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A relation between the Brocard and Miquel Angles

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ABSTRACT. Let XYZ be an inscribed triangle in the reference triangle ABC ($X \in BC, Y \in CA, Z \in AB$) given by its absolute barycentric coordinates: $X = (0, 1-\alpha, \alpha)$, $Y = (\beta, 0, 1-\beta)$, $Z = (1-\gamma, \gamma, 0)$, where α, β, γ are arbitrary real numbers. In this paper we deduce the following simple and beautiful formula between the Brocard and Miquel angles: $S_\omega = S_\theta + a^2\alpha + b^2\beta + c^2\gamma$, where ω is the Brocard and θ the Miquel angle of ABC.

1. PRELIMINARIES

In this paper we will use the barycentric coordinates [1] and the Conway triangle notations [2]. Given a reference triangle ABC whose sides are a, b and c and whose corresponding internal angles A, B and C, then the Conway triangle notations are

$$S = 2 \times Area\, ABC = bc \sin A = ca \sin B = ab \sin C \text{ and } S_\varphi = S \cot \varphi,$$

in particular $S_A = bc \cos A$, $S_B = ca \cos B$, $S_C = ab \cos C$. We will use also some important conditioned trigonometric identities:

$$b^2c^2 - S_A^2 = c^2a^2 - S_B^2 = a^2b^2 - S_C^2 = S^2;$$
$$a^2S_A + S_{BC} = b^2S_B + S_{CA} = c^2S_C + S_{AB} = S^2;$$
$$S_{BC} + S_{CA} + S_{AB} = S^2;$$
$$a^2S_A + b^2S_B + c^2S_C = 2S^2;$$
$$S_\omega = \frac{a^2+b^2+c^2}{2} = S_A + S_B + S_C = a^2 + S_A = b^2 + S_B = c^2 + S_C;$$
$$b^2c^2 + c^2a^2 + a^2b^2 - S_\omega^2 = S^2.$$

We draw a circle through points B and C, tangent to side CA, another circle through points C and A, tangent to side AB and a circle through points A and B, tangent to side BC of the triangle ABC [3]. These three circles have a common point (Ω), the *first Brocard point of triangle ABC* (Figure 1). Moreover, $\measuredangle\Omega AB = \measuredangle\Omega BC = \measuredangle\Omega CA = \omega$, and this angle ω is called the *Brocard angle of the triangle ABC*. The Brocard angle has the property that

$$\cot \omega = \cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{2S}.$$

Similarly we can construct the *second Brocard point* (Ω').

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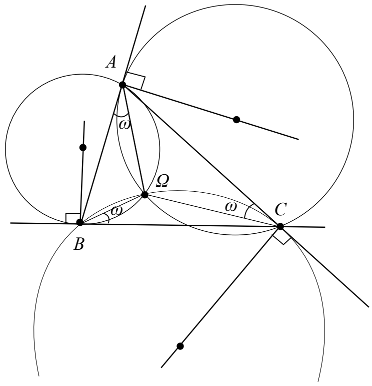

Figure 1.

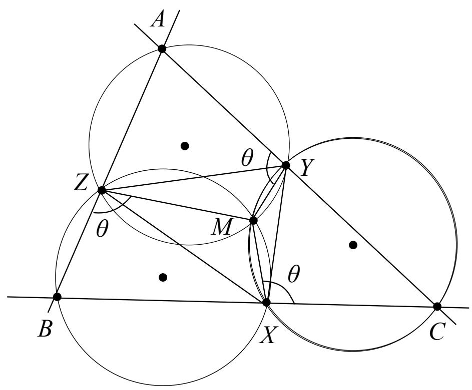

Figure 2.

The three circumcircles to triangles AYZ, BZX and CXY intersect in a single point M, called the *Miquel's point of the triangle* ABC [4]. In addition $\measuredangle AYM = \measuredangle BZM = \measuredangle CXM = \theta$, and this angle θ is called the *Miquel angle of* ABC (Figure 2).

2. Equations of circumcircles of the residual triangles

Firstly we will determine the barycentric coordinates of the circumcenters O_a, O_b, O_c of the residual triangles AYZ, BZX, CXY respectively. Denote with $L(K, PQ)$ the perpendicular from the point K to the line PQ. Consequently $O_a = L(Y_a, AC) \cap L(Z_a, AB)$, where Y_a and Z_a are the midpoints of segments AY and AZ. So $2Y_a = (1+\beta, 0, 1-\beta)$ and $2Z_a = (2-\gamma, \gamma, 0)$ (Figure 3).

