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Approximation by complex Chlodowsky-Szasz-Durrmeyer operators in compact disks

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ABSTRACT. This paper presents a study on the approximation properties of the operators constructed by the composition of Chlodowsky operators and Szasz-Durrmeyer operators. We give the approximation properties and obtain a Voronovskaya-type result for these operators for analytic functions of exponential growth on compact disks. Furthermore, a numerical example with an illustrative graphic is given to compare for the error estimates of the operators.

1. INTRODUCTION

Lorentz [10] was the first person who studied the approximation properties of complex Bernstein polynomials on compact disks. Very recently, the problem of the approximation of complex operators has become an important issue on the approximation theory. In [5], Gal studied the Voronovskaja-type result for complex Bernstein polynomials on compact disks. Various extensions and generalizations of complex Bernstein polynomials have been considered by Gupta [6] and Anastassiou [1]. In [8] Ispir introduced the complex modified Szasz-Mirakjan operators and after that N. Cetin and N. Ispir [3] obtained Voronovskaja type results for these operators. They estimated the exact orders of approximation and also proved that the complex modified Szasz-Mirakjan operators preserve the geometric properties on unit disk. Also, many researchers have studied complex Bernstein-Durrmeyer operators, Szasz-Mirakjan operator and its Durrmeyer variant in complex domain. For details we refer the readers to [2], [4], [7] and [9]. İzgi [9] defined the following operators which are combination Chlodowsky and Szasz-Durrmeyer operators on C[0,∞), as

$$Z_n\left(f,x\right)=\frac{n}{b_n}\sum_{k=0}^{n}p_{n,k}\left(\frac{x}{b_n}\right)\int\limits_0^\infty s_{n,k}\left(\frac{t}{b_n}\right)f\left(t\right)dt,\qquad 0\leq x\leq b_n \tag{1.1}$$

where $p_{n,k}(u)=\begin{cases}\binom{n}{k}u^k(1-u)^{n-k}, 0\leq k\leq n\\ 0, k>n.\end{cases}$, $s_{n,k}(x)=e^{-nx}\dfrac{(nx)^k}{k!}$ and (b_n) is a positive and increasing sequence with properties $\lim_{n\to\infty}b_n=\infty$ and $\lim_{n\to\infty}\dfrac{b_n^2}{n}=0$.

In this study we shall prove theorems giving some approximation properties of the complex Chlodowsky-Szasz-Durrmeyer operators attached to analytic functions having suitable exponential growth on compact disks. Then, we obtain Voronovskaja type result. The complex Chlodowsky-Szasz-Durrmeyer operators are obtained from the real version,

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simply by replacing the real variable x by the complex variable z in the operators defined by (1), which is given below:

$$\mathcal{Z}_n(f,x) = \frac{n}{b_n}\sum_{k=0}^{n} p_{n,k}\left(\frac{z}{b_n}\right)\int_0^\infty s_{n,k}\left(\frac{t}{b_n}\right) f(t)\,dt, \qquad 0 \le x \le b_n$$

where $z \in \mathbb{C}$ is such that $0 \le \mathbb{R}(z) \le b_n$ and (b_n) is a positive and increasing sequence with properties $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \dfrac{b_n}{n} = 0$. Throughout the present article we denote $D_R := \{z \in \mathbb{C} : |z| < R, R > 1\}$. By H_R, we mean the class of all functions satisfying : f: $[R;\infty) \cup \overline{D_R} \to \mathbb{C}$ is continuous in $[R;\infty) \cup \overline{D_R}$, analytic in D_R i.e. $f(z) = \sum\limits_{p=0}^{\infty} c_p z^p$ for all $z \in D_R$.

2. Auxiliary results

In this section, we shall need the following auxiliary results.

