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Generalized semi-open sets via ideals in topological space

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ABSTRACT. In this paper we have introduced a new type of sets termed as $\hat{\mu}$-open sets which unifies semiopen sets and discussed some of its properties. We have also introduced another type of weak open sets termed as $\mathcal{I}_{\hat{\mu}}$-open sets depending on a GT as well as an ideal on a topological space. Finally the concept of weakly $\mathcal{I}_{\hat{\tau}}$-open sets are investigated.

1. INTRODUCTION

The concept of ideal on topological spaces was studied by Kuratowski [11] and Vaidyanat-haswamy [17] which is one of the important area of research in the branch of mathematics. After then different mathematicians applied the concept of ideals in topological spaces (see [2, 8, 9, 10, 14, 16, 17]). In the past few years mathematicians turned their attention towards the generalized open sets (see [3, 4, 6, 15, 16] for details). Our aim in this paper is to use the concept of ideals in the generalized topology introduced by A. Császár. We recall some notions defined in [4].

Let $expX$ denotes the power set of a non-empty set X. A class μ ($\subseteqq expX$) is called a generalized topology [4], (briefly, GT) if $\varnothing \in \mu$ and μ is closed under arbitrary union. The elements of μ are called μ-open sets and the complement of μ-open sets are known as μ-closed sets. A set X with a GT μ on it is known as a generalized topological space (briefly, GTS) and is denoted by (X, μ). A GT μ is said to be a quasi topology (briefly QT) [5] if $M, M^{'} \in \mu$ implies $M \cap M^{'} \in \mu$. The pair (X, μ) is said to be a QTS if μ is a QT on X.

For any $A \subseteqq X$, the generalized μ-closure of A is denoted by $c_{\mu}(A)$ and is defined by $c_{\mu}(A) = \cap\{F : F$ is μ-closed and $A \subseteqq F\}$, similarly $i_{\mu}(A) = \cup\{U : U \subseteqq A$ and $U \in \mu\}$ (see [4, 6]). Throughout the paper μ, λ will always mean GT on the respective sets.

An ideal [11] $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X with the following properties : (i) $A \subseteqq B$ and $B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$ (ii) $A \in \mathcal{I}$, $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$. A topological space (X, τ) with an ideal $\mathcal{I}$ is denoted by $(X, \tau, \mathcal{I})$ and known as an ideal topological space.

2. PROPERTIES OF $\hat{\mu}$-OPEN, $\mathcal{I}_{\hat{\mu}}$-OPEN, AND WEAKLY $\mathcal{I}_{\hat{\tau}}$-OPEN SETS

Definition 2.1. Let μ be a GT on a topological space (X, τ). A subset A of X is called $\hat{\mu}$-open if $A \subseteqq c_{\mu}(int(A))$.

Theorem 2.1. *Let μ be a GT on a topological space (X, τ). A subset A of X is $\hat{\mu}$-open if and only if there exists an open set U such that $U \subseteqq A \subseteqq c_{\mu}(U)$.*

Proof. Let A be a $\hat{\mu}$-open set. Then $A \subseteqq c_{\mu}(int(A))$. Let $U = int(A)$. Then U is an open set and $U \subseteqq A \subseteqq c_{\mu}(int(A)) = c_{\mu}(U)$.

Conversely, let there be an open set U such that $U \subseteqq A \subseteqq c_{\mu}(U)$. Now $U \subseteqq A \Rightarrow U \subseteqq int(A) \Rightarrow c_{\mu}(U) \subseteqq c_{\mu}(int(A))$. Thus $A \subseteqq c_{\mu}(int(A))$. □

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Remark 2.1. Let μ be a GT on a topological space (X,τ). If
(i) $\mu=\tau$, then a $\hat{\mu}$-open set reduces to a semi-open set.
(ii) every open set is $\hat{\mu}$-open.
(iii) If λ be any other GT on X with $\mu \subseteqq \lambda$, then every $\hat{\lambda}$-open set is $\hat{\mu}$-open.

Remark 2.2. Let μ be a GT on a topological space (X,τ). Then the collection of all $\hat{\mu}$-open sets forms a GT on X.