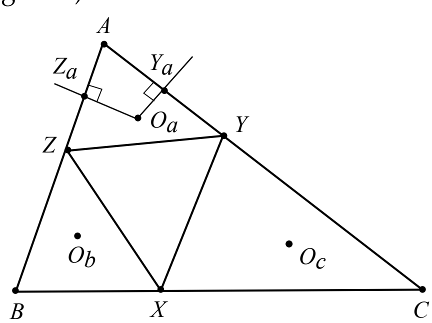

Figure 3.

The equations of the perpendiculars $L(Y_a, AC)$ and $L(Z_a, AB)$ are

$$\begin{vmatrix} x & y & z \\ 1+\beta & 0 & 1-\beta \\ S_C & -b^2 & S_A \end{vmatrix} = 0 \Leftrightarrow b^2(1-\beta)x + (S_C - S_A - b^2\beta)y - b^2(1+\beta)z = 0,$$

$$\begin{vmatrix} x & y & z \\ 2-\gamma & \gamma & 0 \\ S_B & S_A & -c^2 \end{vmatrix} = 0 \Leftrightarrow -c^2\gamma x + c^2(2-\gamma)y + (2S_A - c^2\gamma)z = 0.$$

With the Cramer rule from these equations we can obtain the coordinates of O_a:

$$\delta = \begin{vmatrix} b^2(1-\beta) & S_C - S_A - b^2\beta \\ -c^2\gamma & c^2(2-\gamma) \end{vmatrix} = b^2c^2(1-\beta) - c^2S_A\gamma;$$

$$\delta_x = \begin{vmatrix} b^2(1+\beta) & S_C - S_A - b^2\beta \\ -(2S_A - c^2\gamma) & c^2(2-\gamma) \end{vmatrix} z = \left[2S^2 - b^2S_B(1-\beta) - c^2S_C\gamma\right] z;$$

$$\delta_y = \begin{vmatrix} b^2(1-\beta) & b^2(1+\beta) \\ -c^2\gamma & -(2S_A - c^2\gamma) \end{vmatrix} z = \left[-b^2S_A(1-\beta) + b^2c^2\gamma\right] z.$$

Consequently the coordinates of O_a are

$$O_a = \left(2S^2 - b^2S_B(1-\beta) - c^2S_C\gamma : -b^2S_A(1-\beta) + b^2c^2\gamma : b^2c^2(1-\beta) - c^2S_A\gamma\right).$$

The sum of these coordinates is $2S^2$, so the absolute barycentric coordinates of O_a are

$$O_a = \left(1 - \frac{b^2S_B(1-\beta) + c^2S_C\,\gamma}{2S^2}, \frac{-b^2S_A(1-\beta) + b^2c^2\,\gamma}{2S^2}, \frac{b^2c^2(1-\beta) - c^2S_A\,\gamma}{2S^2}\right).$$

Similarly we obtain the absolute barycentric coordinates of O_b and O_c:

$$O_b = \left(\frac{c^2a^2(1-\gamma) - a^2S_B\alpha}{2S^2}, 1 - \frac{c^2S_C(1-\gamma) + a^2S_A\alpha}{2S^2}, \frac{-c^2S_B(1-\gamma) + c^2a^2\alpha}{2S^2}\right).$$

$$O_c = \left(\frac{-a^2S_C(1-\alpha) + a^2b^2\beta}{2S^2}, \frac{a^2b^2(1-\alpha) - b^2S_C\beta}{2S^2}, 1 - \frac{a^2S_A(1-\alpha) + b^2S_B\beta}{2S^2}\right).$$

Denote with $O_a(R_a)$, $O_b(R_b)$, $O_c(R_c)$ the circumcircles of the residual triangles AYZ, BZX, CXY respectively, where R_a, R_b, R_c are the radii of these circles.

