

Solving a system of integral equations by using some tripled fixed point theorems

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ABSTRACT. A tripled fixed point theorems in ordered metric spaces is used in order to prove the existence and uniqueness of a solution for a class of integral equations. The conditions of the theorem are much weaker than those existing in literature and the theorem is useful for solving some general problems. An example to illustrate our theoretical results is also given.

1. INTRODUCTION

Fixed point theory in partial ordered metric spaces, that is, in metric spaces endowed with a partial ordering, has developed rapidly in recent years. The study of coupled fixed point theory has been considered in 2004 by Ran and Reurings [16] and in 2006 by Bhaskar and Lakshmikantham [10]. A rich literature on the existence of coupled fixed points of mixed monotone, monotone and non-monotone mappings, has been developed ever since publication of that paper (see [7], [8], [14]). Berinde and Borcut [6], introduced the concept of tripled fixed point and proved some related fixed point theorems. After that various results on the existence of tripled fixed points for various classes of mappings have been obtained, see [4], [11], [12], [13], [15]), for a selection of them.

In this paper we obtain the existence of a solution for a class of tripled integral equations in the framework of tripled fixed point theorem on partially ordered metric spaces.

2. PRELIMINARIES

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

Let (X, d) be a partially ordered set and d be a metric on X , with precondition that (X, d) is complete metric space. In addition, we define such a partial order in the product space $X \times X \times X$, for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \iff x \geq u, y \leq v, z \geq w.$$

Definition 2.1. ([11]) Let (X, d) be a partially ordered set, $F : X \times X \times X \rightarrow X$ be a mapping. If $F(x, y, z)$ is monotone non-decreasing in x and z , and non-increasing in y , that is, for any $x, y, z \in X$

$$x_1, x_2 \in X, x_1 \leq x_2 \implies F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \implies F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1, z_2 \in X, z_1 \leq z_2 \implies F(x, y, z_1) \leq F(x, y, z_2),$$

then we say that F has the *mixed-monotone property*.

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Definition 2.2. ([6]) Let $F : X \times X \times X \rightarrow X$ be a mapping. If

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z,$$

for any $(x, y, z) \in X \times X \times X$ then we say that (x, y, z) is a *tripled fixed point of F* .

Definition 2.3. ([6]) Let (X, d) be a metric space. On the product space X^3 , we consider the metric $\bar{d} : X \times X \times X \rightarrow X$, defined by

$$\bar{d}((x, y, z), (u, v, w)) = d(x, u) + d(y, v) + d(z, w), \quad (2.1)$$

for all $(x, y, z), (u, v, w) \in X^3$.

Theorem 2.1. ([6]) Let $F : X \times X \times X \rightarrow X$ be a continuous mapping satisfy the above mixed monotone property on X . Suppose that there exists a $k \in [0, 1)$ with

$$d(F(x, y, z), F(u, v, w)) \leq \frac{k}{3} [d(x, u) + d(y, v) + d(z, w)],$$

for any $(u, v, w) \leq (x, y, z)$. If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0),$$

then, there exists $x, y, z \in X$, such that

$$x = F(x, y, z), \quad y = F(y, x, y), \quad z = F(z, y, x).$$

Theorem 2.2. ([13]) Let (X, d) be a complete metric space, $F : X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property on X .

Suppose either,

1. F is continuous or;
2. X has the following properties:

- (a) if a non decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$, for all n ;
- (b) if a non increasing sequence $y_n \rightarrow y$, then $y_n \geq y$, for all n .

If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0),$$

then, there exists $x, y, z \in X$, such that

$$x = F(x, y, z), \quad y = F(y, x, y), \quad z = F(z, y, x).$$

3. MAIN RESULTS

In this section, inspired by the works [6], [13], we present an existence theorem for a nonlinear tripled system and we give an example to support our theoretical result. Consider the following system of integral equations

$$\begin{cases} x(t) = f(t) + \int_a^b K(t, s, x(s), y(s), z(s)) ds \\ y(t) = f(t) + \int_a^b K(t, s, y(s), x(s), y(s)) ds \\ z(t) = f(t) + \int_a^b K(t, s, z(s), y(s), x(s)) ds \end{cases} \quad (3.2)$$

where $a, b \in \mathbb{R}$, $a < b$, $x, y, z \in C[a, b]$, $f : [a, b] \rightarrow \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are given mapping.

