CREAT. MATH. INFORM.
Volume **29** (2020), No. 2,
Pages 145 - 152

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2020.02.05

Nonlinear elliptic anisotropic problem involving non-local boundary conditions with variable exponent and graph data

ADAMA KABORE and STANISLAS OUARO

ABSTRACT. We study a nonlinear elliptic anisotropic problem involving non-local conditions. We also consider variable exponent and general maximal monotone graph datum at the boundary. We prove the existence and uniqueness of weak solution to the problem.

1. INTRODUCTION AND ASSUMPTIONS

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ such that $\partial\Omega$ is Lipschitz and $\partial\Omega = \Gamma_D \cup \Gamma_{Ne}$ with $\Gamma_D \cap \Gamma_{Ne} = \emptyset$ and $dist(\Gamma_D, \Gamma_{Ne}) > 0$. Our aim is to study the following problem

$$S^{\rho}_{f,d} \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial}{\partial x_i} u) + |u|^{P_M(x)-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \left.\begin{array}{l} \rho(u) + \sum_{i=1}^{N} \int_{\Gamma_{Ne}} a_i(x, \frac{\partial}{\partial x_i} u)\eta_i \ni d \\ u \equiv constant \end{array}\right\} & \text{on } \Gamma_{Ne}, \end{cases} \tag{1.1}$$

where the right-hand side $f \in L^\infty(\Omega)$ and $\eta_i,\ i \in \{1, ..., N\}$ are the components of the outer normal unit vector, ρ a maximal monotone graph on $\mathbb{R}$ such that

$$D(\rho) = \mathbb{R},\ Im(\rho) = \mathbb{R} \text{ and } 0 \in \rho(0). \tag{1.2}$$

For any $\Omega \subset \mathbb{R}^N$, we set

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) :\ \inf_{x\in\Omega} h(x) > 1\}, \tag{1.3}$$

and we denote

$$h^+ = \sup_{x\in\Omega} h(x),\ h^- = \inf_{x\in\Omega} h(x). \tag{1.4}$$

We consider the exponents, $\vec{p}(.) : \bar{\Omega} \to \mathbb{R}^N$ such that $\vec{p}(.) = (p_1(.), ..., p_N(.))$ with $p_i \in C_+(\bar{\Omega})$ for every $i \in \{1, ..., N\}$ and for all $x \in \bar{\Omega}$. We put $p_M(x) = \max\{p_1(x), ..., p_N(x)\}$ and $p_m(x) = \min\{p_1(x), ..., p_N(x)\}$.

We assume that for $i = 1, ..., N$, the function $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and satisfies the following conditions.

- (H_1): $a_i(x, \xi)$ is the continuous derivative with respect to ξ of the mapping $A_i = A_i(x, \xi)$, that is, $a_i(x, \xi) = \frac{\partial}{\partial \xi} A_i(x, \xi)$ such that the following equality holds.

$$A_i(x, 0) = 0, \tag{1.5}$$

Received: 12.11.2019. In revised form: 04.09.2020. Accepted: 11.09.2020
2010 *Mathematics Subject Classification.* 35J05, 35J25, 35J60, 35J66.
Key words and phrases. *Maximal monotone graph, non-local boundary conditions, variable exponent, Leray-Lions operator.*
Corresponding author: Adama Kabore; ouaro@yahoo.fr

for almost every $x \in \Omega$.

- (H_2) : There exists a positive constant C_1 such that

$$|a_i(x,\xi)| \leq C_1(j_i(x) + |\xi|^{p_i(x)-1}), \tag{1.6}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(\cdot)}(\Omega)$, with $\dfrac{1}{p_i(x)} + \dfrac{1}{p_i'(x)} = 1$.

- (H_3) : There exists a positive constant C_2 such that

$$(a_i(x,\xi) - a_i(x,\eta)).(\xi - \eta) \geq \begin{cases} C_2|\xi-\eta|^{p_i(x)} & \text{if } |\xi-\eta| \geq 1, \\ C_2|\xi-\eta|^{p_i^-} & \text{if } |\xi-\eta| < 1, \end{cases} \tag{1.7}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$.