Lemma 2.1. *Let $e_p(z) = z^p$ and $\mathrm{K}_{n,p}(z) = \mathcal{Z}_n(e_p, z)$, for all $e_p = t^p$, $p \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$, then we have that $\mathcal{Z}_n(e_0, z) = 1$ and*

$$\mathrm{K}_{n,p+1}(z) = \frac{z(b_n - z)}{n}\mathrm{K}'_{n,p}(z) + \frac{(nz + (p+1)b_n)}{n}\mathrm{K}_{n,p}(z).$$

Proof. By a simple calculation, we obtain $p'_{n,k}\left(\frac{z}{b_n}\right) = \dfrac{kb_n - zn}{z(b_n - z)} p_{n,k}\left(\frac{z}{b_n}\right)$. It follows that

$$\begin{aligned} K'_{n,p}(z) &= \frac{n}{b_n}\sum_{k=0}^{n} p'_{n,k}\left(\frac{z}{b_n}\right)\int_0^\infty s_{n,k}\left(\frac{t}{b_n}\right) f(t)\,dt \\ z(b_n - z)K'_{n,p}(z) &= \frac{n}{b_n}\sum_{k=0}^{n} p_{n,k}\left(\frac{z}{b_n}\right)\int_0^\infty (k+1)b_n - nt)s_{n,k}\left(\frac{t}{b_n}\right) f(t)\,dt \\ &\quad + nK_{n,p+1}(z) - (b_n + zn)K_{n,p}(z) \end{aligned}$$

Using

$$b_n^2\left(\frac{t}{b_n}s_{n,k}\left(\frac{t}{b_n}\right)\right)' = (k+1)b_n - nt)s_{n,k}\left(\frac{t}{b_n}\right)$$

we obtain

$$\begin{aligned} z(b_n - z)K'_{n,p}(z) &= \frac{n}{b_n}\sum_{k=0}^{n} p_{n,k}\left(\frac{z}{b_n}\right)\int_0^\infty b_n^2\left(\frac{t}{b_n}s_{n,k}\left(\frac{t}{b_n}\right)\right)' f(t)\,dt \\ &\quad + nK_{n,p+1}(z) - (b_n + zn)K_{n,p}(z) \end{aligned}$$

Also, using integration by parts, we have

$$z(b_n - z)K'_{n,p}(z) = -b_n pK_{n,p}(z) + nK_{n,p+1}(z) - (b_n + zn)K_{n,p}(z)$$

So, in conclusion, we have

$$z(b_n - z)K'_{n,p}(z) = nK_{n,p+1}(z) - ((p+1)b_n + zn)K_{n,p}(z)$$

which implies the recurrence in the statement. □

Lemma 2.2. *Let* $f \in [R,+\infty) \cup \overline{D_R}$ *is analytic in* D_R *and there exists* $B, C > 0$ *such that* $|f(x)| \leq Ce^{Bx}$ *, for all* $x \in [R, b_n]$ *. Denoting* $f(z) = \sum_{p=0}^{\infty} c_p z^p$ *for all* $z \in D_R$ *and* $1 \leq r < R$*. Then for all* $|z| \leq r$ *and* $n \in \mathbb{N}$*, we get*

$$\mathcal{Z}_n(f,x) = \sum_{p=0}^{\infty} c_p \mathcal{Z}_n(e_p, z).$$

Proof. For any $m \in \mathbb{N}$, we define

$$f_m(z) = \sum_{p=0}^{m} c_p z^p \text{ if } |z| \leq r \text{ and } f_m(x) = f(x) \text{ if } x \in (r, b_n].$$

Since $|f_m(z)| \leq \sum_{p=0}^{\infty} |c_p| r^p = C_r$ for $|z| \leq r$ and $m \in \mathbb{N}$ and f_m is bounded and integrable on $[0,b_n]$ and this implies that for each fixed $m, n \in \mathbb{N}$, $\frac{n}{b_n} > B$,

$$\mathcal{Z}_n(f_m, z) \leq \frac{n}{b_n} \sum_{k=0}^{n} \left| p_{n,k}\left(\frac{z}{b_n}\right) \right| \int_0^{\infty} s_{n,k}\left(\frac{t}{b_n}\right) |f_m(t)|\, dt < \infty.$$

Therefore $\mathcal{Z}_n(f_m, z)$ is well defined. Similarly, for the function f, it follows that $\mathcal{Z}_n(f, x)$ is also well defined and it is an analytic function of z. Denoting

$$f_{m,p}(z) = c_p e_p(z) \text{ if } |z| \leq r \text{ and } f_{m,p}(x) = \frac{f(x)}{m+1} \text{ if } x \in (r, b_n],\ 1 \leq r < R,$$

it is clear that each $f_{m,p}$ is of exponential growth on $[0,\infty)$ and that $f_m(z) = \sum_{p=0}^{m} f_{m,p}(z)$, by the linearity of $\mathcal{Z}_n$, it follows that

$$\mathcal{Z}_n(f_m, z) = \sum_{p=0}^{m} c_p \mathcal{Z}_n(e_p, z) \text{ for all } |z| \leq r \text{ and } m, n \in \mathbb{N}.$$