Proposition 2.1.

$$(a) \quad R_a = \frac{bc}{2S}\sqrt{b^2(1-\beta)^2 - 2(1-\beta)\gamma S_A + c^2\gamma^2},$$

$$(b) \quad R_b = \frac{ca}{2S}\sqrt{c^2(1-\gamma)^2 - 2(1-\gamma)\alpha S_B + a^2\alpha^2},$$

$$(c) \quad R_c = \frac{ab}{2S}\sqrt{a^2(1-\alpha)^2 - 2(1-\alpha)\beta S_C + b^2\beta^2}.$$

Proof. We have:

$$\begin{aligned} R_a^2 = AO_a^2 &= \left(\frac{b^2S_B(1-\beta) + c^2S_C\gamma}{2S^2}\right)^2 S_A + \left(\frac{-b^2S_A(1-\beta) + b^2c^2\gamma}{2S^2}\right)^2 S_B \\ &+ \left(\frac{b^2c^2(1-\beta) - c^2S_A\gamma}{2S^2}\right)^2 S_C = \frac{1}{4S^4}\Big[b^4(S_B^2S_A + S_A^2S_B + c^4S_C)(1-\beta)^2 \\ &+2b^2c^2S_A(S_BS_C - b^2S_B - c^2S_C)(1-\beta)\gamma + c^4(S_C^2S_A + S_A^2S_C + b^4S_B)\gamma^2\Big] \\ &= \frac{b^2c^2}{4S^2}\left[b^2(1-\beta)^2 - 2(1-\beta)\gamma S_A + c^2\gamma^2\right]. \end{aligned}$$

□

We will introduce the following notations:

$$\lambda = \lambda(\alpha,\beta,\gamma) = a^2\alpha + b^2\beta + c^2\gamma,$$
$$\lambda' = \lambda(1-\alpha, 1-\beta, 1-\gamma) = a^2(1-\alpha) + b^2(1-\beta) + c^2(1-\gamma),$$
$$E = E(\alpha,\beta,\gamma) = S_\omega - \lambda = \frac{1}{2}\left[a^2(1-2\alpha) + b^2(1-2\beta) + c^2(1-2\gamma)\right].$$

Consequently $\lambda' = 2S_\omega - \lambda$ i.e. $\lambda + \lambda' = 2S_\omega = a^2 + b^2 + c^2$ or $S_\omega - \lambda = \lambda' - S_\omega$.

Proposition 2.2.

$$(a) \quad O_bO_c = \frac{a}{2S}\sqrt{E^2+S^2} = \frac{a}{2S}\sqrt{(a^2\alpha + b^2\beta + c^2\gamma - S_\omega)^2 + S^2},$$
$$(b) \quad O_cO_a = \frac{b}{2S}\sqrt{E^2+S^2} = \frac{b}{2S}\sqrt{(a^2\alpha + b^2\beta + c^2\gamma - S_\omega)^2 + S^2},$$
$$(c) \quad O_aO_b = \frac{c}{2S}\sqrt{E^2+S^2} = \frac{c}{2S}\sqrt{(a^2\alpha + b^2\beta + c^2\gamma - S_\omega)^2 + S^2}.$$

Proof. We calculate the differences of coordinates of the points O_b and O_c:

$$\begin{aligned}
x_{O_b} - x_{O_c} &= \frac{1}{2S^2}\left[c^2a^2(1-\gamma) - a^2S_B\alpha + a^2S_C(1-\alpha) - a^2b^2\beta\right] \\
&= \frac{a^2}{2S^2}(c^2 + S_C - a^2\alpha - b^2\beta - c^2\gamma) = \frac{a^2(S_\omega - \lambda)}{2S^2}, \\
y_{O_b} - y_{O_c} &= \frac{1}{2S^2}\left[2S^2 - c^2S_C(1-\lambda) - a^2S_A\alpha - a^2b^2(1-\alpha) + b^2S_C\beta\right] \\
&= \frac{1}{2S^2}\left[S_C(a^2\alpha + b^2\beta + c^2\gamma - S_\omega) + S^2\right] = \frac{S_C(\lambda - S_\omega) + S^2}{2S^2},
\end{aligned}$$

$$\begin{aligned}
z_{O_b} - z_{O_c} &= \frac{1}{2S^2}\left[-c^2S_B(1-\gamma) + c^2a^2\alpha - 2S^2 + a^2S_A(1-\alpha) + b^2S_B\beta\right] \\
&= \frac{1}{2S^2}\left[S_B(a^2\alpha + b^2\beta + c^2\gamma - S_\omega) - S^2\right] = \frac{S_B(\lambda - S_\omega) - S^2}{2S^2}.
\end{aligned}$$