- (H_4) : For almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$,

$$|\xi|^{p_i(x)} \leq a_i(x,\xi).\xi \leq p_i(x)A_i(x,\xi). \tag{1.8}$$

Non-local boundary value problems of various kinds for partial differential equations are of great interest by now in several fields of application. In a typical non-local problem, the partial differential equation (resp. boundary conditions) for an unknown function u at any point in a domain Ω involves not only the local behavior of u in a neighborhood of that point but also the non-local behavior of u elsewhere in Ω. For example, at any point in Ω the partial differential equation and/or the boundary conditions may contains integrals of the unknown u over parts of Ω, values of u elsewhere in D or, generally speaking, some non-local operator on u. Beside the mathematical interest of nonlocal conditions, it seems that this type of boundary condition appears in petroleum engineering model for well modeling in a $3D$ stratified petroleum reservoir with arbitrary geometry (see [3] and [4]).

2. Preliminary and main result

This part is related to anisotropic Lebesgue and Sobolev spaces with variable exponent, some of their properties (for more details see [6] and [7]) and the main result of the paper. Given a measurable function $p(.) : \Omega \to [1,\infty)$. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite.
If the exponent is bounded, i.e, if $p_+ < \infty$, then the expression

$$|u|_{p(.)} := \inf\left\{\lambda > 0 : \rho_{p(.)}(\frac{u}{\lambda}) \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Then, $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$, for all $x \in \Omega$.

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow.

$$W^{1,\vec{p}(\cdot)}(\Omega) := \left\{u \in L^{p_M(\cdot)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega), \text{ for all } i \in \{1,...,N\}\right\}.$$

Endowed with the norm

$$\|u\|_{\vec{p}(.)} := |u|_{p_M(.)} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(.)},$$

the space $\left(W^{1,\vec{p}(.)}(\Omega), \|.\|_{\vec{p}(.)}\right)$ is a reflexive Banach space (see [7], Theorem 2.1 and Theorem 2.2).
As consequence, we have the following.
Let us introduce the following notation:

$$\vec{p}_- = (p_1^-, ..., p_N^-).$$

In the sequel, we consider the following spaces.

$$W_D^{1,\vec{p}(.)}(\Omega) = \{\xi \in W^{1,\vec{p}(.)}(\Omega) \ : \ \xi = 0 \text{ on } \Gamma_D\}$$

and

$$W_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \in W_D^{1,\vec{p}(.)}(\Omega) \ : \ \xi \equiv \text{constant on } \Gamma_{Ne}\}.$$

For any $v \in W_{Ne}^{1,\vec{p}(.)}(\Omega)$, we set $v_N = v_{Ne} := v|_{\Gamma_{Ne}}$.
The concept of solution for $S_{f,d}^{\rho}$ is given as follows.

Definition 2.1. A solution of $S_{f,d}^{\rho}$ is a couple $(u, v) \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \times \mathbb{R}$ satisfying

$$\begin{cases} w = |u|^{p_M(x)-2}u \ a.e. \ in \ \Omega, v \in \rho(u_N), \\ \varphi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^{\infty}(\Omega), \\ \displaystyle\int_{\Omega} \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i}u) \frac{\partial}{\partial x_i}\varphi \right) dx + \int_{\Omega} w\varphi dx = \int_{\Omega} f\varphi dx + (d-v)\varphi_{Ne}. \end{cases} \tag{2.9}$$

Our main result in this paper is the following theorem.