In this context, we consider on $X = C[a, b]$ the metric

$$d(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|.$$

and, correspondingly, on X^3 we shall have the metric \bar{d} defined by (2.1)

$$\bar{d}((x, y, z), (u, v, w)) = d(x, u) + d(y, v) + d(z, w).$$

The main result of this note is the following theorem.

Theorem 3.3. *Consider the nonlinear system (3.2). Suppose that the following conditions hold:*

- i) $f : [a, b] \rightarrow \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are continuous;
- ii) $K(t, s, \cdot, \cdot, \cdot)$ has the generalized mixed monotone property with respect to the last three variables, for all $t, s \in [a, b]$;
- iii) There exists $\alpha, \beta, \gamma : [a, b] \times [a, b] \rightarrow \mathbb{R}_+$ such that, for each $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$ with $x_1 \leq y_1, x_2 \geq y_2, x_3 \leq y_3$ we have

$$\begin{aligned} & |K(t, s, x_1(s), x_2(s), x_3(s)) - K(t, s, y_1(s), y_2(s), y_3(s))| \\ & \leq \alpha(t, s) \cdot |x_1 - y_1| + \beta(t, s) \cdot |x_2 - y_2| + \gamma(t, s) |x_3 - y_3|; \\ \text{iv) } & \max_{t \in [a, b]} \left(\int_a^b \alpha(t, s) ds \right) < \frac{1}{3}, \max_{t \in [a, b]} \left(\int_a^b \beta(t, s) ds \right) < \frac{1}{3}, \max_{t \in [a, b]} \left(\int_a^b \gamma(t, s) ds \right) < \frac{1}{3}; \end{aligned}$$

- v) There exists $x_0, y_0, z_0 \in C[a, b]$ such that

$$\begin{cases} x_0(t) \leq \varphi(t) + \int_a^b K(t, s, x_0(s), y_0(s), z_0(s)) ds \\ y_0(t) \geq \varphi(t) + \int_a^b K(t, s, y_0(s), x_0(s), y_0(s)) ds \\ z_0(t) \leq \varphi(t) + \int_a^b K(t, s, z_0(s), y_0(s), x_0(s)) ds \end{cases} \quad (3.3)$$

or

$$\begin{cases} x_0(t) \geq \varphi(t) + \int_a^b K(t, s, x_0(s), y_0(s), z_0(s)) ds \\ y_0(t) \leq \varphi(t) + \int_a^b K(t, s, y_0(s), x_0(s), y_0(s)) ds \\ z_0(t) \geq \varphi(t) + \int_a^b K(t, s, z_0(s), y_0(s), x_0(s)) ds \end{cases} \quad (3.4)$$

for all $t \in [a, b]$.

Then, there exists a solution $(\bar{x}, \bar{y}, \bar{z})$ for the system (3.2).

Proof. We prove that all assumptions of Theorem 2.1 are satisfied.

Define $F : X \times X \times X \rightarrow X$ by

$$F(x, y, z) = \phi(t) + \int_a^b K(t, s, x(s), y(s), z(s)) ds,$$

for each $t \in [a, b]$.

Then the system (3.2) can be written as a tripled fixed point problem of F :

$$\begin{cases} F(x, y, z) = x \\ F(y, x, y) = y \\ F(z, y, x) = z \end{cases}$$

First, we will prove that F has the mixed monotone property. For $x_1 \leq x_2$ and $t \in [a, b]$, we have:

$$F(x_1, y, z) - F(x_2, y, z) = \int_a^b [K(t, s, x_1(s), y(s), z(s)) - K(t, s, x_2(s), y(s), z(s))] ds \leq 0,$$

due to condition (ii).

Thus, $F(x_1, y, z) \leq F(x_2, y, z)$ for every element $x_1 \leq x_2$, $t \in [a, b]$. Similary, we show that $F(x, y_1, z) \geq F(x, y_2, z)$ for all $y_1 \geq y_2$ and $F(x, y, z_1) \leq F(x, y, z_2)$ for all $z_1 \leq z_2$. Then, for all $x_1 \leq y_1$, $x_2 \geq y_2$, $x_3 \leq y_3$, we have

