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Coefficient bounds for M-fold symmetric analytic bi-Bazilevič functions using by Faber polynomial expansion

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ABSTRACT. A function is said to be bi-univalent in the open unit disc $\mathbb{D}$, if both the function f and its inverse are univalent in the unit disc. Besides, a function is said to be bi-Bazilevič in $\mathbb{D}$, if both the function f and its inverse are Bazilevič there. The behaviour of these types of functions are unpredictable and not much is known about their coefficients. In this study, we determined coefficient estimates for the Taylor Maclaurin coefficients of the class on m-fold symmetric bi-Bazilevič functions. We also, use the Faber Polynomial expansions to obtain these coefficient estimates associated with upper bounds.

1. INTRODUCTION

Let $\mathcal{A}$ indicate the family of functions analytic in the open unit disk $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Furthermore, let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathcal{D}$ (see for details [12]). From the Koebe $1/4$ Theorem (for details, see [12]) every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{D})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \\ &= w + \sum_{k=2}^{\infty} b_k w^k. \end{aligned} \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ denote the class of all bi-univalent functions in $\mathbb{D}$ given by the Taylor-Maclaurin series expansion given by Eq. (1.1).

Detailed information about the class of Σ was given in the references [9], [20], [29], [31] and [33].

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Let $m \in \mathbb{N}$. A domain $\mathbb{E}$ is said to be *m-fold symmetric* if a rotation of $\mathbb{E}$ about the origin through an angle $2\pi/m$ carries $\mathbb{E}$ on itself. It follows that, a function f analytic in $\mathbb{D}$ is said to be m-fold symmetric if

$$f\left(e^{2\pi i/m}z\right) = e^{2\pi i/m} f(z).$$

In particular every f is one-fold symmetric and every odd f is two-fold symmetric. $\mathcal{S}_m$ indicate the class of m-fold symmetric univalent functions in $\mathbb{D}$. $f \in \mathcal{S}_m$ is characterized by having a power series as following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{D},\ m \in \mathbb{N}). \tag{1.3}$$

In [32] Srivastava et al. defined m-fold symmetric bi-univalent function analogues to the concept of m-fold symmetric univalent functions. They introduce some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $(m \in \mathbb{N})$. In addition, they acquired the series expansion for f^{-1} as follows

$$\begin{aligned} g(w) &= w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots \\ &= z + \sum_{k=1}^{\infty} A_{mk+1} z^{mk+1} \end{aligned} \tag{1.4}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{D}$.

For $m = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ. For some examples of m-fold symmetric bi-univalent functions, see [32]. The coefficient problem for m-fold symmetric analytic bi-univalent functions is one of the favorite subjects of geometric function theory in these days, see [16], [32]. Here, in this study, we use the Faber polynomial expansions for a general subclass of m-fold symmetric analytic bi-univalent functions to determine estimates for the general coefficient bounds $|a_{mk+1}|$.

2. Definition of the class $B_{\Sigma,m}(\alpha,\beta)$

Firstly, we consider a class of m-fold symmetric analytic bi-univalent functions defined by Jahangiri and Hamidi [17].

For $0 \le \alpha < 1$ and $0 \le \beta < 1$, $m \in \mathbb{N}$, a function $f \in \Sigma_m$ given by (1.3) is said to be Bazilevič [7] of order α and type β, denoted by $B_{\Sigma,m}(\alpha,\beta)$, if the following conditions are satisfied:

$$Re\left(\frac{zf'(z)}{f(z)^{1-\beta}z^{\beta}}\right) > \alpha; \qquad z \in \mathbb{D} \tag{2.5}$$

and

$$Re\left(\frac{wg'(w)}{g(w)^{1-\beta}w^{\beta}}\right) > \alpha; \qquad w \in \mathbb{D} \tag{2.6}$$