Theorem 2.1. *For any $(f, d) \in L^{\infty}(\Omega) \times \mathbb{R}$, the problem $S_{f,d}^{\rho}$ admits at least one solution (u, v) in the sense of Definition 2.1. Moreover if (u_1, v_1) and (u_2, v_2) are two solutions of $S_{f,d}^{\rho}$, then*

$$(v_1 - v_2)^+ + \int_{\Omega} (w_1 - w_2)^+ dx \leq \int_{\Omega} (f_1 - f_2)^+ dx + (d_1 - d_2)^+, \tag{2.10}$$

where $w_1 = |u_1|^{p_M(x)-2}u_1$ and $w_2 = |u_2|^{p_M(x)-2}u_2$.

3. Proof of the main result

The proof of the main result is done in three steps.
Step 1: Approximated problem for continuous functions. We assume that ρ is a continuous, non-decreasing and onto function on $\mathbb{R}$ such that

$$\rho(0) = 0. \tag{3.11}$$

We define a new bounded domain $\tilde{\Omega}$ in $\mathbb{R}^N$ as follow.
We fix $\theta > 0$ and we set $\tilde{\Omega} = \Omega \cup \{x \in \mathbb{R}^N / dist(x, \Gamma_{Ne}) < \theta\}$. Then, $\partial\tilde{\Omega} = \Gamma_D \cup \tilde{\Gamma}_{Ne}$ is Lipschitz with $\Gamma_D \cap \tilde{\Gamma}_{Ne} = \emptyset$.

Let us consider $\tilde{a}_i(x, \xi)$ Carathéodory and satisfying (1.5), (1.6), (1.7) and (1.8), for all $x \in \tilde{\Omega}$.
We also consider a function $\tilde{d}$ in $L^{\infty}(\tilde{\Gamma}_{Ne})$ such that

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{d} d\sigma = d. \tag{3.12}$$

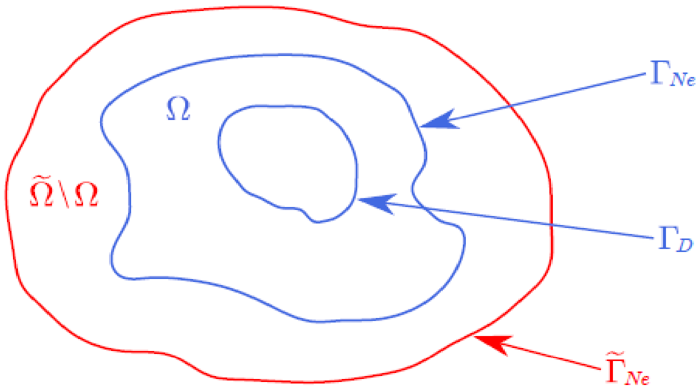

Figure 1: Domains representation

We consider the problem

$$P(\tilde{\rho}, \tilde{f}, \tilde{d}) \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\tilde{a}_i(x, \frac{\partial}{\partial x_i}u) + |u|^{P_M(x)-2}u\chi_{\Omega}(x) = \tilde{f} & \text{in } \tilde{\Omega} \\ u = 0 & \text{on } \Gamma_D \\ \tilde{\rho}(u) + \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\eta_i = \tilde{d} & \text{on } \tilde{\Gamma}_{Ne}, \end{cases} \quad (3.13)$$

where the function $\tilde{\rho}$ is defined as follow.

- $\tilde{\rho}(s) = \frac{1}{|\tilde{\Gamma}_{Ne}|}\rho(s)$, where $|\tilde{\Gamma}_{Ne}|$ denotes the Hausdorff measure of $\tilde{\Gamma}_{Ne}$.
- $\tilde{f}(x) = (f\chi_{\Omega})(x)\ \forall x \in \tilde{\Omega}$.