Proof. Clearly $\varnothing$ is a $\hat{\mu}$-open set. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of $\hat{\mu}$-open sets. Then there exist open sets U_α such that $U_\alpha \subseteqq A_\alpha \subseteqq c_\mu(U_\alpha)$ for each $\alpha \in \Lambda$. Thus $\cup\{U_\alpha : \alpha \in \Lambda\} = U$ (say) $\subseteqq \cup\{A_\alpha : \alpha \in \Lambda\} \subseteqq c_\mu(U)$, where U is open showing that the union of $\hat{\mu}$-open sets is a $\hat{\mu}$-open set. □

Example 2.1. (a) Let $X=\{a,b,c\}, \tau=\{\varnothing,\{a\},\{a,c\},X\}$ and $\mu=\{\varnothing,\{a,b\},\{a,c\},X\}$. Then μ is a GT on the topological space (X,τ). It can be checked easily that $\{a,b\}$ is a $\hat{\mu}$-open set which is not an open set.

(b) Let $X=\{a,b,c,d\}$, $\mu=\{\varnothing,\{a\},\{a,b\},\{a,b,c\}\}$ and $\tau=\{\varnothing,\{a\},\{c\},\{a,c\},\{a,b\}, \{c,d\},\{a,b,c\},\{a,c,d\},X\}$. Then μ is a GT on the topological space (X,τ). It can be easily verified that $\{a,d\}$ and $\{c,d\}$ are both $\hat{\mu}$-open but their intersection $\{d\}$ is not so.

Theorem 2.2. *Let μ be a GT on a topological space (X,τ) and A be a $\hat{\mu}$-open set such that $A \subseteqq B \subseteqq c_\mu(A)$. Then B is also a $\hat{\mu}$-open set.*

Proof. As A is $\hat{\mu}$-open, there exists an open set U such that $U \subseteqq A \subseteqq c_\mu(U)$. Thus $U \subseteqq B$. Also $c_\mu(A) \subseteqq c_\mu(U) \Rightarrow B \subseteqq c_\mu(U)$. Thus $U \subseteqq B \subseteqq c_\mu(U)$. Thus B is $\hat{\mu}$-open. □

Definition 2.2. Let μ be a GT on an ideal topological space $(X,\tau,\mathcal{I})$. A subset A of X is called $\mathcal{I}_{\hat{\mu}}$-open if there exists an open set U such that $U \setminus A \in \mathcal{I}$ and $A \setminus c_\mu(U) \in \mathcal{I}$.

If $A \in \mathcal{I}$, then A is an $\mathcal{I}_{\hat{\mu}}$-open set and also by Theorem 2.1, every $\hat{\mu}$-open set (hence every open set) is $\mathcal{I}_{\hat{\mu}}$-open for any ideal $\mathcal{I}$ on X. Also note that if we take $\mu=\tau$, then $\mathcal{I}_{\hat{\mu}}$-open set reduces to $\mathcal{I}$-semi-open set [13].

Example 2.2. (a) Let $X=\{a,b,c\}$, $\mu=\{\varnothing,\{a\},\{a,b\},\{a,c\},\{b,c\},X\}$, $\tau=\{\varnothing,\{a\},\{a,b\}, \{a,c\},X\}$ and $\mathcal{I}=\{\varnothing,\{a\},\{b\},\{a,b\}\}$. Then μ is a GT on the ideal topological space $(X,\tau,\mathcal{I})$. It can be verified that $\{b\}$ is $\mathcal{I}_{\hat{\mu}}$-open but not $\hat{\mu}$-open.

(b) Let $\mathbb{R}$ be the set of reals, $\mathbb{Q}$ be the set of rationals and $\mathbb{I}$ be the set of irrationals. Consider $\mathcal{I}=\{A \subseteqq \mathbb{R} : A \text{ is finite}\}$ and $\mu=\{\varnothing,\mathbb{I},\mathbb{R}\}$. Then μ is a GT on the ideal topological space $(\mathbb{R},\tau_u,\mathcal{I})$, where τ_u denotes the usual topology on $\mathbb{R}$. We note that for all $x \in \mathbb{Q}$, $\{x\}$ is an $\mathcal{I}_{\hat{\mu}}$-open set as $\{x\} \in \mathcal{I}$ but $\mathbb{Q}=\cup\{\{x\} : x \in \mathbb{Q}\}$ is not $\mathcal{I}_{\hat{\mu}}$-open.