where $m \in \mathbb{N}$ and $f^{-1} = g$ is defined by (1.4). If $f^{-1} = g$ is the inverse of the function $f \in \mathcal{S}$, then g has a Maclaurin series expansion in some disk about the origin [12]. In 1923, Löwner [24] proved that the inverse of the Koebe function $f(z) = z/(1-z)^2$ provides the best upper bounds for the coefficients of the inverses of the functions $f \in \mathcal{S}$. Sharp bounds for the coefficients of the inverses of univalent functions have been obtained in a surprisingly straightforward way, whereas the case for the subclasses of univalent functions turned out to be a challenge. In 1979, Krzyz et al.[19] obtained sharp upper bounds for the first two coefficients of inverses of the functions starlike of order α; $0 \leq \alpha < 1$. In 1982, Libera and Zlotkiewicz [21] found the bounds for the first seven coefficients of the inverse of convex functions. Later, in [22] they obtained the bounds for the first six coefficients of the inverse of f provided $Ref'(z) > 0$, $z \in \mathbb{D}$. In a follow up paper [21] they considered the odd functions $f(z) = z + a_3z^3 + a_5z^5 + \cdots$ and showed that if $Ref'(z) > 0$, $z \in \mathbb{D}$ then $[-z + log((1+z)/(1-z))]^{-1}$ is the extremal function for the inverse of f. In 1986, Juneja and Rajasekaran [18] obtained coefficient estimates for inverses of α-spiral functions. In 1989, Silverman [30] proved that if $f \in \mathcal{S}$ is such that $\sum_{n=2}^{\infty} n|a_n| \leq 1$ then the n-th coefficient of the inverse of f is bounded above by $\frac{1}{n}\binom{2n-3}{n-2}\frac{1}{2^{n-2}}$.

In 1992, Libera and Zlotkiewicz [23] proved that the n-th coefficients of the inverse of starlike functions are bounded above by $[(2n!)/n!(n+1)!]$. Chou [11] in 1994, proved that if $f \in \mathcal{S}$ and $Ref'(z) > 0$, $z \in \mathbb{D}$ then $[-z + log((1+z)/(1-z))]^{-1}$ is the extremal function for the inverse of f. Estimates for the first two coefficients of the inverses of subclasses of starlike functions were also obtained in [11],[12] and [23]. Finding coefficient estimates for the inverses of univalent function becomes even more involved when the bi-univalency condition is imposed on these functions. A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{D}$ if its inverse map $g = f^{-1}$ is also univalent in $\mathbb{D}$. The class of bi-univalent analytic functions was first introduced and studied by Lewin [20] where it was proved that $|a_2| \leq 1.51$. Brannan and Clunie [8] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [29] proved that $|a_2| \leq 4/3$. Brannan and Taha [9] and Taha [33] also investigated certain subclasses of bi-univalent functions and found estimates for their initial coefficients. Recently, Srivastava et al. [31], Frasin and Aouf [13], and Ali et al. [5] found estimates for the first two coefficients of certain subclasses of bi-univalent functions. The bi-univalency requirement makes the behavior of the coefficients of the function f and its inverse $g = f^{-1}$ unpredictable. Not much is known about the higher coefficients of bi-univalent functions as Ali et al.[5] also remarked that finding the bounds for the n-th,$(n \geq 4)$ coefficients of classes of bi-univalent functions is an open problem. Hamidi et al. [14], [15] used Faber polynomial expansions to find coefficient estimates for classes of meromorphic bi-univalent functions. Furthermore, other researchers studied on the coefficient estimates of bi-Bazilevič functions [25], [26], [27], [28].

In this study, we use the Faber polynomial expansions to find upper bound for the n-th, $(n \geq 3)$ coefficient of class of analytic bi-Bazilevič functions. A function is said to be bi-Bazilevič of order α and type β in $\mathbb{D}$, if both f and its inverse $g = f^{-1}$ are Bazilevič of order α and type β in $\mathbb{D}$. We conclude that our paper is an examination of the unexpected behavior of the general and first two coefficients of bi-Bazilevič functions. We hope that the technique presented in this article triggers further interest in applying our approach to other related problems.