It is sufficent to prove that for any fixed $n \in \mathbb{N}$,

$$\lim_{m\to\infty} \mathcal{Z}_n(f_m, z) = \mathcal{Z}_n(f, z)$$

uniformly in compact disk $|z| \leq r$. But this is immediate from

$$\lim_{m\to\infty} \|f_m - f\|_r = 0, \text{ from } \|f_m - f\|_{B[0,\infty)} \leq \|f_m - f\|_r$$

and from the inequality

$$\begin{aligned} |\mathcal{Z}_n(f_m, z) - \mathcal{Z}_n(f, z)| &\leq \frac{n}{b_n} \sum_{k=0}^{n} \left| p_{n,k}\left(\frac{z}{b_n}\right) \right| \int_0^{\infty} s_{n,k}\left(\frac{t}{b_n}\right) |f_m(t) - f(t)|\, dt \\ &\leq \|f_m - f\|_r \left\{ 1 + \sum_{k=0}^{n} \binom{n}{k} \left(\frac{r}{b_n}\right)^k \right\} \\ &= M_{n,r} \|f_m - f\|_r \end{aligned}$$

valid for all $|z| \leq r$, where $\|.\|_{B[0,\infty)}$ denotes the uniform norm on $C[0,\infty)$-the space of all complex-valued bounded functions on $[0,\infty)$. Thus, as $m \to \infty$, we get the required result. □

3. Approximation by complex Chlodowsky-Szasz-Durrmeyer operators

The first main result is expressed by the following upper estimate for $\mathcal{Z}_n(f_m, z)$ in a compact disk.

Theorem 3.1. *Let $f : [R,\infty) \cup \overline{D_R} \to \mathbb{C}$ be continuous in $[R,\infty) \cup D_R$ and analytic in D_R there exists $B, C > 0$ such that $|f(x)| \le Ce^{Bx}$, for all $x \in [R, +\infty)$. Further, let f be bounded and integrable in* [0, ∞). *Suppose that there exists $M > 0$ and $A \in \left(\frac{1}{R}, 1\right)$ with the property $|c_p| \le \frac{MA_p}{(2p)!}$, $\forall p \in N \cup \{0\}$. Let $1 \le r < \frac{1}{A}$ be arbitrary but fixed then for all $|z| \le r$ and $n \ge n_0$, $n_0 \in \mathbb{N}$, we have*

$$|\mathcal{Z}_n(f,z) - f(z)| \le C_{r,A}(f)\frac{(b_n+1)}{n+1} \quad \textit{where } C_{r,A}(f) = M\sum_{p=1}^{\infty}(Ar)^p.$$

Proof. By using the recurrence relation of Lemma 1, we have

$$K_{n,p+1}(z) = \frac{z(b_n - z)}{n}K'_{n,p}(z) + \frac{(nz + (p+1)b_n)}{n}K_{n,p}(z), \forall z \in \mathbb{C}, p, n \in \mathbb{N}.$$

From this we immediately get the recurrence formula

$$\begin{aligned} K_{n,p}(z) - z^p &= \frac{z(b_n - z)}{n}\left(K_{n,p-1}(z) - z^{p-1}\right)' + \frac{(nz + pb_n)}{n}\left(K_{n,p-1}(z) - z^{p-1}\right) \\ &\quad + \frac{((2p-1)b_n - (p-1)z)}{n}z^{p-1}, \forall z \in \mathbb{C}, p, n \in \mathbb{N}. \end{aligned}$$

Now, for $1 \le r < R$, by linear transformation the Bernstein's inequality in the closed unit disk becomes $P'_p(z) \le \frac{p}{r}\|P_p\|_r$, for all $|z| \le r$, where $P_p(z)$ is a polynomial of degree $\le p$. Thus, from the above recurrence relation, we get

$$\begin{aligned} \|K_{n,p}(z) - e_p\|_r &\le \frac{r(b_n - r)}{n}\|K_{n,p-1}(z) - e_{p-1}\|_r\frac{p-1}{r} + \frac{(nr + pb_n)}{n}\|K_{n,p-1}(z) - e_{p-1}\|_r \\ &\quad + \frac{((2p-1)b_n - (p-1)r)}{n}r^{p-1} \\ &\le r\left(1 + \frac{2p(b_n+1)}{n}\right)\|K_{n,p-1}(z) - e_{p-1}\|_r + \frac{2p(b_n+1)}{n}r^p. \end{aligned}$$