We obviously have $\tilde{f} \in L^{\infty}(\tilde{\Omega})$, $\tilde{d} \in L^{\infty}(\tilde{\Gamma}_{Ne})$.
We introduce the following space

$$W_D^{1,\vec{p}(.)}(\tilde{\Omega}) = \{\xi \in W^{1,\vec{p}(.)}(\tilde{\Omega})\ :\ \xi = 0 \text{ on } \Gamma_D\}.$$

Definition 3.2. A measurable function $u : \tilde{\Omega} \to \mathbb{R}$ is a solution to problem $P(\tilde{\rho}, \tilde{f}, \tilde{d})$ if $u \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and

$$\int_{\tilde{\Omega}} \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}\tilde{\varphi}dx + \int_{\Omega} |u|^{P_M(x)-2}u\tilde{\varphi}dx = \int_{\Omega} f\tilde{\varphi}dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d} - \tilde{\rho}(u))\tilde{\varphi}d\sigma, \quad (3.14)$$

for any $\tilde{\varphi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^{\infty}(\Omega)$.

The problem $P(\tilde{\rho}, \tilde{f}, \tilde{d})$ admits at least one solution in the sense of Definition 3.2 (see [8]).
Step 2: The regularized problem corresponding to $S^{\rho}_{f,d}$. For any $\epsilon > 0$, we denote by ρ_{ϵ} the Yosida regularization of ρ.

Now, we set $\tilde{a}_i(x,\xi) = a_i(x,\xi)\chi_\Omega(x) + \frac{1}{\epsilon^{p_i(x)}}|\xi|^{p_i(x)-2}\xi\chi_{\tilde{\Omega}\setminus\Omega}(x)$ for all $(x,\xi) \in \tilde{\Omega}\times\mathbb{R}^N$, $\tilde{\rho}_\epsilon(s) = \frac{1}{|\tilde{\Gamma}_{Ne}|}\rho_\epsilon(s)$ for all $s \in \mathbb{R}$. We consider the following problem $P_\epsilon(\tilde{\rho}_\epsilon, \tilde{f}, \tilde{d})$

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\chi_\Omega(x) + \frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\chi_{\tilde{\Omega}\setminus\Omega}(x)\right) + |u_\epsilon|^{P_M(x)-2}u_\epsilon\chi_\Omega = \tilde{f} & \text{in } \tilde{\Omega} \\ u_\epsilon = 0 & \text{on } \Gamma \\ \tilde{\rho}_\epsilon(u_\epsilon) + \sum_{i=1}^{N}\tilde{a}_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\eta_i = \tilde{d} & \text{on } \tilde{\Gamma} \end{cases} \tag{3.15}$$

$P_\epsilon(\tilde{\rho}_\epsilon, \tilde{f}, \tilde{d})$ has at least one solution (see [8]). So, there exists at least one measurable function $u_\epsilon : \tilde{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \sum_{i=1}^{N}\int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}\tilde{\varphi}\right)dx + \sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}\tilde{\varphi}\right)dx \\ \int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon\tilde{\varphi} = \int_\Omega f\tilde{\varphi}dx + \int_{\tilde{\Gamma}_{Ne}}(\tilde{d} - \tilde{\rho}_\epsilon(u_\epsilon))\tilde{\varphi}d\sigma, \end{cases} \tag{3.16}$$

where $u_\epsilon \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and for all $\tilde{\varphi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$.
Moreover, we have

$$\begin{cases} \tilde{\rho}_\epsilon(u_\epsilon)| \le k_3 := \max\{\|\tilde{d}\|_\infty, (\tilde{\rho}_\epsilon \circ b^{-1})(\|f\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}, \\ |b(u_\epsilon)| \le k_4 := \max\{\|f\|_\infty; (b\circ\rho_\epsilon^{-1})(|\tilde{\Gamma}_{Ne}|\|\tilde{d}\|_\infty)\} \; a.e. \text{ in } \Omega. \end{cases} \tag{3.17}$$

The following result gives a priori estimates on the solution u_ϵ of the problem $P_\epsilon(\tilde{\rho}_\epsilon, \tilde{f}, \tilde{d})$ (see [1, 5]).