(c) Let us consider $X=\{a,b,c,d\}$, $\mu=\{\varnothing,\{a,b,c\},\{b,c,d\},\{a,c,d\},X\}$, $\tau=\{\varnothing,\{a\}, \{a,b\},\{a,c\},\{a,b,c\},X\}$ and $\mathcal{I}=\{\varnothing,\{c\}\}$. It can be checked that $\{a,b,c\}$ and $\{b,c,d\}$ are both $\mathcal{I}_{\hat{\mu}}$-open but their intersection $\{b,c\}$ is not so.

Theorem 2.3. *Let μ be a GT on an ideal topological space $(X,\tau,\mathcal{I})$ where $\mathcal{I}$ is not countably additive. Then $\mathcal{I}$ is a minimal ideal on X i.e., $\mathcal{I}=\{\varnothing\}$ if and only if the concept of $\hat{\mu}$-openness and $\mathcal{I}_{\hat{\mu}}$-openness are the same.*

Proof. Suppose that $\mathcal{I}=\{\varnothing\}$. It is sufficient to show that whenever A is an $\mathcal{I}_{\hat{\mu}}$-open set it is $\hat{\mu}$-open. Indeed, if A is $\mathcal{I}_{\hat{\mu}}$-open, then there exists an open set U such that $U \setminus A$, $A \setminus c_\mu(U) \in \mathcal{I}=\{\varnothing\}$ and so $U \subseteqq A \subseteqq c_\mu(U)$ proving A to be a $\hat{\mu}$-open set (by Theorem 2.1).

Conversely, whenever a set is $\mathcal{I}_{\hat{\mu}}$-open then it is $\hat{\mu}$-open. Let $A \in \mathcal{I}$. Then A is an $\mathcal{I}_{\hat{\mu}}$-open set and hence by the assumption A is a $\hat{\mu}$-open set. Thus there is an open set V_1 such that $V_1 \subseteqq A \subseteqq c_\mu(V_1)$. Then $V_1 \in \mathcal{I}$ (as $V_1 \subseteqq A$ and $A \in \mathcal{I}$). Thus $A \cup V_1 \in \mathcal{I}$. By the similar argument as earlier $A \cup V_1$ is also $\hat{\mu}$-open. Thus there exists an open set V_2 such that $V_2 \subseteqq A \cup V_1 \subseteqq c_\mu(V_2)$. Similarly, there exists an open set V_3 such that $V_3 \subseteqq A \cup V_1 \cup V_2 \subseteqq c_\mu(V_3)$. Continuing in this way we can obtain an infinite sequence of open sets $V_1, V_2, V_3, \ldots$ such that $A \cup V_1 \cup V_2 \cup V_3 \cup \ldots \in \mathcal{I}$. But this is not possible as $\mathcal{I}$ is not countably additive. Thus, it must be the case that $V_1 = \varnothing$ (similarly for the other V_i's). Thus $c_\mu(V_1) = \varnothing$. Thus $A = \varnothing$ (as $V_1 \subseteqq A \subseteqq c_\mu(V_1)$). This shows that $\mathcal{I} = \varnothing$. □

Remark 2.3. Let $\mathcal{I}$ and $\mathcal{I}'$ be two ideals on a topological space (X, τ) and μ be a GT on X. If $\mathcal{I} \subseteqq \mathcal{I}'$, then every $\mathcal{I}_{\hat{\mu}}$-open set is $\mathcal{I}'_{\hat{\mu}}$-open (see Definition 2.2) and hence if A is $(\mathcal{I} \cap \mathcal{I}')_{\hat{\mu}}$-open, then it is $\mathcal{I}_{\hat{\mu}}$-open as well as $\mathcal{I}'_{\hat{\mu}}$-open.

Proposition 2.1. *Let μ be a GT on an ideal topological space $(X, \tau, \mathcal{I})$. The union of finite number of $\mathcal{I}_{\hat{\mu}}$-open sets is an $\mathcal{I}_{\hat{\mu}}$-open set.*

