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An alternative proof of a Tauberian theorem for the weighted mean summability of integrals over $\mathbb{R}_+$

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ABSTRACT. Let $0 \not\equiv q(x)$ be a nondecreasing function on $\mathbf{R}_+ := [0, \infty)$ such that $q(0) = 0$ and

$$\limsup_{x\to\infty} \frac{q(x)}{q(\rho x)} < 1$$

for every $\rho > 1$. Given a real- or complex- valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(u)du$ and its weighted mean as

$$\sigma(s(x)) := \frac{1}{q(x)} \int_0^x s(u)dq(u), \quad x > 0$$

provided $q(t) > 0$.

If the limit $\lim_{x\to\infty} s(x) = l$ exists, then $\lim_{x\to\infty} \sigma(s(x)) = l$ also exists. However, the converse is not true in general. The converse implication might hold under some appropriate conditions. These conditions are called Tauberian conditions and theorems involving these conditions are called Tauberian theorems.

In this paper, we have given an alternative proof of the Tauberian theorem stating that the existence of $\lim_{x\to\infty} s(x) = l$ follows from that of $\lim_{x\to\infty} \sigma(s(x)) = l$ and a Tauberian condition.

If $f(x)$ is a real or complex-valued function, then this Tauberian condition is one-sided or two-sided, respectively.

1. INTRODUCTION

Let $0 \not\equiv q(x)$ be a nondecreasing function on $\mathbf{R}_+ := [0, \infty)$ such that $q(0) = 0$ and

$$\gamma := \limsup_{x\to\infty} \frac{q(x)}{q(\rho x)} < 1 \tag{1.1}$$

for every $\rho > 1$. Let f be a real- or complex-valued function which is integrable in Lebesgue's sense over every finite interval $(0, x)$ for $0 < x < \infty$, in symbol $f \in L^1_{loc}(\mathbf{R}_+)$. We define

$$s(x) := \int_0^x f(u)du \tag{1.2}$$

and its weighted mean as

$$\sigma(s(x)) := \frac{1}{q(x)} \int_0^x s(u)dq(u), \quad x > 0$$

provided $q(x) > 0$.

If

$$\lim_{x\to\infty} \sigma(s(x)) = l, \tag{1.3}$$

then we say that $s(x)$ is said to be summable to l by the weighted mean method determined by the function $q(x)$, in short, $(\overline{N}, q)$ summable to l.

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Recall that condition (1.1) implies that $\lim_{x\to\infty} q(x) = \infty$, which is necessary and sufficient condition for every $(\overline{N}, q)$ summable $s(x)$ to be convergent to the same limit [4].

If the limit

$$\lim_{x\to\infty} s(x) = l \tag{1.4}$$

exists, then (1.3) also exists. However, the converse implication is not true in general. The converse implication might hold under some appropriate conditions. These conditions are called Tauberian conditions and theorems involving these conditions are called Tauberian theorems.

A number of authors such as Fekete and Móricz [2] and Totur and Okur [13] have recently obtained one-sided and two-sided Tauberian conditions for the weighted mean method $(\overline{N}, q)$ of integrals. For more detailed studies related to Tauberian theorems for the weighted mean method $(\overline{N}, q)$, we refer the reader to Özsaraç and Çanak [7], Tietz and Zeller [10], Totur and Çanak [12], Totur et al. [14], etc.

In this paper, we give an alternative proof of a Tauberian theorem stating that the existence of the limit $\lim_{x\to\infty} s(x) = l$ follows from that of $\lim_{x\to\infty} \sigma(s(x)) = l$ and a Tauberian condition.

Tauberian conditions are either one-sided or two-sided if the functions are real-valued or complex-valued, respectively. As special cases, we obtain Tauberian conditions of Landau and Hardy type for weighted mean summable integrals.

2. Preliminaries

A real-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly decreasing [6] if

$$\lim_{\rho\to 1^+} \liminf_{t\to\infty} \min_{t\le x\le \rho t} (s(x) - s(t)) \ge 0. \tag{2.5}$$

It is easy to see that a real-valued function $s(x)$ is slowly decreasing if and only if for every $\epsilon > 0$ there exist $t_0 = t_0(\epsilon) > 0$ and $\rho = \rho(\epsilon) > 1$ such that $s(x) - s(t) \ge -\epsilon$ whenever $t_0 \le t \le x \le \rho t$.