In what follows we prove the result by mathematical induction with respect to p, that this recurrence relation implies

$$\|K_{n,p}(z) - e_p\|_r \le r^p\frac{2^{p+1}p!(b_n+1)}{n}, \text{for all } p \in \mathbb{N}, n \ge n_0, n_0 \in \mathbb{N}.$$

Suppose that it is valid for p, the above recurrence relation implies that

$$\begin{aligned} \|K_{n,p+1}(z) - e_{p+1}\|_r &\le r\left(1 + \frac{2(p+1)(b_n+1)}{n}\right)\|K_{n,p}(z) - e_p\|_r + \frac{2(p+1)(b_n+1)}{n}r^{p+1} \\ &\le r^{p+1}\frac{2^{p+2}(p+1)!(b_n+1)}{n}. \end{aligned}$$

It is easy to see by mathematical induction that this last inequality holds true for all $p \ge 1$ and $n \ge n_0, n_0 \in \mathbb{N}$. From the hypothesis on f, by Lemma 2 we can write

$$\mathcal{Z}_n(f,z) = \sum_{p=0}^{\infty}c_p\mathcal{Z}_n(e_p,z) = \sum_{p=0}^{\infty}c_pK_{n,p}(z), \text{for all } z \in D_R, n \in \mathbb{N},$$

which from the hypothesis on c_p immediately implies for all $|z| \leq r$ with $Re(z) \leq b_n, n \in \mathbb{N}$ with $n \geq n_0, n_0 \in \mathbb{N}$

$$|\mathcal{Z}_n(f,z) - f(z)| \leq \sum_{p=1}^{\infty} \frac{M(Ar)^p(b_n+1)}{n+1} = C_{r,A}(f)\frac{(b_n+1)}{n+1},$$

where $C_{r,A}(f) = M\sum\limits_{p=1}^{\infty}(Ar)^p < \infty$ for all $1 \leq r < \frac{1}{A}$,by ratio test. Thus, the proof is completed. □

4. Voronovskaja-type result

In the following theorem we obtain a quantitative Voronovskaja-type result:

Theorem 4.2. *Let* $f \in H_R$ *and be bounded and integrable on* $[0, \infty)$ *and suppose that there exists* $M > 0$ *and* $A \in \left(\frac{1}{R}, 1\right)$ *with the property* $|c_p| \leq \dfrac{MA^p}{(2p)!}$ *and* $|f(x)| \leq Ce^{Bx}$ *, for all* $x \in [R, +\infty)$ *. Let* $1 \leq r < \dfrac{1}{A}$ *be arbitrary but fixed then for all* $|z| \leq r$ *and* $p \in N \cup \{0\}$*,* $n \geq n_0, n_0 \in \mathbb{N}$*, we have*

$$\left|\mathcal{Z}_n(f,z) - f(z) - \frac{n}{b_n}\left(f^{'}(z) + z\left(1-\frac{z}{2b_n}\right)f^{''}(z)\right)\right|$$
$$\leq \left(\frac{b_n+1}{n}\right)^2\left(\frac{b_n+2}{n}+1\right)M\left(\sum_{p=1}^{[\alpha]}p\,(Ar)^p + 2\frac{1}{(1-Ar)\log\frac{1}{Ar}}\right)$$

Proof. By using Lemma 2, we may write

$$\frac{n}{b_n}\left(f^{'}(z) + z\left(1-\frac{z}{2b_n}\right)f^{''}(z)\right) = \frac{n}{b_n}\sum_{p=0}^{\infty}c_p\left(p^2 e_{p-1} - \frac{p^2-p}{2b_n}e_p\right)$$