Proof. Let A and B be two $\mathcal{I}_{\hat{\mu}}$-open sets. Then there exist two open sets G and H such that $G \setminus A \in \mathcal{I}$, $A \setminus c_\mu(G) \in \mathcal{I}$, $H \setminus B \in \mathcal{I}$, $B \setminus c_\mu(H) \in \mathcal{I}$. Let $U = G \cup H$ and observe that $U \setminus (A \cup B) \subseteqq ((G \setminus A) \setminus B) \cup ((H \setminus B) \setminus A) \in \mathcal{I}$. Also $A \cup B \setminus c_\mu(G \cup H) \subseteqq ((A \setminus c_\mu(G) \setminus c_\mu(H)) \cup ((B \setminus c_\mu(H)) \setminus c_\mu(G)) \in \mathcal{I}$. Thus $A \cup B$ is $\mathcal{I}_{\hat{\mu}}$-open. □

Proposition 2.2. *Let μ be a GT on an ideal topological space $(X, \tau, \mathcal{I})$ and suppose that there exists a μ-dense open subset $A \in \mathcal{I}$. Then every subset B of X is $\mathcal{I}_{\hat{\mu}}$-open.*

Proof. Let B be any subset of X. Note that (as $\mathcal{I}$ is an ideal, $A \in \mathcal{I}$, $A \setminus B \subseteqq A$), we shall have $A \setminus B \in \mathcal{I}$. Put $U = A$. Then $U \setminus B = A \setminus B \in \mathcal{I}$ and $B \setminus c_\mu(U) = B \setminus c_\mu(A) = B \setminus X = \varnothing \in \mathcal{I}$. Consequently, B is $\mathcal{I}_{\hat{\mu}}$-open. □

Example 2.3. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$ be the topology on X, $\mathcal{I} = \{\varnothing, \{c\}\}$ and $\mu = \{\varnothing, \{a, b\}, \{a, c\}, X\}$. Then μ is a GT on the ideal topological space $(X, \tau, \mathcal{I})$. It is easy to check that $c_\mu(\{a\}) = X$ where $\{a\} \notin \mathcal{I}$. It can be checked that $\{b\}$ is not an $\mathcal{I}_{\hat{\mu}}$-open set.

Proposition 2.3. *Let μ be a GT on an ideal topological space $(X, \tau, \mathcal{I})$ and A be an open set such that $A \subseteqq B \subseteqq c_\mu(A)$. Then B is an $\mathcal{I}_{\hat{\mu}}$-open set.*

Proof. Obvious. □

Proposition 2.4. *Let μ be a GT on an ideal topological space $(X, \tau, \mathcal{I})$ where every non-empty open subset is μ-dense in (X, τ). Then for any subset A of X,*
(a) if A is $\mathcal{I}_{\hat{\mu}}$-open with $A \notin \mathcal{I}$, then
(i) $A \subseteqq B$ implies B is $\mathcal{I}_{\hat{\mu}}$-open.
(ii) $A \cup B$ is $\mathcal{I}_{\hat{\mu}}$-open for any subset B of X.
(b) Moreover, if the collection of open subsets of X satisfies finite intersection property and $A, B \notin \mathcal{I}$ be two $\mathcal{I}_{\hat{\mu}}$-open sets, then $A \cap B$ is also an $\mathcal{I}_{\hat{\mu}}$-open set.

Proof. (a)(i) Suppose that A is $\mathcal{I}_{\hat{\mu}}$-open and $A \subseteqq B$. Then there is an open set G such that $G \setminus A \in \mathcal{I}$ and $A \setminus c_\mu(G) \in \mathcal{I}$. We first observe that $G \neq \varnothing$ for otherwise, $c_\mu(G) = \varnothing$ (and $A \in \mathcal{I}$). Since $A \subseteqq B$, we have $G \setminus B \subseteqq G \setminus A \in \mathcal{I}$ and $B \setminus c_\mu(G) = B \setminus X = \varnothing \in \mathcal{I}$. Thus B is $\mathcal{I}_{\hat{\mu}}$-open.
(ii) As $A \subseteqq A \cup B$, (ii) follows directly from (i).
(b) Let A and B be two $\mathcal{I}_{\hat{\mu}}$-open sets. If $A \cap B = \varnothing$, then the proof is trivial. We assume therefore that $A \cap B \neq \varnothing$. By assumption there exist two open sets G and H such that $G \setminus A \in \mathcal{I}$, $A \setminus c_\mu(G) \in \mathcal{I}$, $H \setminus B \in \mathcal{I}$, $B \setminus c_\mu(H) \in \mathcal{I}$. Consider the open set $G \cap H$ which is non-empty. Since $G \cap H \setminus (A \cap B) = ((G \setminus A) \cap H) \cup ((H \setminus B) \cap G) \in \mathcal{I}$, $(A \cap B) \setminus c_\mu(G \cap H) = (A \cap B) \setminus X = \varnothing \in \mathcal{I}$, thus $A \cap B$ is $\mathcal{I}_{\hat{\mu}}$-open. □