An equivalent reformulation of (2.5) can be given as follows:

$$\lim_{\rho\to 1^-} \liminf_{t\to\infty} \min_{\rho t\le x\le t} (s(t) - s(x)) \ge 0. \tag{2.6}$$

It is easy to see that a real-valued function $s(x)$ is slowly decreasing if and only if for every $\epsilon > 0$ there exist $t_1 = t_1(\epsilon) > 0$ and $\rho = \rho(\epsilon)$ with $0 < \rho < 1$ such that $s(t) - s(x) \ge -\epsilon$ whenever $t_1 \le \rho t \le x \le t$.

Note that the concept of slow decrease was introduced by Schmidt [9] for sequences of real numbers.

The concept of regularly varying of index $\alpha > 0$ was introduced by Karamata [5] as follows (see [1] for more details):

A nondecreasing function $q(x)$ defined on $\mathbf{R}_+$ with $q(0) = 0$ is called regularly varying of index $\alpha > 0$ if

$$\lim_{x\to\infty} \frac{q(\rho x)}{q(x)} = \rho^\alpha, \quad \rho > 0. \tag{2.7}$$

Remark 2.1. Condition (2.5) holds if $q(x)$ is regularly varying of index $\alpha > 0$ and there exist constant $C > 0$ and x_0 such that $\frac{q(x)}{q'(x)} f(x) \ge -C$ for all $x > x_0$.

Indeed, for all $x_0 < t < x < \infty$ we have

$$s(x) - s(t) = \int_t^x f(u)du \ge -C \int_t^x \frac{q'(u)}{q(u)} du = -C \ln \frac{q(x)}{q(t)},$$

whence it follows that

$$\min_{t\le x\le \rho t}(s(x)-s(t)) \ge -C\ln\frac{q(\rho t)}{q(t)}$$

for $x > x_0$ and $\rho > 1$. Taking the $\liminf$ of both sides as $t \to \infty$, we have

$$\liminf_{t\to\infty}\min_{t\le x\le \rho t}(s(x)-s(t)) \ge -C\ln\rho^{\alpha}.$$

Choosing ρ sufficiently close to 1, inequality (2.5) is satisfied.

A complex-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly oscillating [9] if

$$\lim_{\rho\to 1^+}\limsup_{t\to\infty}\max_{t\le x\le \rho t}|s(x)-s(t)| = 0. \tag{2.8}$$

It is easy to see that a complex-valued function $s(x)$ is slowly oscillating if and only if for every $\epsilon > 0$ there exist $t_0 = t_0(\epsilon) > 0$ and $\rho = \rho(\epsilon) > 1$ such that $|s(x)-s(t)| \le \epsilon$ whenever $t_0 \le t \le x \le \rho t$.

An equivalent reformulation of (2.5) can be given as follows:

$$\lim_{\rho\to 1^-}\liminf_{t\to\infty}\max_{\rho t\le x\le t}|s(t)-s(x)| = 0. \tag{2.9}$$

It is easy to see that a complex-valued function $s(x)$ is slowly oscillating if and only if for every $\epsilon > 0$ there exist $t_1 = t_1(\epsilon) > 0$ and $\rho = \rho(\epsilon)$ with $0 < \rho < 1$ such that $|s(t)-s(x)| \le \epsilon$ whenever $t_1 \le \rho t \le x \le t$.

Note that the concept of slow oscillation was introduced by Hardy [3] for sequences of real numbers.

Remark 2.2. Condition (2.8) holds if $q(x)$ is regularly varying of index $\alpha > 0$ and there exist constant $C > 0$ and x_0 such that $\frac{q(x)}{q'(x)}|f(x)| \le C$ for all $x > x_0$.

3. An auxiliary result

We need the following lemma for the proof of our main results.