The length of segment O_bO_c is

$$(O_bO_c)^2 = \left(\frac{a^2(S_\omega - \lambda)}{2S^2}\right)^2 S_A + \left(\frac{S_C(\lambda - S_\omega) + S^2}{2S^2}\right)^2 S_B + \left(\frac{S_B(\lambda - S_\omega) - S^2}{2S^2}\right)^2 S_C$$
$$= \frac{1}{4S^4}\left[(a^4S_A + S_BS_C^2 + S_B^2S_C)(\lambda - S_\omega)^2 + (S_B + S_C)S^4\right] = \frac{a^2\left((\lambda - S_\omega)^2 + S^2\right)}{4S^2}.$$

□

Proposition 2.3. *The equations of circumcircles of triangles AYZ, BZX, CXY are:*

$$O_a(R_a) : S_Ax^2 + S_By^2 + S_Cz^2 - (x+y+z)\left[S_Ax - (S_A - c^2\gamma)y + (S_C - b^2\beta)z\right] = 0,$$

$$O_b(R_b) : S_Ax^2 + S_By^2 + S_Cz^2 - (x+y+z)\left[(S_A - c^2\gamma)x + S_By - (S_B - a^2\alpha)z\right] = 0,$$

$$O_c(R_c) : S_Ax^2 + S_By^2 + S_Cz^2 - (x+y+z)\left[-(S_C - b^2\beta)x + (S_B - a^2\alpha)y + S_Cz\right] = 0.$$

Proof. The equation of the circle $O_a(R_a)$ is $PO_a^2 = R_a^2 \Leftrightarrow PO_a^2 = AO_a^2$, where $P = \left(\frac{x}{\mu}, \frac{y}{\mu}, \frac{z}{\mu}\right)$, $\mu = x + y + z \neq 0$. So

$$\left(\frac{x}{\mu} - x_{O_a}\right)^2 S_A + \left(\frac{y}{\mu} - y_{O_a}\right)^2 S_B + \left(\frac{z}{\mu} - z_{O_a}\right)^2 S_C$$
$$= (1 - x_{O_a})^2 S_A + y_{O_a}^2 S_B + z_{O_a}^2 S_C \Leftrightarrow S_A x^2 + S_B y^2 + S_C z^2 -$$
$$-2\mu(S_A x_{O_a} x + S_B y_{O_a} y + S_C z_{O_a} z) - \mu^2(1 - 2x_{O_a})S_A = 0$$

$$S_A x^2 + S_B y^2 + S_C z^2 - \mu\big[S_A x + (S_A - 2S_A x_{O_a} + 2S_B y_{O_a})y+$$
$$+ (S_A - 2S_A x_{O_a} + 2S_C z_{O_a})\big] = 0$$
$$\Leftrightarrow S_A x^2 + S_B y^2 + S_C z^2 - (x + y + z)\big[S_A x - (S_A - c^2\gamma)y + (S_C - b^2\beta)z\big] = 0.$$

□

3. Barycentric coordinates of the Miquel point

Denote with L_{bc}, L_{ca}, L_{ab} the radical axes of the pairs of circles $(O_b(R_b), O_c(R_c))$, $O_c(R_c)$, $O_a(R_a))$, $(O_a(R_a), O_b(R_b))$ respectively.

Proposition 3.4. *The equations of the radical axes L_{bc}, L_{ca}, L_{ab} are*

$$L_{bc} : \big[b^2(1 - \beta) - c^2\gamma\big]x + a^2\alpha y - a^2(1 - \alpha)z = 0,$$
$$L_{ca} : -b^2(1 - \beta)x + \big[c^2(1 - \gamma) - a^2\alpha\big]y + b^2\beta z = 0,$$
$$L_{ab} : c^2\gamma x - c^2(1 - \gamma)y + \big[a^2(1 - \alpha) - b^2\beta\big]z = 0.$$

Proof. We obtain the equation of the radical axis L_{bc} as the difference of the equations of circles $O_b(R_b)$ and $O_c(R_c)$, i.e.

$$(S_A - c^2\gamma)x + S_B y - (S_B - a^2\alpha)z - \big[-(S_C - b^2\beta)x + (S_B - a^2\alpha)y + S_C z\big] = 0$$
$$\Leftrightarrow \big[b^2(1 - \beta) - c^2\gamma\big]x + a^2\alpha y - a^2(1 - \alpha)z = 0.$$

□

Proposition 3.5. *The radical axes L_{bc}, L_{ca}, L_{ab} are concurrent.*

Proof. In the equations of the radical axes L_{bc}, L_{ca}, L_{ab} the sums of the coefficients of x, y respectively z are zero. □

