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Existence of positive solutions for half-linear fractional order BVPs by application of mixed monotone operators

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ABSTRACT. In this paper we developed tripled fixed point theorems of ternary operator on partially ordered metric spaces. As an application we established existence of positive solutions for half-linear fractional order boundary value problem.

1. INTRODUCTION

Fractional differential equations has recently attracted many researchers due to its wide applications [11, 17, 36] in engineering, technology, biology and so on. Establishing existence of solutions for fractional differential equations together with boundary conditions has been carried out by researchers [7, 12, 13, 20–22, 24, 32–35].

The study of turbulent flow through porous media has gained momentum and has wide range of scientific and engineering applications such as fluidized bed combustion, compact heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors [9] and drying of different products such as iron ore [23]. To study such type of problems, Leibenson [18] introduced the following p-Laplacian equation,

$$\big(\Phi_p(u'(t))\big)' = f\big(t, u(t), u'(t)\big),$$

where $\Phi_p(u) = |u|^{p-2}u,\ p > 1$. The operator Φ_p is invertible and its inverse operator is defined by Φ_q, where $q > 1$ is a constant such that $q = p/(p-1)$. Few works has been done for establishing the existence of positive solutions to Caputo fractional boundary value problems involving p-Laplacian operator, see [13, 20–22, 24, 32, 33]. The p-Laplacian operator and fractional calculus arises on many applied fields such as turbulent filtration in porousmedia, blood flow problems, rheology, modeling of viscoplasticity, material science, it is worth studying the fractional differential equations with p-Laplacian operator.

In 1957, Bihari [6] investigated the half-linear differential equation

$$(p(t)u')' + q(t)f(u, p(t)u') = 0 \tag{1.1}$$

where $f(u, v)$ satisfies a Lipschitz condition such that $uf(u, v) > 0$ (for $u \neq 0$) is homogeneous and extended the Strumian theorems to the equation (1.1).

In [10], S. Dhar and Q. Kong studied the following third-order half-linear differential equation

$$\Phi_{\alpha_2}\big(\big(\Phi_{\alpha_1}(u')\big)'\big)' + q(t)\Phi_{\alpha_1\alpha_2}(u) = 0$$

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where $q \in C(\mathbb{R}, \mathbb{R})$, $\Phi_p(u) = |u|^{p-1}u$, $p > 0$ and $\alpha_1, \alpha_2 > 0$, with the boundary conditions $u(a) = u(b) = 0, -\infty < a < b < \infty$ and some additional conditions, derived Lyapunov-type inequalities to the above equation. Gholami and Ghanbari [14], considered the following coupled systems of half-linear fractional order boundary value problem

$$\begin{aligned} \Phi_{\beta_2}\big({}^C D_{a^+}^{\alpha}\big(\Phi_{\beta_1}(u)\big)\big) + \lambda\Phi_{\beta_1\beta_2}\big(f\big(t, v\big)\big) = 0, \\ \Phi_{\gamma_2}\big({}^C D_{a^+}^{\beta}\big(\Phi_{\gamma_1}(v)\big)\big) + \mu\Phi_{\gamma_1\gamma_2}\big(f\big(t, u\big)\big) = 0, \\ u(a) = u(b) = v(a) = v(b) = 0 \end{aligned} \tag{1.2}$$

where $t \in (a, b),\ \alpha, \beta \in (1, 2),\ \Phi_\gamma(u) = |u|^{\gamma-1}u,\ \gamma, \alpha_i, \beta_i \in (0, +\infty), i = 1, 2,\ {}^C D_{a^+}^{\alpha}$ denotes the left sided Caputo fractional derivative of order α, and established existence of positive solutions by using Leray-Schauder and Krasnoselskii-Zabreiko fixed point theorems.

Fixed point theory is one of the most important area of research in Mathematics. In recent years, many results related to fixed point theorems in ordered metric spaces are established in [1–3, 27, 30] and etc. The results in this line was obtained by Ran and Reurings [29]. Subsequently, Nieto and Rodriguez-Lopez [26] extended the results by omitting the continuity hypothesis and applied their result to obtain a unique solution to a first order ordinary differential equation with periodic boundary conditions. Later, Bhaskar and Lakshmikantham [5] established several coupled fixed point theorems for mixed monotone mappings defined on partially ordered complete metric spaces. Recently, their work was extended by Liu, Mao and Shi in [19].

Inspired by the works mentioned above, in this paper, we establish some tripled fixed point theorems using mixed monotone operator on partially ordered complete metric spaces that generalizes [19]. As an application, we study the half-linear fractional order differential equation, for $0 < t < 1$,

$$\Phi_\beta\big({}^C D_{0+}^{r}\big(\Phi_\alpha(u(t))\big)\big) + \Phi_{\alpha\beta}\big(f\big(t, u(t)\big) + g\big(t, u(t)\big) + h\big(t, u(t)\big)\big) = 0, \tag{1.3}$$

satisfying the Sturm-Liouville type boundary conditions

$$\begin{aligned} a(\Phi_\alpha u)(0) - b(\Phi_\alpha u)'(0) = 0, \\ c(\Phi_\alpha u)(1) + d(\Phi_\alpha u)'(1) = 0, \end{aligned} \tag{1.4}$$

where $\Phi_\gamma(u) = |u|^{\gamma-1}u,\ \gamma, \alpha, \beta \in (0, +\infty),\ {}^C D_{0+}^{r}$ is left sided Caputo fractional derivative of order $r,\ 1 < r \le 2,\ f, g, h \in C([0, 1] \times [0, +\infty), [0, +\infty))$, a, b, c, d are real positive constants and then established unique positive solution for (1.3)-(1.4).

The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas which are useful in establishing our results. We construct the Green's function for the homogeneous problem corresponding to (1.3)-(1.4) and also we estimate bounds for the Green's function in Section 3. In Section 4, we establish tripled fixed point theorems. In last section, we establish the boundary value problem (1.3)-(1.4) has unique positive solution by an application of tripled fixed point theorems of ternary operator on partially ordered metric spaces and then finally an example is given to demonstrate our results.

2. Preliminaries

In this section, we provide some definitions and lemmas which are needed in the later discussion.

Definition 2.1. [11] Let $\alpha \in (0, +\infty)$. The operator $I_{a^+}^{\gamma}$ defined on $L_1[a, b]$ by

$$I_{a^+}^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds,$$

for $t \in [a, b]$, is called the left sided Riemann-Liouville fractional integral of order γ. Under same hypotheses, the right-sided Riemann-Liouville fractional integral operator is given by

$$_{b^-}I^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s) ds.$$

Definition 2.2. [11] Suppose $\gamma > 0$ with $n = [\gamma] + 1$. Then the left and right sided Caputo fractional derivatives defined on absolutely continuous functions space $AC^n[a, b]$ are given by

$$\left({}^C D_{a^+}^{\gamma} f\right)(t) = \left(I_{a^+}^{n-\gamma} D^n f\right)(t), \ \left({}_{b^-}^{C} D^{\gamma} f\right)(t) = (-1)^n \left({}_{b^-}I^{n-\gamma} D^n f\right)(t),$$

where $D^n := \frac{d^n}{dt^n}$.