Lemma 3.1. *Let* $0 \not\equiv q : \mathbf{R}_+ \to \mathbf{R}_+$ *be a nondecreasing function such that* $q(0) = 0$. *Then condition* (1.1) *is equivalent with*

$$\delta := \limsup_{x\to\infty}\frac{q(\rho x)}{q(x)} < 1 \tag{3.10}$$

for every $0 < \rho < 1$ *are equivalent.*

Proof. Let $\rho < 1$. Then

$$\limsup_{x\to\infty}\frac{q(\rho x)}{q(x)} = \limsup_{x\to\infty}\frac{q(\rho x)}{q(\frac{1}{\rho}\rho x)}. \tag{3.11}$$

Let $t = \rho x$. Then, since $\dfrac{1}{\rho} > 1$, we have

$$\limsup_{x\to\infty}\frac{q(\rho x)}{q(x)} = \limsup_{t\to\infty}\frac{q(t)}{q(\frac{1}{\rho}t)} < 1 \tag{3.12}$$

by (1.1), which completes the proof of (1.1) $\Rightarrow$ (3.10).

Let $\rho > 1$. Then

$$\limsup_{x\to\infty}\frac{q(x)}{q(\rho x)} = \limsup_{x\to\infty}\frac{q(\frac{1}{\rho}\rho x)}{q(\rho x)}. \tag{3.13}$$

Let $t=\rho x$. Then, since $\dfrac{1}{\rho}<1$, we have

$$\limsup_{x\to\infty}\frac{q(x)}{q(\rho x)}=\limsup_{t\to\infty}\frac{q(\frac{1}{\rho}t)}{q(t)}<1 \tag{3.14}$$

by (3.10), which completes the proof of (3.10) $\Rightarrow$ (1.1). □

4. Main results

An alternative proof of a Tauberian theorem for Cesàro summability of sequences of real numbers was given by Peyerimhoff [8]. Using the proof techniques in [8], we give an alternative proof of the following Tauberian theorems [6] for the weighted mean summability of integrals over $\mathbf{R}_+$ in this work.

First, we consider a real-valued function and prove the following Tauberian theorem in which the Tauberian condition is one-sided.

Theorem 4.1. *Let $0\not\equiv q:\mathbf{R}_+\to\mathbf{R}_+$ be a nondecreasing function such that $q(0)=0$ and (1.1) holds for every $\rho>1$. If a real-valued function $f\in L^1_{loc}(\mathbf{R}_+)$ is such that (1.3) holds and its integral function $s(x)$ is slowly decreasing, then (1.4) holds.*

Proof. Since the weighted mean method is regular, we may assume that $l=0$. Otherwise we consider $s(x)-l$, which is slowly decreasing. Suppose that $s(x)$ is not convergent to 0 as $x\to\infty$.

Then either $\limsup_{x\to\infty}s(x)>0$ or $\liminf_{x\to\infty}s(x)<0$. First we consider the first case. Then there exist $\alpha>0$ and a sequence (n_i) such that $s(x_{n_i})\geq\alpha$ for all nonnegative integers i. If we take $\epsilon=\frac{\alpha}{2}$ in the equivalent form of (2.5), then we find $\rho>1$ and $t_0\geq0$ such that $s(x_m)\geq s(x_{n_i})-\frac{\alpha}{2}\geq\frac{\alpha}{2}$ for $t_0\leq x_{n_i}<x_m\leq x_{m_i}=\rho x_{n_i}$. From

$$\begin{aligned}\sigma(s(x_{m_i}))-\frac{q(x_{n_i})}{q(x_{m_i})}\sigma(s(x_{n_i})) &= \frac{1}{q(x_{m_i})}\int_0^{x_{m_i}}s(u)dq(u)\\ &-\frac{q(x_{n_i})}{q(x_{m_i})}\frac{1}{q(x_{n_i})}\int_0^{x_{n_i}}s(u)dq(u)\\ &=\frac{1}{q(x_{m_i})}\int_{x_{n_i}}^{x_{m_i}}s(u)dq(u),\end{aligned}$$