3. Coefficient estimates

Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as, 3.7,

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)\, w^n, \tag{3.7}$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3$$

$$+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4$$
$$+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5+(-n+2)a_3^2\right]$$
$$+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)a_3a_4\right]+\sum_{j\geqq 7}a_2^{n-j}V_j,$$

where $V_j \quad (7 \leqq j \leqq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$ (see, for details, [4]) and [3]. Especially, the first few terms of K_{n-1}^{-n} are given below:

$$K_1^{-2} = -2a_2$$

$$K_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

and

$$K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, for any $p \in \mathbb{Z}$, ($\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}$) an expansion of K_n^p is given by (see, for details, [3])

$$K_n^p = pa_n + \frac{p(p-1)}{2}E_n^2 + \frac{p!}{(p-3)!3!}E_n^3 + \cdots + \frac{p!}{(p-n)!n!}E_n^n \qquad (p \in \mathbb{Z}) \tag{3.8}$$

where $E_n^p = E_n^p\left(a_2, a_3, \cdots\right)$. By [1],

$$E_n^m(a_1, a_2, \ldots a_n) = \sum_{m=1}^{\infty} \frac{m!(a_1)^{\mu_1}\ldots(a_n)^{\mu_n}}{\mu_1!\ldots\mu_n!},$$

where $a_1 = 1$ and the sum is taken over all nonnegative integers $\mu_1, \mu_2, \ldots, \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \ldots + \mu_n = m \\ \mu_1 + 2\mu_2 + \ldots + n\mu_n = n. \end{cases}$$

In [2], the following equation is clear $E_n^n(a_1, a_2, \ldots a_n) = a_1^n$.

Similarly, using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.3), that is,

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} = z + \sum_{k=1}^{\infty} K_k^{\frac{1}{m}}(a_2, a_3, \cdots, a_{k+1}) z^{mk+1},$$

the coefficients of its inverse map $g = f^{-1}$ may be expressed as:

$$g(w) = f^{-1}(w) = w + \sum_{k=1}^{\infty} \frac{1}{mk+1} K_k^{-(mk+1)}(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}) w^{mk+1}. \tag{3.9}$$

Our first theorem giving by Theorem 3.1 shows an upper bound for $|a_{mk+1}|$ of m-fold symmetric analytic bi-univalent functions in the class $B_{\Sigma,m}(\alpha, \beta)$.

Theorem 3.1. *For $0 \le \alpha < 1$ and $0 \le \beta < 1$, $m \in \mathbb{N}$, let the function $f \in B_{\Sigma,m}(\alpha, \beta)$ be given by (1.3). If $a_{mj+1} = 0$,$(1 \le j \le k-1)$, then*

$$|a_{mk+1}| \le \frac{2(1-\alpha)}{\beta + mk}, \qquad (k \ge 3).$$

Proof. For the function $f \in B_{\Sigma,m}(\alpha,\beta)$ and for its inverse function $g = f^{-1} \in B_{\Sigma,m}(\alpha,\beta)$ of the form (1.3), there exist positive real part functions

$$\mathfrak{p}(z) = 1 + \sum_{k=1}^{\infty} c_k z^{mk} \in \mathcal{A}, \;\; and \;\; \mathfrak{q}(w) = 1 + \sum_{k=1}^{\infty} d_k w^{mk} \in \mathcal{A}$$

where

$$Re(\mathfrak{p}(z)) > 0, \;\; and \;\; Re(\mathfrak{q}(w)) > 0$$

in $\mathbb{D}$. So that,

$$\left(\frac{zf'(z)}{f(z)^{1-\beta} z^{\beta}}\right) = \alpha + (1-\alpha)\mathfrak{p}(z) = 1 + (1-\alpha)\sum_{k=1}^{\infty} K_k^1\left(c_1, c_2, \cdots, c_k\right) z^{mk}, \tag{3.10}$$

and

$$\left(\frac{wg'(w)}{g(w)^{1-\beta} w^{\beta}}\right) = \alpha + (1-\alpha)\mathfrak{q}(w) = 1 + (1-\alpha)\sum_{k=1}^{\infty} K_k^1\left(d_1, d_2, \cdots, d_k\right) w^{mk}. \tag{3.11}$$

Throughout the rest of this study, we shall use the inequalities, according to the Caratheodary lemma [12], $|c_k| \leq 2$ and $|d_k| \leq 2$, $k \in \mathbb{N}$. Using the Faber Polynomial expansions given by ([1], Eqs. (3.10) and (3.11)), we have

$$\begin{aligned}\left(\frac{zf'(z)}{f(z)^{1-p} z^{p}}\right) &= 1 - \sum_{k=2}^{\infty} F_{k-1}^{k+p-1}(a_2, a_3, \cdots, a_n) z^{k-1} \\ &= 1 + \sum_{k=2}^{\infty} \left\{1 + \frac{k-1}{p}\right\} K_{k-1}^{p}(a_2, a_3, \cdots, a_n) z^{k-1}\end{aligned}$$