Lemma 2.1. [17] *Let $\gamma > 0$. Then*

(i) *for $f(t) \in L_1(a, b)$, we have*

$$\left({}^C D_{a^+}^{\gamma} I_{a^+}^{\gamma} f\right)(t) = f(t), \ \left({}_{b^-}^{C} D^{\gamma}{}_{b^-}I^{\gamma} f\right)(t) = f(t).$$

(ii) *for $f(t) \in AC^n[a, b]$, we have*

$$\left(I_{a^+}^{\gamma}\, {}^C D_{a^+}^{\gamma} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k,$$

$$\left({}_{b^-}I^{\gamma}\, {}_{b^-}^{C} D^{\gamma} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!} (b-t)^k.$$

Definition 2.3. [4] Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Consider on the product space $X \times X \times X (= X^3$, in short) the following partial order, for $(x, y, z), (u, v, w) \in X^3$,

$$(x, y, z) \leq (u, v, w) \iff x \leq u, \ y \geq v, \ z \leq w.$$

Definition 2.4. [4] Let $(X, \leq)$ be a partially ordered set and $A : X^3 \to X$. We say that A has the mixed monotone property if $A(x, y, x)$ is monotone nondecreasing in x and z and is monotone non decreasing in y, i.e., for any $x, y, z \in X$,

$$x_1, x_2 \in X, \ x_1 \leq x_2 \implies A(x_1, y, z) \leq A(x_2, y, z),$$

$$y_1, y_2 \in X, \ y_1 \leq y_2 \implies A(x, y_1, z) \geq A(x, y_2, z),$$

$$z_1, z_2 \in X, \ z_1 \leq z_2 \implies A(x, y, z_1) \leq A(x, y, z_2).$$

Definition 2.5. [4] An element $(x, y, z) \in X^3$ is called a tripled fixed point of $A : X^3 \to X$ if

$$A(x, y, z) = x, \ A(y, x, y) = y, \ A(z, y, x) = z.$$

Definition 2.6. [16] A function $\phi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following conditions are satisfied:

(i) ϕ is continuous and nondecreasing,

(ii) $\phi(t) = 0 \iff t = 0$.

3. Green's function and bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1.3)-(1.4) and estimate bounds for the Green's function.

Lemma 3.2. *Let $h \in C(\mathbb{R})$. Then the boundary value problem*

$$\Phi_\beta\big({}^C D^r_{0+}\big(\Phi_\alpha(u(t))\big)\big) + h(t) = 0,\ 0 < t < 1, \tag{3.5}$$

$$\begin{aligned} a(\Phi_\alpha u)(0) - b(\Phi_\alpha u)'(0) = 0,\\ c(\Phi_\alpha u)(1) + d(\Phi_\alpha u)'(1) = 0, \end{aligned} \tag{3.6}$$

has a unique solution

$$u(t) = \Phi_{\alpha^{-1}}\Big(\int_0^1 G(t,s)\Phi_{\beta^{-1}}\big(h(s)\big)ds\Big), \tag{3.7}$$

where

$$G(t,s) := \begin{cases} G_1(t,s),\ 0 \le s \le t \le 1,\\ G_2(t,s),\ 0 \le t \le s \le 1, \end{cases} \tag{3.8}$$

$$G_1(t,s) = G_2(t,s) - \frac{(t-s)^{r-1}}{\Gamma(r)},$$

$$G_2(t,s) = \frac{\Delta}{\Gamma(r)}[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}](at+b),$$

and $\Delta = (ac+ad+bc)^{-1}$.

Proof. From Lemma 2.1, the equation (3.5) transforms to the fractional integral equation

$$\Phi_\alpha(u)(t) = A + Bt - \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}\Phi_{\beta^{-1}}\big(h(s)\big)ds.$$

By the boundary conditions (3.6), one can determine A and B as

$$A = \frac{\Delta b}{\Gamma(r)}\int_0^1 [c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}]\Phi_{\beta^{-1}}\big(h(s)\big)ds,$$

$$B = \frac{\Delta a}{\Gamma(r)}\int_0^1 [c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}]\Phi_{\beta^{-1}}\big(h(s)\big)ds.$$

Thus, we have

$$\begin{aligned}(\Phi_\alpha u)(t) &= \frac{\Delta}{\Gamma(r)}\int_0^1 [c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}](at+b)\Phi_{\beta^{-1}}\big(h(s)\big)ds\\ &\qquad - \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}\Phi_{\beta^{-1}}\big(h(s)\big)ds\\ &= \int_0^1 G(t,s)\Phi_{\beta^{-1}}\big(h(s)\big)ds.\end{aligned}$$

Therefore,

$$u(t) = \Phi_{\alpha^{-1}}\Big(\int_0^1 G(t,s)\Phi_{\beta^{-1}}\big(h(s)\big)ds\Big).$$

□

Lemma 3.3. *The Green's function $G(t,s)$ has the following properties:*

(i) *$G(t,s)$ is continuous on $[0,1]\times[0,1]$,*

(ii) *for $r > \dfrac{2a+b}{a+b}$, $G(t,s) > 0$ for any $t, s \in [0,1]$,*

(iii) *for* $r > \dfrac{2a+b}{a+b}$, $G(t,s) \le G(s,s)$ *for* $t, s \in [0,1]$,

(iv) *there exists* $\xi > 0$ *such that* $\xi G(s,s) \le G(t,s)$ *for* $t, s \in [0,1]$.

Proof. One can easily establish the property (i). To establish (ii), let $s,\ t \in [0,1]$ with $s \le t$, then we have

$$\frac{\partial G_1(t,s)}{\partial t} = \frac{a\Delta}{\Gamma(r)}[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}] - \frac{(r-1)(t-s)^{r-2}}{\Gamma(r)}$$

and

$$\frac{\partial^2 G_1(t,s)}{\partial t^2} = \frac{(r-1)(2-r)(t-s)^{r-3}}{\Gamma(r)} \ge 0.$$

This shows that $\frac{\partial G_1(t,s)}{\partial t}$ is increasing on $t \in [s,1]$. So, by $r > \frac{2a+b}{a+b}$,

$$\begin{aligned}\frac{\partial G_1(t,s)}{\partial t} &\le \frac{\partial G_1(1,s)}{\partial t}\\ &= \frac{a\Delta}{\Gamma(r)}[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}] - \frac{(r-1)(1-s)^{r-2}}{\Gamma(r)}\\ &\le \frac{ac\Delta + (ad\Delta - 1)(r-1)(1-s)^{r-2}}{\Gamma(r)} \le 0.\end{aligned}$$