Defining $K_{n,p}(z) = \mathcal{Z}_n(e_p, z)$, we get

$$\left|\mathcal{Z}_n(f,z) - f(z) - \frac{n}{b_n}\left(f^{'}(z) + z\left(1-\frac{z}{2b_n}\right)f^{''}(z)\right)\right|$$
$$\leq \sum_{p=0}^{\infty}|c_p|\left|K_{n,p}(z) - e_p(z) - \frac{n}{b_n}\left(f^{'}(z) + z\left(1-\frac{z}{2b_n}\right)f^{''}(z)\right)\right|,$$

for all $z \in D_R, n \in \mathbb{N}$. Now, by applying Lemma 1, we get the following recurrence relation

$$K_{n,p}(z) = \frac{z(b_n-z)}{n}K^{'}_{n,p-1}(z) + \frac{(nz+pb_n)}{n}K_{n,p-1}(z)$$

Let us denote $\lambda_{n,p}(z) = K_{n,p}(z) - e_p(z) - \frac{n}{b_n}\left(f^{'}(z) + z\left(1-\frac{z}{2b_n}\right)f^{''}(z)\right)$ .Thus

$$\lambda_{n,p}(z) = \frac{z(b_n-z)}{n}\lambda^{'}_{n,p-1}(z) + \frac{(nz+pb_n)}{n}\lambda_{n,p-1}(z) + \beta_{n,p}(z),$$

where $|\beta_{n,p}(z)| \leq \left(\dfrac{b_n+1}{n}\right)^2(p^3+5p)\,r^p, \forall n \in \mathbb{N}$.It is immediate that $\beta_{n,p}(z)$ is a polynomial in z of degree $\leq p$ and that
$\beta_{n,0}(z) = 0$. Combining (2) and (3), we have

$$|\lambda_{n,p}(z)| \leq \frac{z(b_n-z)}{n}\left|\lambda^{'}_{n,p-1}(z)\right| + \frac{(nz+pb_n)}{n}|\lambda_{n,p-1}(z)| + \left(\frac{b_n+1}{n}\right)^2(p^3+5p)\,r^p.$$

Now, we shall find the estimate of $\lambda_{n,p-1}^{'}(z)$ for $p \geq 1$. Taking into account the fact that $\lambda_{n,p-1}(z)$ is a polynomial of degree $\leq p-1$, we have

$$\begin{aligned}\left|\lambda_{n,p-1}^{'}(z)\right| &\leq \frac{p-1}{r}\left\|\lambda_{n,p-1}(z)\right\|_r \\ &\leq \left(\left(\frac{b_n+1}{n}\right)^2\left((p-1)^4+5(p-1)^2\right)+\frac{b_n}{n}(p-1)^3+\frac{(p-2)^2(p-1)}{2n}\right)r^{p-2}, \forall n\in\mathbb{N}.\end{aligned}$$

Thus for $\forall n \in \mathbb{N}$,

$$\frac{rb_n+r^2}{n^2}\left|\lambda_{n,p-1}^{'}(z)\right| \leq \frac{(p-1)^3}{n^3}(1+rb_n)\left(\frac{(b_n+1)^2}{n}\left(p^2-2p\right)+b_n(p-1)+\frac{(p-1)}{2}\right)r^{p-2}$$

and for all $|z| \leq r$ and $n \in \mathbb{N}, 1 \leq p \leq \dfrac{n}{b_n}$,

$$|\lambda_{n,p}(z)| \leq (r+1)\left|\lambda_{n,p-1}(z)\right| + \left(\frac{b_n+1}{n}\right)^2 2^{(p+3)}p!\left(\frac{b_n+2}{n}+1\right)r^p.$$

We easily obtain step by step the following

$$\begin{aligned}|\lambda_{n,p}(z)| &\leq \left(\frac{b_n+1}{n}\right)^2\left(\frac{b_n+2}{n}+1\right)r^p\sum_{j=1}^{p}2^{(j+3)}j! \\ &\leq \left(\frac{b_n+1}{n}\right)^2\left(\frac{b_n+2}{n}+1\right)r^p p 2^{(p+3)}p!\end{aligned}$$

Denoting by $[\alpha]$ the integral part of , it follows that

$$\begin{aligned}&\left|\mathcal{Z}_n(f,z)-f(z)-\frac{n}{b_n}\left(f^{'}(z)+z\left(1-\frac{z}{2b_n}\right)f^{''}(z)\right)\right| \\ \leq\ & \sum_{p=1}^{[\alpha]}|c_p|\left(\frac{b_n+1}{n}\right)^2\left(\frac{b_n+2}{n}+1\right)r^p p 2^{(p+3)}p! + \sum_{p=[\alpha]+1}^{\infty}|c_p|\,|\lambda_{n,p}(z)| \\ \leq\ & \left(\frac{b_n+1}{n}\right)^2\left(\frac{b_n+2}{n}+1\right)M\sum_{p=1}^{[\alpha]}p\,(Ar)^p + \sum_{p=[\alpha]+1}^{\infty}|c_p|\,|\lambda_{n,p}(z)|\end{aligned}$$