where K_n^p is defined by (3.8). Therefore, the left hand sides of equations (3.10) and (3.11) can be expressed by

$$\left(\frac{zf'(z)}{f(z)^{1-\beta} z^{\beta}}\right) = 1 + \sum_{k=1}^{\infty}\left(1 + \frac{mk}{\beta}\right) K_k^{\beta}\left(a_{k+1}, a_{2k+1}, \cdots, c_{mk+1}\right) z^{mk}, \tag{3.12}$$

and

$$\left(\frac{wg'(w)}{g(w)^{1-\beta} w^{\beta}}\right) = 1 + \sum_{k=1}^{\infty}\left(1 + \frac{mk}{\beta}\right) K_k^{\beta}\left(A_{k+1}, A_{2k+1}, \cdots, A_{mk+1}\right) w^{mk} \tag{3.13}$$

where by

$$A_{mk+1} = \frac{1}{mk+1} K_k^{-(mk+1)}\left(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}\right) z^{mk}, \qquad (k \geq 1).$$

Comparing the corresponding coefficients of Eqs.(3.10) and (3.12) (for any $k \geq 2$) we obtain,

$$\left(1 + \frac{mk}{\beta}\right) K_k^{\beta}\left(a_{k+1}, a_{2k+1}, \cdots, c_{mk+1}\right) = (1-\alpha)c_k, \tag{3.14}$$

similarly, from Eqs.(3.11) and (3.15), we can find

$$\left(1 + \frac{mk}{\beta}\right) K_k^{\beta}\left(A_{k+1}, A_{2k+1}, \cdots, A_{mk+1}\right) = (1-\alpha)d_k. \tag{3.15}$$

Note that for $a_{mj+1} = 0, \;\; (1 \leq j \leq k-1)$, we have

$$A_{mk+1} = -a_{mk+1}$$

and so the equations (3.14) and (3.15), respectively imply

$$(\beta + mk)a_{mk+1} = (1-\alpha)c_k \tag{3.16}$$

and

$$-(\beta + mk)a_{mk+1} = (1-\alpha)d_k. \tag{3.17}$$

Now by solving either of the equations (3.16) or (3.17) for a_{mk+1} and taking the absolute values we can obtain

$$|a_{mk+1}| \leq \frac{2(1-\alpha)}{\beta + mk}.$$

Relaxing the coefficient restrictions imposed on Theorem 3.1, we experience the unpredictable behavior of the coefficients of bi-Bazilevič functions. □

For $\alpha = 0$, $k = n-1$ and one-fold case, we can obtain Corollary 3.1 as follows:

Corollary 3.1. [6] *For $0 \leq \beta < 1$, let the function $f \in B_{\Sigma,1}(0,\beta) = B_{\Sigma}(\beta,\varphi)$ be given by (1.3). If $a_j = 0$,$(2 \leq j \leq n-1)$, then*

$$|a_n| \leq \frac{2}{\beta + (n-1)}, \qquad (n \geq 4).$$

Theorem 3.2. *For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, $m \in \mathbb{N}$, let the function $f \in B_{\Sigma,m}(\alpha,\beta)$ be bi-Bazilevič in $\mathbb{D}$. Then,*

$$(i) \quad |a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\alpha)}{(\beta+2m)(\beta+m)}}; & 0 \leq \alpha < \frac{m}{\beta+2m}, \quad (\beta+m, \beta+2m \neq 0) \\ \frac{2(1-\alpha)}{(\beta+m)}; & \frac{m}{\beta+2m} \leq \alpha < 1, \quad (\beta+m, \beta+2m \neq 0) \end{cases}$$

and

$$(ii) \quad \left| \frac{\beta+2m+1}{2} a_{m+1}^2 - a_{2m+1} \right| \leq \frac{2(1-\alpha)}{\beta+2m}.$$

Proof. The equation (3.14) for $k=1$ and $k=2$, respectively implies

$$(\beta + m)a_{m+1} = (1-\alpha)c_1 \tag{3.18}$$

and

$$(\beta+2m)\left[\frac{\beta-1}{2}a_{m+1}^2 + a_{2m+1}\right] = (1-\alpha)c_2. \tag{3.19}$$