Then $G_1(t,s)$ is decreasing with respect to t on $[s,1]$,

$$G_1(1,s) \le G_1(t,s) \le G_1(s,s).$$

Further,

$$\begin{aligned}G_1(1,s) &= \frac{\Delta}{\Gamma(r)}[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}](a+b) - \frac{(1-s)^{r-1}}{\Gamma(r)}\\ &= \frac{(1-s)^{r-2}}{\Gamma(r)}[-ad\Delta(1-s) + \Delta d(a+b)(r-1)] \ge \frac{\Delta(1-s)^{r-2}}{\Gamma(r)} ads > 0.\end{aligned}$$

When $0 \le t \le s \le 1$, we have

$$\frac{\partial G_2(t,s)}{\partial t} = \frac{a\Delta}{\Gamma(r)}[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}] \ge 0,$$

from this

$$0 < G_2(0,s) \le G_2(t,s) \le G_2(s,s).$$

From the proof of (ii), we have $G(t,s) \le G(s,s)$. Moreover, we have

$$\kappa(s) \le G(t,s) \le G(s,s),$$

where

$$\kappa(s) := \begin{cases} G_1(1,s), & 0 \le s < \dfrac{ad(2-r)+bc}{ad+bc},\\ G_2(0,s), & \dfrac{ad(2-r)+bc}{ad+bc} \le s < 1.\end{cases}$$

Since

$$\begin{aligned}\frac{G_1(1,s)}{G(s,s)} &= \frac{a+b}{as+b} - \frac{(1-s)^{r-1}}{\Delta[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}](as+b)}\\ &\le 1 - \frac{1}{\Delta[c+d(r-1)](as+b)} \le 1 - \frac{1}{\Delta[c+d(r-1)]b} \le \frac{bd(r-1) - a(c+d)}{bc+bd(r-1)} := \xi_1\end{aligned}$$

and

$$\frac{G_2(0,s)}{G(s,s)} = \frac{[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}]b}{[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}](as+b)} = \frac{b}{as+b} \ge \frac{b}{a+b} := \xi_2,$$

taking $\xi = \min\{\xi_1, \xi_2\}$, we get $\xi G(s,s) \leq G(t,s)$. □

4. Tripled fixed point theorems

In this section, we establish tripled fixed point theorems which will be useful in establishing our main results. Denote

$$\Psi = \{\psi \in C([0,+\infty),[0,+\infty)) : \psi(0) = 0 \text{ and for any } t > 0, \psi(t) > 0\}.$$

If ϕ is an altering distance function, then $\phi \in \Psi$.
We assume the following:

(H_1) $\psi \in \Psi$ (ψ is not necessarily altering distance function),

(H_2) $A : X^3 \to X$ is a mixed monotone mapping, there exists a constant $\lambda \in (0,1)$ such that

$$\begin{aligned}\phi\big(d(A(u,v,w),A(x,y,z)) + d(A(v,u,v),A(y,x,y)) + d(A(w,v,u),A(z,y,x))\big)\\ \leq \lambda\phi\big(d(u,x)+d(v,y)+d(w,z)\big) - \psi\big(\lambda(d(u,x)+d(v,y)+d(w,z))\big)\end{aligned}$$

and for each $u \geq x,\ v \leq y,\ w \geq z,\ \phi$ is an altering distance function which satisfies

$$\phi(t+s) \leq \phi(t) + \phi(s),\ t,s \in [0,+\infty),$$

(H_2') there exists $\psi \in \Psi,\ \lambda \in (0,1)$ and for $u \geq x,\ v \leq y,\ w \geq z,$

$$\begin{aligned}d(A(u,v,w),A(x,y,z)) + d(A(v,u,v),A(y,x,y)) + d(A(w,v,u),A(z,y,x))\\ \leq \lambda\big(d(u,x)+d(v,y)+d(w,z)\big) - \psi\big(\lambda(d(u,x)+d(v,y)+d(w,z))\big),\end{aligned}$$

(H_2'') there exists $\lambda \in (0,1)$ and for $u \geq x,\ v \leq y,\ w \geq z,$

$$\begin{aligned}d(A(u,v,w),A(x,y,z)) + d(A(v,u,v),A(y,x,y)) + d(A(w,v,u),A(z,y,x))\\ \leq \frac{\lambda}{3}\big[d(u,x)+d(v,y)+d(w,z)\big],\end{aligned}$$

(H_3) there exists $(u_0,v_0,w_0) \in X^3$ such that

$$u_0 \leq A(u_0,v_0,w_0), v_0 \geq A(v_0,u_0,v_0), w_0 \leq A(w_0,v_0,u_0),$$

(H_4) (a) A is continuous or
(b) X has the following properties:
(i) If a sequence $\{u_n\}$ is nondecreasing and converges to u, then $u_n \leq u, \forall n$,
(ii) If a sequence $\{v_n\}$ is nonincreasing and converges to v, then $v_n \geq v, \forall n$,
(iii) If a sequence $\{w_n\}$ is nondecreasing and converges to w, then $w_n \leq w, \forall n$.

(H_5) for every (u,v,w), $(u^*,v^*,w^*) \in X^3$, there exists $(x,y,z) \in X^3$ which is comparable to (u,v,w) and (u^*,v^*,w^*),

(H_6) every triple of elements in X has either a lower bound or an upper bound.

Theorem 4.1. *Let $(X,\leq)$ be a partially ordered set and let d be a metric on X such that (X,d) is a complete metric space. Assume that (H_1), (H_2), (H_3) and (H_4) hold. Then there exists $(u,v,w) \in X^3$ such that*

$$A(u,v,w) = u,\ A(v,u,v) = v \text{ and } A(w,v,u) = w.$$

Proof. Denote

$$u_1 = A(u_0,v_0,w_0) \geq u_0,\ v_1 = A(v_0,u_0,v_0) \leq v_0 \text{ and } w_1 = A(w_0,v_0,u_0) \geq w_0.$$

For $n \geq 1$, denote $u_n = A(u_{n-1}, v_{n-1}, w_{n-1})$, $v_n = A(v_{n-1}, u_{n-1}, v_{n-1})$ and $w_n = A(w_{n-1}, v_{n-1}, u_{n-1})$. Then, by mixed monotone property of A, it can be easily proved that

$$\begin{cases} u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots , \\ v_0 \geq v_1 \geq \cdots \geq v_n \geq \cdots , \\ w_0 \leq w_1 \leq \cdots \leq w_n \leq \cdots . \end{cases} \tag{4.9}$$