But

$$\begin{aligned}\sum_{p=[\alpha]+1}^{\infty}|c_p|\,|\lambda_{n,p}(z)| &\leq \sum_{p=[\alpha]+1}^{\infty}|c_p|\left(|K_{n,p}(z)-e_p(z)|+\frac{b_n}{n}\left|p^2e_{p-1}(z)-\frac{(p^2-p)}{2b_n}e_p(z)\right|\right) \\ &\leq 2M\left(\frac{b_n+1}{n}\right)\sum_{p=1}^{[\alpha]}(Ar)^p \leq 2M\left(\frac{b_n+1}{n}\right)\frac{(Ar)^{\alpha}}{(1-Ar)},\end{aligned}$$

where $\forall n \geq n_0, n_0 \in \mathbb{N}$. By elementary calculations for all $|z| \leq r$ and $n \geq n_0, n_0 \in \mathbb{N}$, we get

$$\sum_{p=[\alpha]+1}^{\infty}|c_p|\,|\lambda_{n,p}(z)| \leq 2M\left(\frac{b_n+1}{n}\right)\left(\frac{b_n+2}{n}+1\right)\frac{1}{(1-Ar)\log\frac{1}{Ar}}.$$

Finally, we obtain

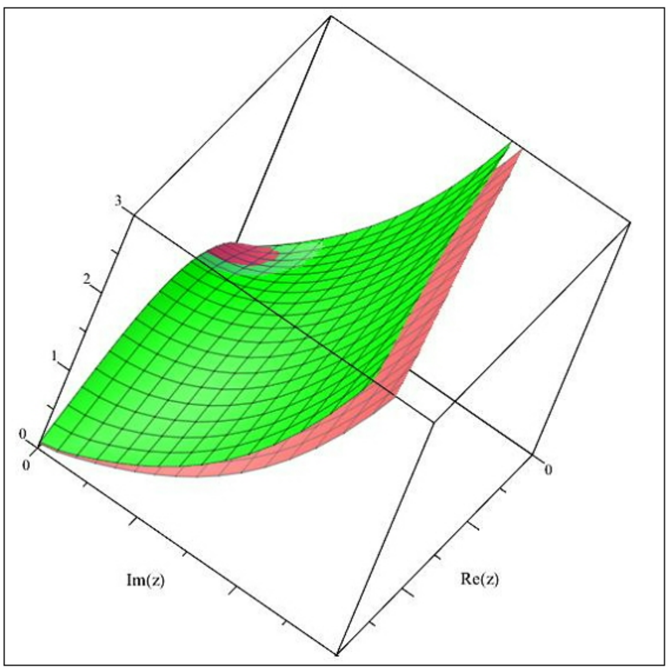
$$\left| \mathcal{Z}_n(f,z) - f(z) - \frac{n}{b_n}\left(f'(z) + z\left(1 - \frac{z}{2b_n}\right)f''(z)\right)\right|$$
$$\leq \left(\frac{b_n+1}{n}\right)^2\left(\frac{b_n+2}{n}+1\right)M\left(\sum_{p=1}^{[\alpha]} p\,(Ar)^p + 2\frac{1}{(1-Ar)\log\frac{1}{Ar}}\right)$$

where for $rA < 1$, by ratio test the above series is convergent. This completes the proof of the theorem. □

5. Example

In the following examples we show a comparison for the error estimates of the function $f(x)$ and the operator $\mathcal{Z}_n(f,z)$ by using the software "MAPLE 17 ".

Example 5.1. Choosing $f(z) = e^{(-z/5)}\sin(z)$; we compute the error estimations of the complex Chlodowsky-Szasz-Durrmeyer operators $\mathcal{Z}_n(f,z)$ given in (2) and $f(z)$. Here we take $b_n = ln(1+n)$, $n = 55$.



Curves for the error estimates of $\mathcal{Z}_n(f,z)$ (green) and $f(z)$

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