Example 2.4. Let $X = \{a, b, c\}$, $\mathcal{I} = \{\varnothing, \{b\}\}$, $\tau = \{\varnothing, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mu = \{\varnothing, \{a, c\}, X\}$. Then μ is a GT on the ideal topological space (X, τ) such that every non-empty open set is μ-dense in (X, τ). It can be checked that $\{b\}$ is an $\mathcal{I}_{\hat{\mu}}$-open set but $\{b, c\}$ is not so.

Proposition 2.5. *Let μ be a QT on an ideal topological space $(X, \tau, \mathcal{I})$ with $\tau \subseteqq \mu$ and every non-empty open subset is μ-dense in (X, τ). A subset A which is not μ-dense is $\mathcal{I}_{\hat{\mu}}$-open if and only if $c_\mu(A)$ is $\mathcal{I}_{\hat{\mu}}$-open.*

Proof. Let A be $\mathcal{I}_{\hat{\mu}}$-open. Then as $A \subseteqq c_\mu(A)$, by Proposition 2.4(i), $c_\mu(A)$ is also $\mathcal{I}_{\hat{\mu}}$-open (if $A \notin \mathcal{I}$). For $A \in \mathcal{I}$, we proceed as follows: As A is $\mathcal{I}_{\hat{\mu}}$-open, there exists an open set U such that $U \setminus A$ and $A \setminus c_\mu(U)$ are both in $\mathcal{I}$ which implies that $(U \setminus A) \cup A = U \cup A \in \mathcal{I}$. Thus $U \setminus c_\mu(A) \in \mathcal{I}$ (as $U \setminus c_\mu(A) \subseteqq U \subseteqq U \cup A$). Also, $c_\mu(A) \setminus c_\mu(U) = c_\mu(A) \setminus X = \varnothing \in \mathcal{I}$.

Conversely, suppose that $c_\mu(A)$ is $\mathcal{I}_{\hat{\mu}}$-open. Then there exists an open set G such that $G \setminus c_\mu(A) \in \mathcal{I}$ and $c_\mu(A) \setminus c_\mu(G) \in \mathcal{I}$. If $G = \varnothing$, then $c_\mu(A) \setminus c_\mu(G) = c_\mu(A) \in \mathcal{I} \Rightarrow A \in \mathcal{I}$. Thus A is $\mathcal{I}_{\hat{\mu}}$-open. If G is non-empty, consider the μ-open set $H = G \setminus c_\mu(A) = G \cap (X \setminus c_\mu(A)) \in \mathcal{I}$. Again, $H \setminus A = G \cap (X \setminus c_\mu(A)) \cap (X \setminus A) \subseteqq G \cap (X \setminus c_\mu(A)) \in \mathcal{I}$. Thus $H \setminus A \in \mathcal{I}$ and $A \setminus c_\mu(H) = A \setminus c_\mu(G \cap (X \setminus c_\mu(A))) = A \setminus X = \varnothing$. This shows that A is $\mathcal{I}_{\hat{\mu}}$-open. □

Theorem 2.4. *Let μ be a GT on an ideal topological space $(X, \tau, \mathcal{I})$. Then $X \setminus A$ is $\mathcal{I}_{\hat{\mu}}$-open if and only if there exists a closed set F such that $i_\mu(F) \setminus A \in \mathcal{I}$ and $A \setminus F \in \mathcal{I}$.*

Proof. First suppose that $X \setminus A$ is $\mathcal{I}_{\hat{\mu}}$-open. Then there exists an open set G such that $G \setminus (X \setminus A) = A \setminus (X \setminus G) \in \mathcal{I}$ and $(X \setminus A) \setminus c_\mu(G) = i_\mu(X \setminus G) \setminus A \in \mathcal{I}$. Let $F = X \setminus G$. Then F is closed and the rest follows. The converse part can be done similarly by taking $G = X \setminus F$. □

Definition 2.3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is called weakly $\mathcal{I}_{\hat{\tau}}$-open if $A = \varnothing$ or if $A \neq \varnothing$, there exists a non-empty open set U such that $U \setminus A \in \mathcal{I}$. The complement of a weakly $\mathcal{I}_{\hat{\tau}}$-open set is termed as weakly $\mathcal{I}_{\hat{\tau}}$-closed set.