For simplicity, we denote $D_n^x = d(x_{n-1}, x_n),\ x = u, v, w$. Then by (H_1), we have

$$\begin{aligned} \phi\big(D_{n+1}^u + D_{n+1}^v + D_{n+1}^w\big) &\leq \lambda\phi\big(D_n^u + D_n^v + D_n^w\big) - \psi\big(\alpha(D_n^u + D_n^v + D_n^w)\big) \\ &\leq \lambda\phi\big(D_n^u + D_n^v + D_n^w\big). \end{aligned}$$

Now we claim that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are Cauchy sequences. For $m > n$, we have

$$\begin{aligned} &\phi\big(d(u_m, u_n) + d(v_m, v_n) + d(w_m, w_n)\big) \\ &\quad\leq\ \phi\big(d(u_m, u_{m-1}) + \cdots + d(u_{n+1}, u_n) + d(v_m, v_{m-1}) + \cdots + d(v_{n+1}, v_n) \\ &\qquad\qquad + d(w_m, w_{m-1}) + \cdots + d(w_{n+1}, w_n)\big) \\ &= \phi(D_m^u + D_{m-1}^u + \cdots + D_{n+1}^u + D_m^v + D_{m-1}^v + \cdots + D_{n+1}^v \\ &\qquad\qquad + D_m^w + D_{m-1}^w + \cdots + D_{n+1}^w) \\ &\leq \phi(D_m^u + D_m^v + D_m^w) + \phi(D_{m-1}^u + D_{m-1}^v + D_{m-1}^w) + \cdots \\ &\qquad\qquad + \phi(D_{n+1}^u + D_{n+1}^v + D_{n+1}^w) \\ &\leq \lambda^{m-1}\phi(D_1^u + D_1^v + D_1^w) + \lambda^{m-2}\phi(D_1^u + D_1^v + D_1^w) + \cdots \\ &\qquad\qquad + \lambda^n\phi(D_1^u + D_1^v + D_1^w) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})\phi(D_1^u + D_1^v + D_1^w) \\ &= \lambda^n\left(\frac{1 - \lambda^{m-n}}{1 - \lambda}\right)\phi(D_1^u + D_1^v + D_1^w). \end{aligned}$$

Since $\phi \in C([0, \infty), [0, \infty))$, $\phi(t) = 0 \iff t = 0$ and $\lambda \in (0, 1)$, it follows that

$$d(u_m, u_n) + d(v_m, v_n) + d(w_m, w_n) \to 0 \text{ as } n, m \to \infty.$$

Hence $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are Cauchy sequences. Since X is complete metric space, there exists $u, v, w \in X$ such that

$$u_n \to u,\ v_n \to v,\ w_n \to w \text{ as } n \to \infty. \tag{4.10}$$

Case(a): If A is continuous. Then

$$u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} A(u_{n-1}, v_{n-1}, w_{n-1}) = A\big(\lim_{n\to\infty} u_{n-1}, \lim_{n\to\infty} v_{n-1}, \lim_{n\to\infty} w_{n-1}\big) = A(u, v, w).$$

Similarly, we can establish other two identities,

$$v = A(v, u, v) \text{ and } w = A(w, v, u).$$

Hence, $(u, v, w) \in X^3$ is a tripled fixed point of A.
Case(b): Suppose X has the property (b) of (H_4). Then by (4.9) and (4.10), we have

$$u_n \geq u,\ v_n \leq v \text{ and } w_n \geq w \text{ for } n = 1, 2, \cdots . \tag{4.11}$$

From triangle inequality, we have

$$\begin{aligned} d\big(u, A(u, v, w)\big) &\leq d(u, u_{n+1}) + d\big(u_{n+1}, A(u, v, w)\big) \\ &= d(u, u_{n+1}) + d\big(A(u_n, v_n, w_n), A(u, v, w)\big). \end{aligned}$$

Since ϕ is nondecreasing, from $(H_1), (H_2)$, (4.11) and above inequality,

$$\begin{aligned}\phi\big(d(u, A(u,v,w))\big) &\leq \phi\big(d(u,u_{n+1}) + d(A(u_n,v_n,w_n), A(u,v,w))\big)\\ &\leq \phi\big(d(u,u_{n+1})\big) + \phi\big[d\big(A(u_n,v_n,w_n), A(u,v,w)\big)\big]\\ &\leq \phi\big(d(u,u_{n+1})\big) + \phi\big[d\big(A(u_n,v_n,w_n), A(u,v,w)\big)\\ &\quad + d\big(A(v_n,u_n,v_n), A(v,u,v)\big)\\ &\quad + d\big(A(w_n,v_n,u_n), A(w,v,u)\big)\big]\\ &\leq \phi\big(d(u,u_{n+1})\big) + \lambda\phi\big(d(u_n,u) + d(v_n,v) + d(w_n,w)\big)\\ &\qquad\qquad - \psi\big(\lambda(d(u_n,u) + d(v_n,v) + d(w_n,w))\big)\\ &\to 0 \quad \text{as } n \to \infty\end{aligned}$$

Since $\phi(t) = 0 \iff t = 0$, it follows that $u = A(u,v,w)$. Similarly, one can establish the other two identities . □

Corollary 4.1. *Let $(X, \leq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Assume $A : X^3 \to X$ is a mixed monotone operator and $(H_2'), (H_3)$ and (H_4) hold. Then A has tripled fixed point.*

Proof. Define a function $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t$ for $t \in [0,\infty)$. Then ϕ is an altering distance function and satisfies $\phi(t+s) = \phi(t) + \phi(s)$ for all $t, s \in [0,\infty)$. Hence, the corollary follows from Theorem 4.1.

□

Corollary 4.2. *Let $(X, \leq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Assume $A : X^3 \to X$ is a mixed monotone operator and $(H_2''), (H_3)$ and (H_4) hold. Then A has tripled fixed point.*

Proof. Define a function $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t/3$ for $t \in [0,\infty)$. Then ϕ is an altering distance function and satisfies $\phi(t+s) = \phi(t) + \phi(s)$ for all $t, s \in [0,\infty)$. Hence, the corollary follows from the Corollary 4.1. □

Theorem 4.2. *In addition to the hypothesis of Theorem 4.1, assume either (H_5) or (H_6) holds. Then the tripled fixed point of A is unique.*

Proof. From Theorem 4.1, the set of tripled fixed points of A is nonempty. Assume that (u,v,w) and $(u^*,v^*,w^*) \in X^3$ are two tripled fixed points of A.
Case(i): If (x,y,z) is comparable to (u,v,w) and (u^*,v^*,w^*). We define the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ as follows:

$$\begin{aligned} x_0 &= x, y_0 = y, z_0 = z,\\ x_{n+1} &= A(x_n,y_n,z_n),\ y_{n+1} = A(y_n,x_n,y_n),\ z_{n+1} = A(z_n,y_n,x_n),\ n \geq 0.\end{aligned}$$