Proposition 3.1. *Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\rho}_\epsilon, \tilde{f}, \tilde{d})$. Then, the following statements hold.*

(i) There exists C a positive constant independent of ϵ such that

$$\sum_{i=1}^{N}\int_\Omega\left(\frac{\partial}{\partial x_i}|u_\epsilon|\right)^{p_i(x)}dx + \sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon}|\frac{\partial}{\partial x_i}u_\epsilon|\right)^{p_i(x)}dx \le C\left(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}\right).$$

(ii)

$$\int_\Omega |u_\epsilon|^{P_M(x)-1}dx + \int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}_\epsilon(u_\epsilon)|dx \le (\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}).$$

The following result states useful convergences results (see [1, 5]).

Proposition 3.2. *As $\epsilon \to 0$ we have*

(i) $u_\epsilon \to u$ a.e. in Ω and a.e. on $\tilde{\Gamma}_{Ne}$ with $u \in W_D^{1,(p_1^-,\dots,p_N^-)}(\tilde{\Omega})$;

(ii) for all $i = 1,\dots N$, $\frac{\partial u_\epsilon}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i} = 0$ in $L^{p_i^-}(\tilde{\Omega}\setminus\Omega)$ with $\frac{\partial u}{\partial x_i} = 0$ in $\tilde{\Omega}\setminus\Omega$;

(iii) $a_i(x, \frac{\partial u_\epsilon}{\partial x_i}) \rightharpoonup a_i(x, \frac{\partial u}{\partial x_i})$ weakly in $L^1(\Omega)$ and a.e.in Ω.

Step 3: Proof of Theorem 2.1. Thanks to Proposition 3.2,
$\forall i = 1,\dots,N$, $\frac{\partial u}{\partial x_i} = 0$ in $\tilde{\Omega}\setminus\Omega$, then
$u = constant\ a.e.$

on $\tilde{\Omega} \setminus \Omega$ so that, we conclude that $u \in W_{Ne}^{1,\vec{p}(.)}(\Omega)$.

To show that u is a solution of $P(\rho, f, d)$, we only have to prove the equality (2.9). The sequences $(\tilde{\rho}_\epsilon(u_\epsilon))_{\epsilon>0}$ is uniformly bounded in $L^\infty(\tilde{\Gamma}_{Ne})$. Hence, there exists $v_1 \in L^\infty(\tilde{\Gamma}_{Ne})$ such that, as $\epsilon \to 0$,

$$\tilde{\rho}_\epsilon(u_\epsilon) \rightharpoonup^* v_1 \; in \; L^\infty(\tilde{\Gamma}_{Ne}). \tag{3.18}$$

Let $\varphi \in W_D^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$. we consider the function $\varphi_1 \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$, such that

$$\varphi_1 = \varphi\chi_\Omega + \varphi_N \chi_{\tilde{\Omega}\setminus\Omega}.$$

Then, $\varphi_1 = constant$ on $\tilde{\Omega} \setminus \Omega$. Such function φ_1 in the equality (3.16) gives us

$$\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i} u_\epsilon). \frac{\partial}{\partial x_i}\varphi \right) dx + \int_\Omega |u_\epsilon|^{P_M(x)-2} u_\epsilon \varphi dx = \int_\Omega f\varphi dx + \left(d - \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}_\epsilon(u_\epsilon) d\sigma \right) \varphi_N. \tag{3.19}$$

Passing to the limit in (3.19) as $\epsilon \to 0$ and using the convergences in Proposition 3.2, one has

$$\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i}\varphi \right) dx + \lim_{\epsilon\to 0} \int_\Omega b(u_\epsilon)\varphi \;\; = \;\; \int_\Omega f\varphi dx + d\varphi_N - \left(\lim_{\epsilon\to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}_\epsilon(u_\epsilon) d\sigma \right) \varphi_N$$

By Proposition 3.2 and Lebesgue dominated convergence theorem, we deduce that

$$b(u_\epsilon) \to b(u) \text{ in } L^1(\Omega). \tag{3.20}$$

Thanks to (3.18) and (3.20), we deduce that

$$\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i}\varphi \right) dx + \int_\Omega b(u)\varphi \;\; = \;\; \int_\Omega f\varphi dx + d\varphi_N - \left(\int_{\tilde{\Gamma}_{Ne}} v_1 d\sigma \right) \varphi_N.$$