Let M be the point of concurrence of the radical axes L_{bc}, L_{ca}, L_{ab}. This point is called the *radical center* of circles $O_a(R_a)$, $O_b(R_b)$, $O_c(R_c)$ or the *Miquel point determined by the inscribed triangle XYZ* (Figure 4):

$$M = L_{bc} \cap L_{ca} \cap L_{ab} = O_a(R_a) \cap O_b(R_b) \cap O_c(R_c).$$

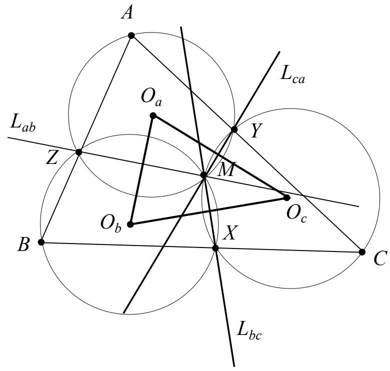

Figure 4.

From the equations of radical axis L_{bc} and L_{ca}, applied the Cramer rule we can determine the barycentric coordinates of Miquel point:

$$\begin{aligned}\delta &= \begin{vmatrix} b^2(1-\beta)-c^2\gamma & a^2\alpha \\ -b^2(1-\beta) & c^2(1-\gamma)-a^2\alpha \end{vmatrix} = \begin{vmatrix} b^2(1-\beta)-c^2\gamma & a^2\alpha \\ -c^2\gamma & c^2(1-\gamma) \end{vmatrix} \\ &= c^2\big[a^2\alpha+(1-\gamma)(b^2-a^2\alpha-b^2\beta-c^2\gamma)\big] = c^2\big[(b^2-\lambda)(1-\gamma)+a^2\alpha\big]; \\ \delta_x &= \begin{vmatrix} a^2(1-\alpha) & a^2\alpha \\ -b^2\beta & c^2(1-\gamma)-a^2\alpha \end{vmatrix} z = a^2\big[(c^2-\lambda)(1-\alpha)+b^2\beta\big]z; \\ \delta_y &= \begin{vmatrix} b^2(1-\beta)-c^2\gamma & a^2(1-\alpha) \\ -b^2(1-\beta) & -b^2\beta \end{vmatrix} z = b^2\big[(a^2-\lambda)(1-\beta)+c^2\gamma\big]z.\end{aligned}$$

Therefore

$$M=\left(a^2\big[(c^2-\lambda)(1-\alpha)+b^2\beta\big] : b^2\big[(a^2-\lambda)(1-\beta)+c^2\gamma\big] : c^2\big[(b^2-\lambda)(1-\gamma)+a^2\alpha\big]\right).$$

We denote with μ_P the sum of coordinates of an arbitrary point P.
Therefore:

$$\begin{aligned}\mu_M &= b^2c^2+c^2a^2+a^2b^2-\lambda\big[a^2(1-\alpha)+b^2(1-\beta)+c^2(1-\gamma)\big] \\ &= b^2c^2+c^2a^2+a^2b^2-\lambda\lambda' = b^2c^2+c^2a^2+a^2b^2+\lambda(\lambda-2S_\omega) \\ &= b^2c^2+c^2a^2+a^2b^2+(\lambda-S_\omega)^2-S_\omega^2=(\lambda-S_\omega)^2+S^2=(S_\omega-\lambda')^2+S^2.\end{aligned}$$

Let x_M, y_M, z_M be the absolute barycentric coordinates of M:

$$\begin{aligned}x_M &= \frac{a^2\big[(c^2-\lambda)(1-\alpha)+b^2\beta\big]}{(\lambda-S_\omega)^2+S^2} = \frac{a^2\big[c^2(1-\gamma)+(b^2-\lambda')\alpha\big]}{(\lambda-S_\omega)^2+S^2}, \\ y_M &= \frac{b^2\big[(a^2-\lambda)(1-\beta)+c^2\gamma\big]}{(\lambda-S_\omega)^2+S^2} = \frac{b^2\big[a^2(1-\alpha)+(c^2-\lambda')\beta\big]}{(\lambda-S_\omega)^2+S^2}, \\ z_M &= \frac{c^2\big[(b^2-\lambda)(1-\gamma)+a^2\alpha\big]}{(\lambda-S_\omega)^2+S^2} = \frac{c^2\big[b^2(1-\beta)+(a^2-\lambda')\gamma\big]}{(\lambda-S_\omega)^2+S^2}.\end{aligned}$$

4. Main results

The lines MX, MY, MZ and the corresponding sides BC, CA, AB have the same angle named the *Miquel angle determined by the inscribed triangle XYZ*, which is a direct measured angle and change between 0 and π (Figure 2).