Further, set $u_0 = u, v_0 = v, w_0 = w, u_0^* = u^*, v_0^* = v^*, w_0^* = w^*$ and define the sequences $\{u_n\}, \{v_n\}, \{w_n\}, \{u_n^*\}, \{v_n^*\}$ and $\{w_n^*\}$ as follows: For $n \geq 1$,

$$\begin{aligned} u_{n+1} &= A(u_n,v_n,w_n),\ v_{n+1} = A(v_n,u_n,v_n),\ w_{n+1} = A(w_n,v_n,u_n),\\ u_{n+1}^* &= A(u_n^*,v_n^*,w_n^*),\ v_{n+1}^* = A(v_n^*,u_n^*,v_n^*),\ w_{n+1}^* = A(w_n^*,v_n^*,u_n^*).\end{aligned}$$

Since (x,y,z) is comparable to (u,v,w), we may assume $(u,v,w) \geq (x,y,z) = (x_0,y_0,z_0)$. It is easy to prove by induction that

$$(u,v,w) \geq (x_n,y_n,z_n),\ n \geq 1.$$

From (H_2), we have

$$\begin{aligned}
&\phi\big(d(u, x_{n+1}) + d(v, y_{n+1}) + d(w, z_{n+1})\big)\\
&= \phi\big(d(A(u, v, w), A(x_n, y_n, z_n)) + d(A(v, u, v), A(y_n, x_n, y_n))\\
&\qquad\qquad + d(A(w, v, u), A(z_n, y_n, x_n))\big)\\
&\le \lambda\phi\big(d(u, x_n) + d(v, y_n) + d(w, z_n)\big) - \psi\big(\lambda[d(u, x_n) + d(v, y_n) + d(w, z_n)]\big)\\
&\le \lambda\phi\big(d(u, x_n) + d(v, y_n) + d(w, z_n)\big).
\end{aligned}$$

Similarly, we can prove

$$\phi\big(d(u^*, x_{n+1}) + d(v^*, y_{n+1}) + d(w^*, z_{n+1})\big) \le \lambda\phi\big(d(u^*, x_n) + d(v^*, y_n) + d(w^*, z_n)\big).$$

Since ϕ is nondecreasing and by triangle inequality, it follows that

$$\begin{aligned}
&\phi\big(d(u, u^*) + d(v, v^*) + d(w, w^*)\big)\\
&\le \phi\big(d(u, x_{n+1}) + d(v, y_{n+1}) + d(w, z_{n+1})\big)\\
&\qquad + \phi\big(d(u^*, x_{n+1}) + d(v^*, y_{n+1}) + d(w^*, z_{n+1})\big)\\
&\le \lambda\phi\big(d(u, x_n) + d(v, y_n) + d(w, z_n)\big)\\
&\qquad + \lambda\phi\big(d(u^*, x_n) + d(v^*, y_n) + d(w^*, z_n)\big)\\
&\le \lambda^n\phi\big(d(u, x_1) + d(v, y_1) + d(w, z_1)\big)\\
&\qquad + \lambda^n\phi\big(d(u^*, x_1) + d(v^*, y_1) + d(w^*, z_1)\big)\\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since ϕ is altering distance function, it follows that

$$d(u, u^*) + d(v, v^*) + d(w, w^*) = 0.$$

Hence,

$$u = u^*,\ v = v^*,\ w = w^*.$$

Case(ii): Let (x, y, z) be either upper or lower bound for (u, v, w) and (u^*, v^*, w^*). Then (x, y, z) is comparable to (u, v, w) and (u^*, v^*, w^*). This completes the proof. □

Corollary 4.3. *In addition to the hypothesis of Corollary 4.1, assume either (H_5) or (H_6) holds. Then the tripled fixed point of A is unique.*

Theorem 4.3. *In addition to the hypothesis of Theorem 4.2, assume that every triple of elements u, v, w of X are comparable. Then $u = v = w$, i.e., $u = A(u, u, u)$.*

Proof. From Theorem 4.2, A has unique tripled fixed point (u, v, w). Since u, v, w are comparable, $u = A(u, v, w), v = A(v, u, v), w = A(w, v, u)$ are comparable. Since ϕ is nondecreasing function, it follows that

$$\begin{aligned}
\phi\big(2d(u, w)\big) &= \phi\big(d(u, w) + d(w, u)\big) = \phi\big(d(A(u, v, w), A(w, v, u)) + d(A(w, v, u), A(u, v, w))\big)\\
&\le \phi\big(d(A(u, v, w), A(w, v, u)) + d(A(v, u, v), A(v, u, v)) + d(A(w, v, u), A(u, v, w))\big)\\
&\le \lambda\phi\big(d(u, w) + 0 + d(w, u)\big) - \psi\big(\lambda(d(u, w) + 0 + d(w, u))\big) \le \lambda\phi\big(2d(u, w)\big).
\end{aligned}$$

Since $0 < \lambda < 1,\ d(u, w) = 0, i.e., u = w$. Similarly,

$$\begin{aligned}
\phi\big(3d(u, v)\big) &= \phi\big(d(u, v) + d(v, u) + d(u, v)\big) = \phi\big(d(A(u, v, w), A(v, u, v))\\
&+ d(A(v, u, v), A(u, v, w)) + d(A(u, v, w), A(v, u, v))\big) = \phi\big(d(A(u, v, u), A(v, u, v))\\
&+ d(A(v, u, v), A(u, v, u)) + d(A(u, v, u), A(v, u, v))\big) \le \lambda\phi\big(d(u, v) + d(v, u) + d(u, v)\big)
\end{aligned}$$

$$-\psi\big(\lambda(d(u,v)+d(v,u)+d(u,v))\big) \le \lambda\phi\big(3d(u,v)\big).$$

Since $0<\lambda<1,\ d(u,v)=0, i.e., u=v$. Hence $u=v=w$. $\square$

Corollary 4.4. *In addition to the hypothesis of Corollary 4.3, assume that every triple of elements u,v,w of X are comparable. Then $u=v=w, i.e., u=A(u,u,u)$.*

Theorem 4.4. *In addition to the hypothesis of Theorem 4.2, assume that $u_0,v_0,w_0\in X$ are comparable. Then $u=v=w, i.e., u=A(u,u,u)$.*

Proof. Recall that $u_0,v_0,w_0\in X$ are such that

$$A(u_0,v_0,w_0)\ge u_0,\ A(v_0,u_0,v_0)\le v_0,\ A(w_0,v_0,u_0)\ge w_0.$$

Now, if $u_0\le v_0$ and $w_0\le v_0$, we claim that $u_n\le v_n$ and $w_n\le v_n$ for all $n\in\mathbb{N}$. Indeed, by the mixed monotone property of A,

$$u_1=A(u_1,v_1,w_1)\le A(v_1,u_1,v_1)=v_1$$

and

$$w_1=A(w_1,v_1,u_1)\le A(v_1,u_1,v_1)=v_1.$$

Assume that $u_n\le v_n$ and $w_n\le v_n$ for some n. Then,

$$\begin{aligned}u_{n+1}&=A^{n+1}(u_0,v_0,w_0)\\&=A\big(A^n(u_0,v_0,w_0),A^n(v_0,u_0,v_0),A^n(w_n,v_n,u_n)\big)\\&=A(u_n,v_n,w_n)\le A(v_n,u_n,v_n)=v_{n+1}.\end{aligned}$$