we have

$$\begin{aligned}\sigma(s(x_{m_i}))-\frac{q(x_{n_i})}{q(x_{m_i})}\sigma(s(x_{n_i})) &= \frac{1}{q(x_{m_i})}\int_{x_{n_i}}^{x_{m_i}}s(u)dq(u)\\ &\geq\frac{\alpha}{2}\frac{1}{q(x_{m_i})}\int_{x_{n_i}}^{x_{m_i}}q(u)du\\ &=\frac{\alpha}{2}\frac{1}{q(x_{m_i})}(q(x_{m_i})-q(x_{n_i}))\\ &=\frac{\alpha}{2}\left(1-\frac{q(x_{n_i})}{q(x_{m_i})}\right)\end{aligned}$$

for $t_0\leq x_{n_i}<x_m\leq x_{m_i}=\rho x_{n_i}$. By (1.1), we obtain $0\geq\frac{\alpha}{2}(1-\gamma)$ which is a contradiction. Hence, we have

$$\limsup_{x\to\infty}s(x)\leq0. \tag{4.15}$$

Next we consider the second case. Then there exist $\beta<0$ and a sequence (x_{n_i}) such that $s(x_{n_i})\leq\beta<0$ for all nonnegative integers i. If we take $\epsilon=-\dfrac{\beta}{2}$ in the equivalent form of (2.5), then we find $0 < \rho < 1$ and $t_1 = t_1(\epsilon)$ such that $s(x_n) \leq s(x_{n_i}) - \frac{\beta}{2} \leq \frac{\beta}{2}$ for $t_1 \leq x_{m_i} = \rho x_{n_i} < x_n \leq x_{n_i}$. From

$$\begin{aligned} \sigma(s(x_{n_i})) - \frac{q(x_{m_i})}{q(x_{n_i})}\sigma(s(x_{m_i})) &= \frac{1}{q(x_{n_i})}\int_0^{x_{n_i}} s(u)dq(u) \\ &- \frac{q(x_{m_i})}{q(x_{n_i})}\frac{1}{q(x_{m_i})}\int_0^{x_{m_i}} s(u)dq(u) \\ &= \frac{1}{q(x_{n_i})}\int_{x_{m_i}}^{x_{n_i}} s(u)dq(u), \end{aligned}$$

we have

$$\begin{aligned} \sigma(s(x_{n_i})) - \frac{q(x_{m_i})}{q(x_{n_i})}\sigma(s(x_{m_i})) &= \frac{1}{q(x_{n_i})}\int_{x_{m_i}}^{x_{n_i}} s(u)dq(u) \\ &\leq \frac{\beta}{2}\frac{1}{q(x_{n_i})}\int_{x_{m_i}}^{x_{n_i}} dq(u) \\ &= \frac{\beta}{2}\frac{1}{q(x_{n_i})}(q(x_{n_i}) - q(x_{m_i})) \\ &= \frac{\beta}{2}\left(1 - \frac{q(x_{m_i})}{q(x_{n_i})}\right) \end{aligned}$$

for $t_1 \leq x_{m_i} = \rho x_{n_i} \leq x_n \leq x_{n_i}$. By (3.10), we obtain $0 \leq \frac{\beta}{2}(1-\delta)$ which is a contradiction. Hence, we have

$$\liminf_{x\to\infty} s(x) \geq 0. \tag{4.16}$$

Combining (4.15) and (4.16) yields convergence of $s(x)$ to 0 as $x \to \infty$. □

A real-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly increasing if $-s$ is slowly decreasing. It is trivial that a real-valued function $s(x)$ defined on $\mathbf{R}_+$ is slowly oscillating if and only if it is both slowly decreasing and slowly increasing.

Remark 4.3. Theorem 4.1 remains true if slow decrease of $s(x)$ is replaced by slow increase of $s(x)$.

Second, we consider a complex-valued function and prove the following Tauberian theorem in which the Tauberian condition is two-sided.