We consider $w = b(u) \in L^1(\Omega)$ and $v = \int_{\tilde{\Gamma}_{Ne}} v_1 d\sigma \in \mathbb{R}$ to obtain from the above equality

$$\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i} u). \frac{\partial}{\partial x_i}\varphi \right) dx + \int_\Omega w\varphi dx \;\; = \;\; \int_\Omega f\varphi dx + (d - v)\, \varphi_N.$$

To conclude that (u, v) is a solution of $S_{f,d}^\rho$, it remain to show that

$$v \in \rho(u_N).$$

We have $\tilde{\rho}_\epsilon(u_\epsilon) \rightharpoonup^* v_1$ in $L^\infty(\tilde{\Gamma}_{Ne})$ as $\epsilon \to 0$. So $\tilde{\rho}_\epsilon(u_\epsilon) \rightharpoonup v_1$ in $L^{p_m^-}(\tilde{\Gamma}_{Ne})$ as $\epsilon \to 0$.
We also have $u_\epsilon \to u$ in $L^{p_m^-}(\tilde{\Gamma}_{Ne})$ as $\epsilon \to 0$ and $\tilde{\rho}_\epsilon \to \dfrac{1}{|\tilde{\Gamma}_{Ne}|}\rho$ in the sense of graph. Then (see [2]), $v_1 \in \dfrac{1}{|\tilde{\Gamma}_{Ne}|}\rho(u)$ a.e. on $\tilde{\Gamma}_{Ne}$ and $v_2 = |\tilde{\Gamma}_{Ne}| v_1 \in \rho(u)$ a.e. on $\tilde{\Gamma}_{Ne}$.
We know that $u \equiv constant$ in $\tilde{\Omega} \setminus \Omega$ so $u \equiv constant$ on $\tilde{\Gamma}_{Ne}$ and we get $v_2 \in \rho(u_N)$.
Using the fact that $\mathcal{D}(\rho) = \mathbb{R}$ either $\rho(u_N) = s$ or $\rho(u_N) = [r, s]$ with $(r, s) \in \mathbb{R}^2$ such that $r < s$, it yields that $\dfrac{1}{\tilde{\Gamma}_{Ne}} \int_{\tilde{\Gamma}_{Ne}} v_2 d\sigma \in \rho(u_N)$ and $v \in \rho(u_N)$. □ Let us prove now the inequality (2.10) of Theorem 2.1.

Proof. We have

$$\begin{cases} w_1 = b(u_1), \; w_2 = b(u_2) \\ v_1 \in \rho((u_1)_{Ne}), \; v_2 \in \rho((u_2)_{Ne}), \end{cases}$$

and for any $\varphi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u_1) \frac{\partial}{\partial x_i} \varphi \right) dx + \int_\Omega w_1 \varphi dx = \int_\Omega f_1 \varphi dx + (d_1 - v_1)\varphi_{Ne} \tag{3.21}$$

and

$$\int_\Omega \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u_2) \frac{\partial}{\partial x_i} \varphi \right) dx + \int_\Omega w_2 \varphi dx = \int_\Omega f_2 \varphi dx + (d_2 - v_2)\varphi_{Ne}. \tag{3.22}$$

Subtracting (3.21) from (3.22), one has

$$\begin{cases} \int_\Omega \sum_{i=1}^N \left(a_i(x, \frac{\partial}{\partial x_i} u_1) - a_i(x, \frac{\partial}{\partial x_i} u_2) \right) \frac{\partial}{\partial x_i} \varphi dx + \int_\Omega (w_1 - w_2)\varphi dx \\ +(v_1 - v_2)\varphi_{Ne} = \int_\Omega (f_1 - f_2)\varphi dx + (d_1 - d_2)\varphi_{Ne}. \end{cases} \tag{3.23}$$