$$\begin{aligned} & d(F(x_1, x_2, x_3), F(y_1, y_2, y_3)) = \\ & \max_{t \in [a, b]} \left| \int_a^b [K(t, s, x_1(s), x_2(s), x_3(s)) - K(t, s, y_1(s), y_2(s), y_3(s))] ds \right| \leq \\ & \leq \max_{t \in [a, b]} \int_a^b |K(t, s, x_1(s), x_2(s), x_3(s)) - K(t, s, y_1(s), y_2(s), y_3(s))| ds \leq \\ & \leq \max_{t \in [a, b]} \int_a^b \alpha(t, s) ds \cdot d(x_1, y_1) + \max_{t \in [a, b]} \int_a^b \beta(t, s) ds \cdot d(x_2, y_2) + \max_{t \in [a, b]} \int_a^b \gamma(t, s) ds \cdot d(x_3, y_3). \end{aligned}$$

Using (iv), we find that all the assumptions of Theorem 3.3 are satisfied and hence the system (3.2) has a unique solution. \square

Example 3.1. For the system (3.2), let $a = 0$, $b = 1$ and

$$\begin{aligned} f(t) &= \frac{1}{2} + \frac{t}{2}, \\ \alpha(t, s) &= ts^3, \\ \beta(t, s) &= ts^4, \\ \gamma(t, s) &= ts^5, \end{aligned}$$

and

$$K(t, s, x(s), y(s), z(s)) = \alpha(t, s) \cdot x(s) - \beta(t, s) \cdot y(s) + \gamma(t, s) \cdot z(s).$$

Then (3.2) becomes

$$\begin{cases} x(t) = \frac{1}{2} + \frac{t}{2} + \int_0^1 [ts^3 x(s) - ts^4 y(s) + ts^5 z(s)] ds \\ y(t) = \frac{1}{2} + \frac{t}{2} + \int_0^1 [ts^3 y(s) - ts^4 x(s) + ts^5 y(s)] ds \\ z(t) = \frac{1}{2} + \frac{t}{2} + \int_0^1 [ts^3 z(s) - ts^4 y(s) + ts^5 x(s)] ds \end{cases}$$

$$F(x, y, z) = \frac{1}{2} + \frac{t}{2} + \int_0^1 \alpha(t, s) \cdot x(s) - \beta(t, s) \cdot y(s) + \gamma(t, s) \cdot z(s)$$

Obviously,

$$\begin{aligned} \max_{t \in [0,1]} \int_0^1 ts^3 ds &= \max_{t \in [0,1]} t \cdot \frac{s^4}{4} \Big|_0^1 = \max_{t \in [0,1]} \frac{t}{4} \leq \frac{1}{4} < \frac{1}{3}, \\ \max_{t \in [0,1]} \int_0^1 ts^4 ds &= \max_{t \in [0,1]} t \cdot \frac{s^5}{5} \Big|_0^1 = \max_{t \in [0,1]} \frac{t}{5} \leq \frac{1}{5} < \frac{1}{3}, \\ \max_{t \in [0,1]} \int_0^1 ts^5 ds &= \max_{t \in [0,1]} t \cdot \frac{s^6}{6} \Big|_0^1 = \max_{t \in [0,1]} \frac{t}{6} \leq \frac{1}{6} < \frac{1}{3}. \end{aligned}$$

Let $x_0 = 0$, $y_0 = 2$, $z_0 = 0$, we have

$$\begin{cases} x_0 \leq F(x_0, y_0, z_0) = \frac{1}{2}, \\ y_0 \geq F(x_0, y_0, z_0) = \frac{1}{2} + \frac{t}{10}, \\ z_0 \leq F(x_0, y_0, z_0) = \frac{1}{2} + \frac{t}{10}, \end{cases} \quad \text{for any } t \in [0, 1].$$

As a results, all the conditions of Theorem 3.3 are satisfied. This means that (3.2) has a triplet unique solution

$$\begin{cases} x^* = F(x^*, y^*, z^*), \\ y^* = F(y^*, x^*, y^*), \\ z^* = F(z^*, y^*, x^*). \end{cases} \quad (x^*, y^*, z^*) \in C[0, 1] \times C[0, 1] \times C[0, 1].$$

4. CONCLUSIONS

In this paper, we obtained a tripled fixed point theorem which is suitable for a class of systems of integral equations. It is based on a fixed point theorem on partially ordered metric spaces. Moreover, we find that all the conditions of the theorem become weaker than those existing in literature. It is also shown that we can obtain much weaker conditions for some types of integral equations.

A concrete example to illustrate the theoretical results is also considered in all computational details.

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