Similarly, the equation (3.15) yields:

$$(\beta+m)A_{m+1} = (1-\alpha)d_1$$

and

$$(\beta+2m)\left[\frac{\beta-1}{2}A_{m+1}^2 + A_{2m+1}\right] = (1-\alpha)d_2,$$

and for suitable values of A_2 and A_3 we deduce

$$-(\beta+m)a_{m+1} = (1-\alpha)d_1 \tag{3.20}$$

and

$$(\beta+2m)\left[\frac{\beta-1}{2}a_{m+1}^2 + ((m+1)a_{m+1}^2 - a_{2m+1})\right] = (1-\alpha)d_2. \tag{3.21}$$

Solve either of the equations (3.18) or (3.19) for a_{m+1} and take the absolute values to obtain

$$|a_{m+1}| \leq \frac{2(1-\alpha)}{\beta+m}.$$

On the other hand , by adding the equations (3.19) and (3.21) we obtain

$$(\beta+2m)(\beta+m)a_{m+1}^2=(1-\alpha)(c_2+d_2).$$

Solving the above equation for a_{m+1} and taking the absolute values we obtain

$$|a_{m+1}|\leq\sqrt{\frac{4(1-\alpha)}{(\beta+m)(\beta+2m)}}.$$

Now, the bounds given for $|a_{m+1}|$ can be justified upon noting that

$$\sqrt{\frac{4(1-\alpha)}{(\beta+m)(\beta+2m)}}<\frac{2(1-\alpha)}{(\beta+m)}\qquad if\qquad 0\leq\alpha<\frac{m}{\beta+2m}.$$

Lastly, from (3.21), we deduce the inequality as follows:

$$\left|\frac{\beta+2m+1}{2}a_{m+1}^2-a_{2m+1}\right|\leq\frac{2(1-\alpha)}{\beta+2m}.$$

This evidently completes the proof of Theorem 3.2. □

Letting $\beta=0$ in Theorem 3.2 and one-fold case we have the following Corollary 3.2 for analytic bi-starlike functions of order $\alpha(0\leq\alpha<1)$.

Corollary 3.2. [17] *Let f be the bi-starlike function class of order $\alpha(0\leq\alpha<1)$ in $\mathbb{D}$. Then*

$$(i)\quad |a_2|\leq\begin{cases}\sqrt{2(1-\alpha)}; & 0\leq\alpha<\frac{1}{2};\\ 2(1-\alpha); & \frac{1}{2}\leq\alpha<1.\end{cases}$$

and

$$(ii)\quad \left|3/2a_2^2-a_3\right|\leq 1-\alpha\quad for\quad 0\leq\alpha<1.$$

For $\beta=1$ and one-fold case we can obtain Corollary 3.3 as follows:

Corollary 3.3. [17] *For $0\leq\alpha<1$, let $f\in B_\Sigma(\alpha,1)$ be bi-Bazilevič in $\mathbb{D}$. Then*

$$|a_2|\leq\begin{cases}\sqrt{\frac{2(1-\alpha)}{3}}; & 0\leq\alpha<\frac{1}{3};\\ 1-\alpha; & \frac{1}{3}\leq\alpha<1.\end{cases}$$

For $\beta=1$ and m-fold case we can obtain Corollary 3.4 as follows:

Corollary 3.4. [10] *For $m\in\mathbb{N}$ and $0\leq\alpha<1$, let the function $f\in H_{\Sigma,m}(\alpha)$ be given by (1.3). Then one has the following inequalities:*

$$(i)\quad |a_{m+1}|\leq\begin{cases}\sqrt{\frac{4(1-\alpha)}{(1+m)(1+2m)}}; & 0\leq\alpha<\frac{m}{1+2m};\\ \frac{2(1-\alpha)}{1+m}; & \frac{m}{1+2m}\leq\alpha<1.\end{cases}$$

$$(ii)\quad \left|a_{2m+1}-(m+1)a_{m+1}^2\right|\leq\frac{2(1-\alpha)}{1+2m}.$$

Remark 3.1. For different values of α and β, Theorem 3.2 demonstrates the fluctuation of the early coefficients of the bi-Bazileviĉ functions. Determination of extremal functions for bi-univalent functions (in general) and for especially bi-Bazilevič functions remain a challenge.

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