In (3.23) we take $\varphi = H_\epsilon(u_1 - u_2 + \epsilon\xi)$ where ξ is any function in $W^{1,\vec{p}(.)}(\Omega)$. After calculus we obtain

$$\begin{cases} \int_\Omega (w_1 - w_2)\xi \chi_{[u_1 = u_2]} dx + \int_\Omega (w_1 - w_2) sign_0^+ \chi_{[u_1 \neq u_2]} dx + (v_1 - v_2)(\xi_{Ne})\chi_{[(u_1)_{Ne} = (u_2)_{Ne}]} \\ +(v_1 - v_2) sign_0^+((u_1)_{Ne} - (u_2)_{Ne})\chi_{[(u_1)_{Ne} \neq (u_2)_{Ne}]} \leq \int_\Omega (f_1 - f_2)^+ dx + (d_1 - d_2)^+. \end{cases} \tag{3.24}$$

Now, we consider the function ξ_0 defined as follows.

$$\xi_0 = \begin{cases} sign_0^+(w_1 - w_2) & \text{in } [u_1 = u_2] \\ sign_0^+(v_1 - v_2) & \text{on } \Gamma_{Ne} \\ 0 & \text{in } \mathbb{R}^N \setminus \{[u_1 = u_2]\}. \end{cases}$$

Replacing ξ by ξ_0 in (3.24), we get

$$\begin{cases} \int_\Omega (w_1 - w_2)\xi_0 \chi_{[u_1 = u_2]} dx + \int_\Omega (w_1 - w_2) sign_0^+ \chi_{[u_1 \neq u_2]} dx + (v_1 - v_2)(\xi_0)_{Ne}\chi_{[(u_1)_{Ne} = (u_2)_{Ne}]} \\ +(v_1 - v_2) sign_0^+((u_1)_{Ne} - (u_2)_{Ne})\chi_{[(u_1)_{Ne} \neq (u_2)_{Ne}]} \leq \int_\Omega (f_1 - f_2)^+ dx + (d_1 - d_2)^+. \end{cases} \tag{3.25}$$

Taking into account the definition of ξ_0, one gets from (3.25)

$$\begin{cases} \int_\Omega (w_1 - w_2) sign_0^+(w_1 - w_2)\chi_{[u_1 = u_2]} dx + \int_\Omega (w_1 - w_2) sign_0^+(u_1 - u_2)\chi_{[u_1 \neq u_2]} dx \\ +(v_1 - v_2) sign_0^+(v_1 - v_2)\chi_{[(u_1)_{Ne} = (u_2)_{Ne}]} \\ +(v_1 - v_2) sign_0^+((u_1)_{Ne} - (u_2)_{Ne})\chi_{[(u_1)_{Ne} \neq (u_2)_{Ne}]} \leq \int_\Omega (f_1 - f_2)^+ dx + (d_1 - d_2)^+; \end{cases}$$

which is equivalent to say

$$\begin{cases} \int_\Omega (w_1 - w_2)^+ \chi_{[u_1 = u_2]} dx + \int_\Omega (w_1 - w_2)^+ \chi_{[u_1 \neq u_2]} dx + (v_1 - v_2)^+ \chi_{[(u_1)_{Ne} = (u_2)_{Ne}]} \\ +(v_1 - v_2)^+ \chi_{[(u_1)_{Ne} \neq (u_2)_{Ne}]} \leq \int_\Omega (f_1 - f_2)^+ dx + (d_1 - d_2)^+; \end{cases}$$

and

$$(v_1 - v_2)^+ + \int_\Omega (w_1 - w_2)^+ dx \leq \int_\Omega (f_1 - f_2)^+ dx + (d_1 - d_2)^+;$$

which correspond to (2.10). □

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Université Joseph KI-ZERBO
UFR. Sciences Exactes et Appliquées
Laboratoire de Mathématiques et Informatique (LAMI)
03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso
Email address: ouaro@yahoo.fr
Email address: kaboreadama59@yahoo.fr