Theorem 4.2. *Let $0 \not\equiv q : \mathbf{R}_+ \to \mathbf{R}_+$ be a nondecreasing function such that $q(0) = 0$ and (1.1) holds for every $\rho > 1$. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is is such that (1.3) holds and its integral function $s(x)$ is slowly oscillating, then (1.4) holds.*

Remark 4.4. We note that if a complex-valued function $s(x)$ is slowly oscillating, then $\Re s(x)$ and $\Im s(x)$ are slowly oscillating, where $\Re$ and $\Im$ denote the real and the imaginary parts of a complex-valued function $s(x)$, respectively. Since one can easily follow the lines of the previous theorem for the real and the imaginary parts of a complex-valued function $s(x)$ to prove Theorem 4.2, we omit the proof of it.

If $q(x) = x$, then $(\overline{N}, q)$ summability method reduces to Cesàro summability method (see [4] and [11]) and we immediately have the following corollaries for Theorems 4.1 and 4.2, respectively.

Corollary 4.1. ([4]) *If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that $s(x)$ is Cesàro summable to a finite number l and is slowly decreasing, then (1.4) holds.*

Corollary 4.2. ([4]) *If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that $s(x)$ is Cesàro summable to a finite number l and slowly oscillating, then (1.4) holds.*

5. CONCLUSION

In a book written by Peyerimhoff [8], a Tauberian theorem for Cesàro summability of sequences of real numbers has been proved. Móricz [6] has studied the weighted mean summability of integrals over $\mathbf{R}_+$ and obtained necessary and sufficient Tauberian conditions for this method over $\mathbf{R}_+$. In the present work, we have given an alternative proof of a Tauberian theorem [4] for the weighted mean summability of integrals over $\mathbf{R}_+$ inspired by a proof techniques of Peyerimhoff [8]. As a continuation of this work, one can easily obtain alternative proofs of Tauberian theorems for the weighted mean summability of integrals of functions of two variables over $\mathbf{R}_+ \times \mathbf{R}_+$.

REFERENCES

[1] Bingham, N. H., Goldie, C. M. and Teugels, J. L., *Regular Variation*, Cambridge University Press, Camdridge, 1987

[2] Fekete, Á. and Móricz, F., *Necessary and sufficient Tauberian conditions in the case of weighted mean summable integrals over R_+. II*, Publ. Math. Debrecen., **67** (2005), No. 1-2, 65–78

[3] Hardy, G. H., *Theorems relating to the summability and convergence of slowly oscillating series*, Proc. London Math. Soc., (2)., **8** (1910), 310–320

[4] Hardy, G. H., *Divergent Series*, Chelsea Publishing Company, New York, 1991

[5] Karamata, J., *Sur un mode de croissance régulière. Théorèmes fondamentaux*, Bull. Soc. Math. France., **61** (1933), 55–62

[6] Móricz, F., *Necessary and sufficient Tauberian conditions in the case of weighted mean summable integrals over $\mathbf{R}_+$*, Math. Inequal. Appl., **7** (2004), No. 1, 87–93

[7] Özsaraç, F. and Çanak, İ., *Tauberian theorems for iterations of weighted mean summable integrals*, Positivity, **23** (2019), No. 1, 219–231

[8] Peyerimhoff, A., *Lectures on summability*, Springer, Berlin, 1969

[9] Schmidt, R., *Über divergente Folgen und lineare Mittelbildungen*, Math. Z., **22** (1925), 89–152

[10] Tietz, H. and Zeller, K., *Tauber-Sätze für bewichtete Mittel*, Arch. Math. (Basel), **68** (1997), No. 3, 214–220

[11] Titchmarsh, E. C., *Introduction to the theory of Fourier Integrals*, Clarendon Press, Oxford, 1937

[12] Totur, Ü. and Çanak, İ., *Some general Tauberian conditions for the weighted mean summability method*, Comput. Math. Appl., **63** (2012), No. 5, 999–1006

[13] Totur, Ü. and Okur, M. A., *Alternative proofs of some classical Tauberian theorems for the weighted mean method of integrals*, Filomat, **29** (2015), No. 10, 2281–2287

[14] Totur, Ü., Okur, M. A. and Çanak, İ., *One-sided Tauberian conditions for the $(\overline{N}, p)$ summability of integrals*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **80** (2018), No. 3, 65–74

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