Similarly, we can prove $w_{n+1}\le v_{n+1}$. Since ϕ is nondecreasing and by triangle inequality,

$$\begin{aligned}
&\phi\big(d(u,v)\big)\le\phi\Big(d(u,A^{n+1}(u_0,v_0,w_0))+d(A^{n+1}(u_0,v_0,w_0),v)\Big)\\
&\le\phi\big(d(u,A^{n+1}(u_0,v_0,w_0))+d(A^{n+1}(u_0,v_0,w_0),A^{n+1}(v_0,u_0,v_0))\\
&\qquad\qquad\qquad\qquad+d(v,A^{n+1}(u_0,v_0,w_0))\Big)\\
&\le\phi\Big(d(u,A^{n+1}(u_0,v_0,w_0))+d(A^{n+1}(u_0,v_0,w_0),A^{n+1}(v_0,u_0,v_0))\\
&\quad+d(A^{n+1}(v_0,u_0,v_0),A^{n+1}(u_0,v_0,u_0))\\
&\quad+d(A^{n+1}(w_0,v_0,u_0),A^{n+1}(v_0,u_0,v_0))+d(v,A^{n+1}(u_0,v_0,w_0))\Big)\\
&\le\phi\Big(d(u,A^{n+1}(u_0,v_0,w_0))\\
&\qquad+d\big(A(A^n(u_0,v_0,w_0),A^n(v_0,u_0,v_0),A^n(w_0,v_0,u_0)),\\
&\qquad\qquad A(A^n(v_0,u_0,v_0),A^n(u_0,v_0,u_0),A^n(v_0,u_0,v_0))\big)\\
&\qquad+d\big(A(A^n(v_0,u_0,v_0),A^n(u_0,v_0,u_0),A^n(v_0,u_0,v_0)),\\
&\qquad\qquad A(A^n(u_0,v_0,u_0),A^n(v_0,u_0,v_0),A^n(u_0,v_0,u_0))\big)\\
&\qquad+d\big(A(A^n(w_0,v_0,u_0),A^n(v_0,w_0,v_0),A^n(u_0,v_0,w_0)),\\
&\qquad\qquad A(A^n(v_0,u_0,v_0),A^n(u_0,v_0,u_0),A^n(v_0,u_0,v_0))\big)\\
&\qquad+d(v,A^{n+1}(u_0,v_0,w_0))\Big).
\end{aligned}$$

Since $\phi(t+s) \le \phi(t) + \phi(s)$ for $t, s \in [0, +\infty)$ and by contraction condition, we have

$$
\begin{aligned}
\phi\big(d(u,v)\big) \le\ & \phi\big(d(u, A^{n+1}(u_0,v_0,w_0))\big) + \lambda\phi\Big(d\big(A^n(u_0,v_0,w_0), A^n(v_0,u_0,v_0)\big) \\
& + d\big(A^n(v_0,u_0,v_0), A^n(u_0,v_0,u_0)\big) + d\big(A^n(w_0,v_0,u_0), A^n(v_0,u_0,v_0)\big)\Big) \\
& - \psi\Big(\lambda\big[d\big(A^n(u_0,v_0,w_0), A^n(v_0,u_0,v_0)\big) + d\big(A^n(v_0,u_0,v_0), A^n(u_0,v_0,u_0)\big) \\
& + d\big(A^n(w_0,v_0,u_0), A^n(v_0,u_0,v_0)\big)\big]\Big) + \phi\big(d(v, A^{n+1}(v_0,u_0,v_0))\big) \\
\le\ & \phi\big(d(u, A^{n+1}(u_0,v_0,w_0))\big) + \lambda\phi\Big(d\big(A^n(u_0,v_0,w_0), A^n(v_0,u_0,v_0)\big) \\
& + d\big(A^n(v_0,u_0,v_0), A^n(u_0,v_0,u_0)\big) + d\big(A^n(w_0,v_0,u_0), A^n(v_0,u_0,v_0)\big)\Big) \\
& + \phi\big(d(v, A^{n+1}(v_0,u_0,v_0))\big) \\
& \vdots \\
\le\ & \phi\big(d(u, A^{n+1}(u_0,v_0,w_0))\big) + \lambda^{n+1}[2d(u_0,v_0) + d(w_0,v_0)] \\
& + \phi\big(d(v, A^{n+1}(v_0,u_0,v_0))\big).
\end{aligned}
$$

Since ϕ is continuous and $\phi(0) = 0$, we have $d(u,v) = 0$ and hence $u = v$. Similarly, we can prove $d(u,w) = 0$ and $d(v,w) = 0$. The other cases for u_0, v_0, w_0 are similar.

Corollary 4.5. *In addition to the hypothesis of Corollary 4.3, assume that $u_0, v_0, w_0 \in X$ are comparable. Then $u = v = w$, i.e., $u = A(u,u,u)$.*

□

5. Main results

In this section we derive the necessary conditions for the existence and uniqueness of positive solution for the problem (1.3)-(1.4) as an application of tripled fixed point theorems established in Section 4.

Let $X = C([0,1], \mathbb{R})$ be a partially ordered set such that for $u, v \in X$,

$$u \le v \iff u(t) \le v(t) \text{ for all } t \in [0,1].$$

If X is endowed with the supremum metric:

$$d(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|,\ u, v \in X,$$

then (X,d) is a complete metric space. Similarly, the corresponding metric d on X^3 is defined by

$$
\begin{aligned}
& d\big((u_1,v_1,w_1),(u_2,v_2,w_2)\big) \\
& \quad = \frac{1}{3}\left[\sup_{t\in[0,1]} |u_1(t) - u_2(t)| + \sup_{t\in[0,1]} |v_1(t) - v_2(t)| + \sup_{t\in[0,1]} |w_1(t) - w_2(t)|\right]
\end{aligned}
$$

and then the partial order relation on X^3 is

$$
\begin{aligned}
(u_1,v_1,w_1) \le (u_2,v_2,w_2) \iff & \text{ for } t \in [0,1], \\
& u_1(t) \le u_2(t),\ v_1(t) \ge v_2(t) \text{ and } w_1(t) \le w_2(t).
\end{aligned}
$$

Theorem 5.5. *Assume that*

(S_1) $f, g, h : [0,1] \times [0,+\infty) \to [0,+\infty)$ *are continuous functions;*

