(r)} \geq \rho^{(m,n)}(g,\varphi). \tag{2.15}$$

Similarly, we get from (2.2) that

$$\limsup_{r\to\infty}\frac{\log^{[p+m-q]}\widehat{|h|}(|f\circ g|(r))}{\log^{[n]}\varphi(r)} \geq \lambda^{(m,n)}(g,\varphi). \tag{2.16}$$

Therefore from (2.14) and (2.15) , we obtain for $\lambda_h^{(p,q)}(f,\varphi_1) > 0$ that

$$\rho_h^{(p+m-q,n)}(f\circ g,\varphi) = \rho^{(m,n)}(g,\varphi). \tag{2.17}$$

Similarly, from (2.14) and (2.16) we get for $\lambda^{(m,n)}(g,\varphi) > 0$ that

$$\lambda^{(m,n)}(g,\varphi) \leq \rho_h^{(p+m-q,n)}(f\circ g,\varphi) \leqslant \rho^{(m,n)}(g,\varphi). \tag{2.18}$$

So from (2.17) and (2.18) one can easily verify that

$$\rho_h^{(p+m-q-1,n)}(f\circ g,\varphi) = \infty, \quad \rho_h^{(p+m-q,n-1)}(f\circ g,\varphi) = 0$$

and

$$\rho_h^{(p+m-q+1,n+1)}(f\circ g,\varphi) = 1,$$

and therefore we obtain that the relative index-pair of $f\circ g$ is $(p+m-q,n)$-φ when $q<m$ and either $\lambda_h^{(p,q)}(f,\varphi_1) > 0$ or $\lambda^{(m,n)}(g,\varphi) > 0$ and thus the third part of the theorem is established. □

In the line of Theorem 2.1 one can easily deduce the conclusion of the following theorem and so its proof is omitted.

Theorem 2.2. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$. Also let the relative index pair of f with respect to h and the index pair of g be (p,q)-φ_1 and (m,n)-φ respectively. Then*

$$(i)\ \lambda_h^{(p,q)}(f,\varphi_1)\,\lambda^{(m,n)}(g,\varphi) \le \lambda_h^{(p,n)}(f\circ g,\varphi) \leqslant \min\left\{\rho_h^{(p,q)}(f,\varphi_1)\,\lambda^{(m,n)}(g,\varphi),\lambda_h^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)\right\}$$
$$\text{if } q=m,\ \lambda_h^{(p,q)}(f,\varphi_1)>0 \text{ and } \lambda^{(m,n)}(g,\varphi)>0;$$

$$(ii)\ \lambda_h^{(p,q+n-m)}(f\circ g,\varphi) = \lambda_h^{(p,q)}(f,\varphi_1) \quad \text{if } q>m,\ \lambda_h^{(p,q)}(f,\varphi_1)>0 \text{ and } \lambda^{(m,n)}(g,\varphi)>0$$

and

$$(iii)\ \lambda_h^{(p+m-q,n)}(f\circ g,\varphi) = \lambda^{(m,n)}(g,\varphi) \quad \text{if } q<m,\ \lambda_h^{(p,q)}(f,\varphi_1)>0 \text{ and } \lambda^{(m,n)}(g,\varphi)>0.$$

Corollary 2.1. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$. Also let the relative index pair of f with respect to h and the index pair of g be $(p-l,m-l)$-φ_1 and (m,n)-φ respectively such that $p-l>0$ and $m-l>0$. Then*

$$\rho_h^{(p,n)}(f\circ g,\varphi)=\rho^{(m,n)}(g,\varphi) \text{ and } \lambda_h^{(p,n)}(f\circ g,\varphi)=\lambda^{(m,n)}(g,\varphi).$$

Proof. In view of Definition 1.6,

$$\rho_h^{(p,m)}(f,\varphi_1)=\lambda_h^{(p,m)}(f,\varphi_1)=1.$$

Therefore the conclusion of the corollary immediately follows from the first part of Theorem 2.1 and Theorem 2.2. □

Corollary 2.2. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$. Also let the relative index pair of f with respect to h and the index pair of g be $(p-1,m-1)$-φ_1 and (m,n)-φ respectively such that $p>1$ and $m>1$. Then*

$$(i)\ \lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\sigma^{(m,n)}(g,\varphi)\le\sigma_h^{(p,n)}(f\circ g,\varphi)\le\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\sigma^{(m,n)}(g,\varphi)$$

$$(ii)\ \lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\sigma}^{(m,n)}(g,\varphi)\le\overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\le\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\sigma}^{(m,n)}(g,\varphi)$$

$$(iii)\ \lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\tau^{(m,n)}(g,\varphi)\le\tau_h^{(p,n)}(f\circ g,\varphi)\le\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\tau^{(m,n)}(g,\varphi)$$

and

$$(iv)\ \lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\tau}^{(m,n)}(g,\varphi)\le\overline{\tau}_h^{(p,n)}(f\circ g,\varphi)\le\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\tau}^{(m,n)}(g,\varphi).$$

Proof. In view of Lemma 2.1 and Corollary 2.1, we get that

$$\sigma_h^{(p,n)}(f\circ g,\varphi)\le\limsup_{r\to\infty}\frac{\log^{[p-1]}\widehat{|h|}(|f|(|g|(r)))}{\log^{[m-1]}|g|(r)}\cdot\limsup_{r\to\infty}\frac{\log^{[m-1]}|g|(r)}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho^{(m,n)}(g,\varphi)}}$$

$$i.e.,\ \sigma_h^{(p,n)}(f\circ g,\varphi)\le\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\sigma^{(m,n)}(g,\varphi). \tag{2.19}$$

Similarly

$$\sigma_h^{(p,n)}(f\circ g,\varphi)\ge\liminf_{r\to\infty}\frac{\log^{[p-1]}\widehat{|h|}(|f|(|g|(r)))}{\log^{[m-1]}|g|(r)}\cdot\limsup_{r\to\infty}\frac{\log^{[m-1]}|g|(r)}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho^{(m,n)}(g,\varphi)}}$$

$$i.e.,\ \sigma_h^{(p,n)}(f\circ g,\varphi)\ge\lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\sigma^{(m,n)}(g,\varphi). \tag{2.20}$$

Hence the first part of corollary follows from (2.19) and (2.20). Further

$$\overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\geq \liminf_{r\to\infty}\frac{\log^{[p-1]}\widehat{|h|}(|f|(|g|(r)))}{\log^{[m-1]}|g|(r)}\cdot\liminf_{r\to\infty}\frac{\log^{[m-1]}|g|(r)}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho^{(m,n)}(g,\varphi)}}$$

Again

$$i.e.,\ \overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\geq \lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\sigma}^{(m,n)}(g,\varphi). \tag{2.21}$$

Also

$$\overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\leq \limsup_{r\to\infty}\frac{\log^{[p-1]}\widehat{|h|}(|f|(|g|(r)))}{\log^{[m-1]}|g|(r)}\cdot\liminf_{r\to\infty}\frac{\log^{[m-1]}|g|(r)}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho^{(m,n)}(g,\varphi)}}$$

$$i.e.,\ \overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\leq \rho_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\sigma}^{(m,n)}(g,\varphi). \tag{2.22}$$

Therefore the second part of corollary follows from (2.21) and (2.22). Reasoning similarly as in the proofs of above, one can easily deduce the conclusion of third and fourth part of corollary, and so its proofs are omitted. Thus the corollary follows. □

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