Proposition 4.6.

$$(a)\quad AM = \frac{bc\sqrt{b^2(1-\beta)^2-2(1-\beta)\gamma S_A+c^2\gamma^2}}{\sqrt{E^2+S^2}} = \frac{aR_a}{O_bO_c},$$

$$(b)\quad BM = \frac{ca\sqrt{c^2(1-\gamma)^2-2(1-\gamma)\alpha S_B+a^2\alpha^2}}{\sqrt{E^2+S^2}} = \frac{bR_b}{O_cO_a},$$

$$(c)\quad CM = \frac{ab\sqrt{a^2(1-\alpha)^2-2(1-\alpha)\beta S_C+b^2\beta^2}}{\sqrt{E^2+S^2}} = \frac{cR_c}{O_aO_b}.$$

Proof. We have:

$$\begin{aligned}AM^2 &= (1-x_M)^2S_A+y_M^2S_B+z_M^2S_C=(y_M+z_M)^2S_A+y_M^2S_B+z_M^2S_C \\ &= c^2y_M^2+2y_Mz_MS_A+b^2z_M^2=\frac{b^2c^2}{(E^2+S^2)^2}\left\{b^2\big[(a^2-\lambda)(1-\beta)+c^2\gamma\big]^2\right.\end{aligned}$$

$$+2\left[(a^2-\lambda)(1-\beta)+c^2\gamma\right]\left[b^2(1-\beta)+(a^2-\lambda')\gamma\right]S_A+c^2\left[b^2(1-\beta)+(a^2-\lambda')\gamma\right]^2\Big\}$$

$$=\frac{b^2c^2}{(E^2+S^2)^2}\Big\{b^2\left[(S_\omega-\lambda-S_A)(1-\beta)+c^2\gamma\right]^2$$
$$+2[(S_\omega-\lambda-S_A)(1-\beta)+c^2\gamma][b^2(1-\beta)+(S_\omega-\lambda'-S_A)\gamma]S_A$$
$$+c^2\left[b^2(1-\beta)+(S_\omega-\lambda'-S_A)\gamma\right]^2\Big\}$$

$$=\frac{b^2c^2}{(E^2+S^2)^2}\Big\{b^2[(E-S_A)(1-\beta)+c^2\gamma]^2$$
$$+2\left[(E-S_A)(1-\beta)+c^2\gamma\right]\left[b^2(1-\beta)-(E+S_A)\gamma\right]S_A$$
$$+c^2[b^2(1-\beta)-(E+S_A)\gamma]^2\Big\}$$

$$=\frac{b^2c^2}{(E^2+S^2)^2}\Big\{b^2[(E-S_A)^2+2(E-S_A)S_A+b^2c^2](1-\beta)^2$$
$$+2[b^2c^2(E-S_A)-(E+S_A)(E-S_A)S_A+b^2c^2S_A-b^2c^2(E+S_A)](1-\beta)\gamma$$
$$+c^2[b^2c^2-2(E+S_A)S_A+(E+S_A)^2]\gamma^2\Big\}=$$

$$=\frac{b^2c^2}{(E^2+S^2)^2}\Big\{b^2(E^2+S^2)(1-\beta)^2-2(E^2+S^2)S_A(1-\beta)\gamma+c^2(E^2+S^2)\gamma^2\Big\}$$
$$=\frac{b^2c^2}{E^2+S^2}\left[b^2(1-\beta)^2-2(1-\beta)\gamma S_A+c^2\gamma^2\right].$$

□

From the law of sinus we have $2\sin\theta=\dfrac{AM}{R_a}=\dfrac{BM}{R_b}=\dfrac{CM}{R_c}$ and from the Proposition 4.1 $\dfrac{AM}{R_a}=\dfrac{a}{O_bO_c}$, $\dfrac{BM}{R_b}=\dfrac{b}{O_cO_a}$, $\dfrac{CM}{R_c}=\dfrac{c}{O_aO_b}$.