It follows that for an ideal topological space $(X, \tau, \mathcal{I})$ with a GT μ on X, any $\mathcal{I}_{\hat{\mu}}$-open set (hence open set) is weakly $\mathcal{I}_{\hat{\tau}}$-open but the converse is false follows from the next example.

Example 2.5. (a) Consider $X = \{a, b, c, d\}$, $\tau = \mu = \{\varnothing, \{a, b\}, \{c, d\}, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space. It is easy to see that $\{b, c\}$ is a weakly $\mathcal{I}_{\hat{\tau}}$-open set but not $\mathcal{I}_{\hat{\mu}}$-open.

(b) Consider $X = \{a, b, c, d\}$, $\tau = \{\varnothing, \{a, b\}, \{c, d\}, X\}$ and $\mathcal{I} = \{\varnothing, \{d\}\}$. Then $\mathcal{I}$ is an ideal on the topological space (X, τ). It is easy to see that $\{a, b\}$ and $\{a, c\}$ are two weakly $\mathcal{I}_{\hat{\tau}}$-open sets but their intersection $\{a\}$ is not so.

Proposition 2.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the collection of all weakly $\mathcal{I}_{\hat{\tau}}$-open sets form a GT on X.*

Proof. $\varnothing$ is clearly a weakly $\mathcal{I}_{\hat{\tau}}$-open set. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of weakly $\mathcal{I}_{\hat{\tau}}$-open sets. Then for each $\alpha \in \Lambda$, there exists a non-empty open set U_α such that $U_\alpha \setminus A_\alpha \in \mathcal{I}$. But $U_\alpha \setminus \cup\{A_\alpha : \alpha \in \Lambda\} \subseteqq U_\alpha \setminus A_\alpha$. Thus $U_\alpha \setminus \cup\{A_\alpha : \alpha \in \Lambda\} \in \mathcal{I}$. Hence $\cup\{A_\alpha : \alpha \in \Lambda\}$ is a weakly $\mathcal{I}_{\hat{\tau}}$-open set. □

Theorem 2.5. *Let $(X, \tau, \mathcal{I})$ an ideal topological space. Then a non-empty subset A of X is weakly $\mathcal{I}_{\hat{\tau}}$-open if and only if there exist a non-empty open set U and a set C in $\mathcal{I}$ such that $U \setminus C \subseteqq A$.*

Proof. Let A be a non-empty weakly $\mathcal{I}_{\hat{\tau}}$-open subset of X. Then there exists a non-empty open set such that $U \setminus A \in \mathcal{I}$. Let $C = U \setminus A = U \cap (X \setminus A)$. Then $U \setminus C \subseteqq A$.

Conversely, let there exist an open set U and C in $\mathcal{I}$ such that $U \setminus C \subseteqq A$. Then $U \setminus A \subseteqq U \cap C \in \mathcal{I}$ (as $C \in \mathcal{I}$). Thus $U \setminus A \in \mathcal{I}$. □

Theorem 2.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then if a subset A of X is weakly $\mathcal{I}_{\hat{\tau}}$-closed, then $A \subseteqq K \cup B$ for some closed set K of X and $B \in \mathcal{I}$.*

Proof. Let A be a weakly $\mathcal{I}_{\hat{\tau}}$-closed set. Then $X \setminus A$ is a weakly $\mathcal{I}_{\hat{\tau}}$-open set. If $X \setminus A = \varnothing$, then $A = X$. Thus $A = X \cup \varnothing$. If $A \neq X$, then there is a non-empty open set U and $B \in \mathcal{I}$ such that $U \setminus B \subseteqq X \setminus A$. So $A \subseteqq X \setminus (U \setminus B) = (X \setminus U) \cup B = K \cup B$ where $K = X \setminus U$ which is a closed set and $B \in \mathcal{I}$. □

Example 2.6. Let $X = \{a, b, c, d\}, \mathcal{I} = \{\varnothing, \{d\}\}$ and $\tau = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$. Then $\{a, d\} \subseteqq X \cup \{d\}$, where X is μ-closed and $\{d\} \in \mathcal{I}$ but $\{a, d\}$ is not a weakly $\mathcal{I}_{\hat{\tau}}$-closed set.