(S_2) *For fixed* $t \in [0,1]$,
(i) the function $u \longmapsto f(t,u)$ *is increasing,*
(ii) the function $v \longmapsto g(t,v)$ *is decreasing,*
(iii) the function $w \longmapsto h(t,w)$ *is increasing;*

(S_3) *Denote*

$$\|G\|_\alpha = \Big[\int_0^1 |G(s,s)|^\alpha ds\Big]^{\frac{1}{\alpha}}$$

and let $\rho : [0,+\infty) \to [0,+\infty)$ *be nondecreasing function such that*

$$\rho(u) = \frac{u}{3} - \psi\Big(\frac{u}{3}\Big),\ u \in [0,+\infty),$$

where $\psi \in \Psi$ *satisfies* $\psi(t+s) \le \psi(t)+\psi(s),\ t,s \in [0,+\infty)$. *Further, there exist positive constants* $\kappa_1,\kappa_2,\kappa_3$ *satisfying*

$$\kappa_1 + 2\kappa_2 + \kappa_3 \le \frac{1}{\|G\|_\alpha}$$

such that, for $(u_1,v_1,w_1),(u_2,v_2,w_2) \in X^3$ *with* $u_1 \ge u_2,\ v_1 \le v_2,\ w_1 \ge w_2$ *implies*

$$\begin{aligned}|f(t,u_1) - f(t,u_2)|+&|g(t,v_1) - g(t,v_2)| + |h(t,w_1) - h(t,w_2)|\\ &\le \kappa_1\rho(u_1-u_2) + \kappa_2\rho(v_2-v_1) + \kappa_3\rho(w_1-w_2);\end{aligned}$$

(S_4) *There exist* $u_0,v_0,w_0 \in X$ *such that* $u_0 \le v_0, w_0 \le v_0$ *and*

$$\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,u_0(s)\big) + g\big(s,v_0(s)\big) + h\big(s,w_0(s)\big)\Big)ds \ge \Phi_\alpha(u_0),$$
$$\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,v_0(s)\big) + g\big(s,u_0(s)\big) + h\big(s,v_0(s)\big)\Big)ds \le \Phi_\alpha(v_0),$$
$$\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,w_0(s)\big) + g\big(s,v_0(s)\big) + h\big(s,u_0(s)\big)\Big)ds \ge \Phi_\alpha(w_0),$$

for all $0 \le t \le 1$.

Then the fractional order boundary value problem (1.3)-(1.4) *has a unique positive solution.*

Proof. From Lemma 3.2, the fractional order boundary value problem (1.3)-(1.4) has an integral formulation given by

$$u(t) = \Phi_{\alpha^{-1}}\bigg(\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,u(s)\big) + g\big(s,u(s)\big) + h\big(s,u(s)\big)\Big)ds\bigg).$$

Define an operator $A : X^3 \to X$ by

$$A(u,v,w) = \Phi_{\alpha^{-1}}\bigg(\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,u(s)\big) + g\big(s,v(s)\big) + h\big(s,w(s)\big)\Big)ds\bigg).$$

Under hypothesis (S_1), the operators A is well defined. So, it is easy to prove that u is the solution of the problem (1.3)-(1.4) if and only if $u = A(u,u,u)$. Further, it follows from hypothesis (S_2) that A is mixed monotone ternary operator.

Let $(u, v, w), (x, y, z) \in X^3$ with $u \geq x, v \leq y, w \geq z$, we have

$$
\begin{aligned}
& d\big(A(u,v,w), A(x,y,z)\big) \\
& = \sup_{t\in[0,1]} \Bigg| \Phi_{\alpha^{-1}}\bigg(\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,u(s)\big) + g\big(s,v(s)\big) + h\big(s,w(s)\big)\Big)ds\bigg) \\
& \qquad - \Phi_{\alpha^{-1}}\bigg(\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,x(s)\big) + g\big(s,y(s)\big) + h\big(s,z(s)\big)\Big)ds\bigg)\Bigg| \\
& \leq \sup_{t\in[0,1]} \Bigg| \int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,u(s)\big) + g\big(s,v(s)\big) + h\big(s,w(s)\big)\Big)ds \\
& \qquad - \int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,x(s)\big) + g\big(s,y(s)\big) + h\big(s,z(s)\big)\Big)ds\Bigg|^{\frac{1}{\alpha}} \\
& \leq \bigg[\int_0^1 G(s,s)\Big[|f(s,u(s)) - f(s,x(s))| + |g(s,v(s)) - g(s,y(s))| \\
& \qquad + |h(s,w(s)) - h(s,z(s))|\Big]^\alpha ds\bigg]^{\frac{1}{\alpha}} \\
& \leq \bigg[\int_0^1 G(s,s)\Big[\kappa_1\rho(u(s) - x(s)) + \kappa_2\rho(y(s) - v(s)) \\
& \qquad + \kappa_3\rho(w(s) - z(s))\Big]^\alpha ds\bigg]^{\frac{1}{\alpha}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\rho(u(s) - x(s)) &\leq \rho(d(u,x)) \\
\rho(y(s) - v(s)) &\leq \rho(d(y,v)) = \rho(d(v,y)) \\
\rho(w(s) - z(s)) &\leq \rho(d(w,z)),
\end{aligned}
$$

it follows that,

$$
d\big(A(u,v,w), A(x,y,z)\big) \leq \|G\|_\alpha \big[\kappa_1\rho(d(u,x)) + \kappa_2\rho(d(v,y)) + \kappa_3\rho(d(w,z))\big].
$$

Similarly, we can establish

$$
d\big(A(v,u,v), A(y,x,y)\big) \leq \|G\|_\alpha \big[(\kappa_1 + \kappa_3)\rho(d(v,y)) + \kappa_2\rho(d(u,x))\big]
$$

and

$$
d\big(A(w,v,u), A(z,y,x)\big) \leq \|G\|_\alpha \big[\kappa_1\rho(d(w,z)) + \kappa_2\rho(d(v,y)) + \kappa_3\rho(d(u,x))\big].
$$

$\square$

Adding above three inequalities, we get

$$\begin{aligned}
d(A(u,v,w),&A(x,y,z)) + d(A(v,u,v),A(y,x,y)) + d(A(w,v,u),A(z,y,x))\\
&\leq \|G\|_\alpha\Big[(\kappa_1+\kappa_2+\kappa_3)\rho(d(u,x)) + (\kappa_1+2\kappa_2+\kappa_3)\rho(d(v,y))\\
&\qquad\qquad\qquad\qquad\qquad\qquad + (\kappa_1+\kappa_3)\rho(d(w,z))\Big]\\
&\leq \|G\|_\alpha(\kappa_1+2\kappa_2+\kappa_3)\Big[\rho(d(u,x)) + \rho(d(v,y)) + \rho(d(w,z))\Big]\\
&\leq \rho(d(u,x)) + \rho(d(v,y)) + \rho(d(w,z))\\
&\leq \frac{1}{3}\Big[d(u,x)+d(v,y)+d(w,z)\Big]\\
&\qquad\qquad - \psi\Big[\frac{1}{3}(d(u,x)+d(v,y)+d(w,z))\Big],
\end{aligned}$$

which satisfies contraction condition (H_6) with $\lambda = \frac{1}{3}$.