Therefore, the triangles ABC and $O_aO_bO_c$ are similar and its similarity ratio is $2\sin\theta=\dfrac{a}{O_bO_c}=\dfrac{2S}{\sqrt{E^2+S^2}}$, from where $\sin\theta=\dfrac{S}{\sqrt{E^2+S^2}}$ (Figure 4).

Proposition 4.7. *Between the Miquel and Brocard angles exist the following relation:*

$$S_\omega=S_\theta+a^2\alpha+b^2\beta+c^2\gamma.$$

Proof. We have: $\cos^2\theta=1-\sin^2\theta=1-\dfrac{S^2}{E^2+S^2}=\dfrac{E^2}{E^2+S^2}$. So $\cos\theta=\dfrac{S_\omega-\lambda}{\sqrt{E^2+S^2}}$ and $S_\theta=S\cot\theta=S\dfrac{S_\omega-\lambda}{S}=S_\omega-(a^2\alpha+b^2\beta+c^2\gamma)$. □

Remark 4.1. The Miquel angle is right if and only if the triangle XYZ is pedal triangle.

5. Isotomic inscribed triangles

If $X'Y'Z'$ and XYZ are isotomic inscribed triangles then $X'=(0,\alpha,1-\alpha)$, $Y'=(1-\beta,0,\beta)$, $Z'=(\gamma,1-\gamma,0)$. Consequently all results referred to the triangle $X'Y'Z'$ can be obtained from above results replacing α,β,γ by $1-\alpha$, $1-\beta$, $1-\gamma$ respectively. For example

$$O_b'O_c'=\frac{a}{2S}\sqrt{(\lambda'-S_\omega)^2+S^2}=\frac{a}{2S}\sqrt{(S_\omega-\lambda)^2+S^2}=O_bO_c,$$

$$O'_cO'_a = \frac{b}{2S}\sqrt{(\lambda' - S_\omega)^2 + S^2} = \frac{b}{2S}\sqrt{(S_\omega - \lambda)^2 + S^2} = O_cO_a,$$
$$O'_aO'_b = \frac{c}{2S}\sqrt{(\lambda' - S_\omega)^2 + S^2} = \frac{c}{2S}\sqrt{(S_\omega - \lambda)^2 + S^2} = O_aO_b.$$

It follows that the triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are congruent.

Proposition 5.8. *If θ' is the Miquel angle of the isotomic inscribed triangle $X'Y'Z'$ then*

$$(a) \quad S_\omega = -S_{\theta'} + a^2\alpha + b^2\beta + c^2\gamma$$
$$(b) \quad \theta + \theta' = \pi.$$

Proof. We have:

$$(a) \quad S_{\theta'} = S_\omega - \left[a^2(1-\alpha) + b^2(1-\beta) + c^2(1-\gamma)\right) = -S_\omega + a^2\alpha + b^2\beta + c^2\gamma$$

$$(b) \quad S_\theta + S'_\theta = S\cot\theta + S\cot\theta' = 0. \quad \text{Therefore } \frac{\sin(\theta+\theta')}{\sin\theta\cdot\sin\theta'} = 0,$$

i.e. $\theta + \theta' = \pi$ [5]. □

Remark 5.2. For $\alpha = \beta = \gamma = 0$ the Miquel point coincide with the 1^{st} Brocard point $\Omega = (c^2a^2 : a^2b^2 : b^2c^2)$ (Figure 1) and for $\alpha = \beta = \gamma = 1$ we obtain the 2^{nd} Brocard point $\Omega' = (a^2b^2 : b^2c^2 : c^2a^2)$ [6].

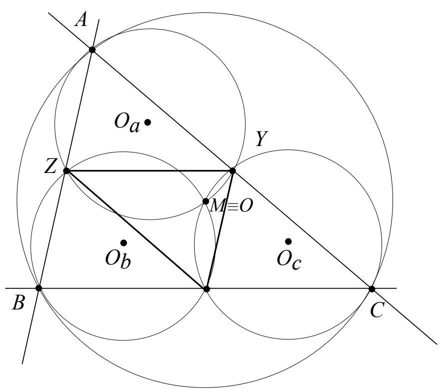

Figure 5.

If $\alpha = \beta = \gamma = \frac{1}{2}$ the Miquel point coincide with the circumcenter $O = (a^2S_A : b^2S_B : c^2S_C)$ of the reference triangle ABC (Figure 5) [6].

References

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