Next, consider a monotone nondecreasing sequences $\{u_n\}$, $\{w_n\}$ of X converging to u, w of X, respectively. Then, for every $t \in [0,1]$, the sequences of real numbers

$$u_1(t) \leq u_2(t) \leq u_3(t) \leq \cdots u_n(t) \leq \cdots \longrightarrow u(t)$$

and

$$w_1(t) \leq w_2(t) \leq w_3(t) \leq \cdots w_n(t) \leq \cdots \longrightarrow w(t).$$

So, for all $t \in [0,1], n \in \mathbb{N}$, $u_n(t) \leq u(t)$ and $w_n(t) \leq w(t)$. Hence, $u_n \leq u, w_n \leq w$ for all n. Similarly, we can prove that $v(t)$ is the limit of monotone nondecreasing sequence $\{v_n\}$ in X is a lower bound for all the elements in the sequence. That is $v \leq v_n$ for all n. Therefore, it follows from Corollary 4.1 that A has a tripled fixed point $(u_0, v_0, w_0) \in X^3$.
Let

$$M(t) = \max\big\{p(t), q(t), l(t)\big\},\ m(t) = \min\big\{p(t), q(t), l(t)\big\},\ t \in [0,1],$$

be in X for any $p, q, l \in X$. Then $M(t)$, $m(t)$ are the upper and lower bounds of p, q, l respectively. Then, by Corollary 4.3, A has a unique tripled fixed point. Finally, from the hypothesis (4), we can easily show that (H_3) and the condition in Corollary 4.5 are satisfied. Hence, the conclusion of Theorem 5.5 follows from Corollary 4.5.

Example 5.1. Consider the following half-linear fractional order boundary value problem, for $0 < t < 1$,

$$\begin{aligned}
\Phi_\beta\Big({}^C D^r_{0+}\big(\Phi_\alpha(u(t))\big)\Big) + \Phi_{\alpha\beta}\Big(f\big(t,u(t)\big) + g\big(t,u(t)\big) + h\big(t,u(t)\big)\Big) &= 0,\\
a(\Phi_\alpha u)(0) - b(\Phi_\alpha u)'(0) &= 0,\\
c(\Phi_\alpha u)(1) + d(\Phi_\alpha u)'(1) &= 0,
\end{aligned} \tag{5.12}$$

where $\alpha = 1.5, \beta = 2, a = 0, b = c = d = 1, f(t,u) = \dfrac{1}{20}t(1-t)^{1/3}u$ $g(t,v) = \dfrac{(1-t)^{1/3}}{50(1+v)}$ and $h(t,w) = \dfrac{1}{300}(1-t)^{1/3}w$. After certain calculations we get $\xi_1 = \dfrac{1}{3}, \xi_2 = 1,$ so $\xi = \dfrac{1}{3}$ and $\|G\|_\alpha = 1.421668665$. Let $\psi(u) = \dfrac{u}{2}$ and $\kappa_1 = \dfrac{3}{10}, \kappa_2 = \dfrac{3}{25}, \kappa_3 = \dfrac{1}{50}$ then $\rho(u) = \dfrac{u}{6}$ and $\|G\|_\alpha(\kappa_1 + 2\kappa_2 + \kappa_3) < 1$.

Now, let $(u_1, v_1, w_1), (u_2, v_2, w_2) \in X^3$ with $u_1 \geq u_2,\ v_1 \leq v_2,\ w_1 \geq w_2$ implies

$$|f(t,u_1) - f(t,u_2)| + |g(t,v_1) - g(t,v_2)| + |h(t,w_1) - h(t,w_2)|$$

$$\leq \frac{1}{20}|t(1-t)^{1/3}||u_1-u_2| + \frac{1}{50}|(1-t)^{1/3}|\Big|\frac{1}{1+v_1} - \frac{1}{1+v_2}\Big|$$
$$+\frac{1}{300}|(1-t)^{1/3}||w_1-w_2| \leq \frac{3}{10}\Big(\frac{u_1-u_2}{6}\Big) + \frac{3}{25}\Big(\frac{v_2-v_1}{6}\Big) + \frac{1}{50}\Big(\frac{w_1-w_2}{6}\Big)$$
$$= \kappa_1\rho(u_1-u_2) + \kappa_2\rho(v_2-v_1) + \kappa_3\rho(w_1-w_2).$$

Let we set now $u_0 = 0, w_0 = 0, v_0 = 1$. Then $\Phi_\alpha(u_0) = \Phi_\alpha(w_0) = 0, \Phi_\alpha(v_0) = 1$, and

$$\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,u_0(s)\big) + g\big(s,v_0(s)\big) + h\big(s,w_0(s)\big)\Big)ds$$
$$\geq \int_0^1 \xi G(s,s)\Phi_\alpha\Big(f\big(s,u_0(s)\big) + g\big(s,v_0(s)\big) + h\big(s,w_0(s)\big)\Big)ds = 0.000188063 \geq \Phi_\alpha(u_0),$$
$$\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,v_0(s)\big) + g\big(s,u_0(s)\big) + h\big(s,v_0(s)\big)\Big)ds$$
$$\leq \int_0^1 G(s,s)\Phi_\alpha\Big(f\big(s,v_0(s)\big) + g\big(s,u_0(s)\big) + h\big(s,v_0(s)\big)\Big)ds = 0.006197707 \leq \Phi_\alpha(v_0),$$
$$\int_0^1 G(t,s)\Phi_\alpha\Big(f\big(s,w_0(s)\big) + g\big(s,v_0(s)\big) + h\big(s,u_0(s)\big)\Big)ds$$
$$\geq \int_0^1 \xi G(s,s)\Phi_\alpha\Big(f\big(s,w_0(s)\big) + g\big(s,v_0(s)\big) + h\big(s,u_0(s)\big)\Big)ds = 0.000188063 \geq \Phi_\alpha(w_0).$$

Since all the hypotheses of Corollary 4.5 are satisfied we get that the fractional order boundary value problem (5.12) has a unique positive solution in $[0,1]$.

6. CONCLUSION

In this paper we consider the existence of a tripled fxed point for mixed monotone mapping satisfying a new contractive inequality which involves an altering distance function in partially ordered metric spaces. We established some existence results for tripled fixed points, as well as the uniqueness of fixed points of mixed monotone operators. The obtained results generalizes the results available in the literature. In addition as an application, we established existence and uniqueness of positive solutions for half-linear fractional order boundary value problem and finally we verified our results with example.

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