Remark 2.4. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\varnothing \neq A \subsetneqq B$ and A is a weakly $\mathcal{I}_{\hat{\tau}}$-open set, then so is B (by Definition 2.3). Thus if A is a weakly $\mathcal{I}_{\hat{\tau}}$-open set so is $A \cup B$, for any subset B of X. In particular, $cl(A)$ is a weakly $\mathcal{I}_{\hat{\tau}}$-open set if A is so but the converse is not true as is seen from the following example.

Example 2.7. Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, \{a, c\}, \{b, d\}, X\}$ and $\mathcal{I} = \{\varnothing, \{b\}\}$. It is easy to verify that $\{a\}$ is not a weakly $\mathcal{I}_{\hat{\tau}}$-open set though $cl(\{a\})$ is weakly $\mathcal{I}_{\hat{\tau}}$-open.

Theorem 2.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $\{a\} \in \tau \cap \mathcal{I}$ for some $a \in X$. Then every subset of X is weakly $\mathcal{I}_{\hat{\tau}}$-open.*

Proof. Suppose $\{b\} \subseteqq X$. Then either $\{a\} \setminus \{b\} = \varnothing \in \mathcal{I}$ (if $b = a$) or $\{a\} \setminus \{b\} = \{a\} \in \mathcal{I}$ (if $a \neq b$), where $\{a\} \in \tau$. Thus $\{b\}$ is weakly $\mathcal{I}_{\hat{\tau}}$-open. Thus by Proposition 2.6, any subset of X is weakly $\mathcal{I}_{\hat{\tau}}$-open. □

Theorem 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space $(X, \tau, \mathcal{I})$. Then the collection of weakly $\mathcal{I}_{\hat{\tau}}$-open sets form a topology on X.*

Proof. Due to Proposition 2.6, we have only to show that X is weakly $\mathcal{I}_{\hat{\tau}}$-open and that the intersection of two weakly $\mathcal{I}_{\hat{\tau}}$-open sets is so. Since $X \in \tau$, X is weakly $\mathcal{I}_{\hat{\tau}}$-open. Let A and B be two weakly $\mathcal{I}_{\hat{\tau}}$-open sets. Then there exist non-empty open sets U and V such that $U \setminus A \in \mathcal{I}$ and $V \setminus B \in \mathcal{I}$. Then $(U \cap V) \setminus (A \cap B) = [(U \setminus A) \cap V] \cup [U \cap (V \setminus B)] \in \mathcal{I}$. Thus $A \cap B$ is a weakly $\mathcal{I}_{\hat{\tau}}$-open set. □

Theorem 2.9. *Let $(X, \tau, \mathcal{I})$ an ideal topological space such that the open sets of X satisfies finite intersection property, where $\mathcal{I} \neq \{\varnothing\}$. Let A be a subset of X such that $cl(A) \neq X$. Then A is weakly $\mathcal{I}_{\hat{\tau}}$-open if and only if $cl(A)$ is so.*

Proof. We first observe that if $A(\neq \varnothing)$ is weakly $\mathcal{I}_{\hat{\tau}}$-open and $A \subseteqq B$, then B is weakly $\mathcal{I}_{\hat{\tau}}$-open and thus $cl(A)$ is weakly $\mathcal{I}_{\hat{\tau}}$-open (as $A \subseteqq cl(A)$).

Conversely, suppose that $cl(A)$ is a weakly $\mathcal{I}_{\hat{\tau}}$-open set. If $cl(A) = \varnothing$, then $A = \varnothing$. Thus A is a weakly $\mathcal{I}_{\hat{\tau}}$-open set. If $cl(A) \neq \varnothing$, then $U \setminus cl(A) \in \mathcal{I}$ for some non-empty open set U. Let $V = U \setminus cl(A)$. Then $V \setminus A = U \setminus cl(A) \in \mathcal{I}$. It is now sufficient to show that $V \neq \varnothing$. We note that $X \setminus cl(A)$ and U are non-empty open sets. Thus $(X \setminus cl(A)) \cap U = V \neq \varnothing$. Thus A is weakly $\mathcal{I}_{\hat{\tau}}$-open. □

Example 2.8. Let $X = \{a, b\}$, $\tau = \{\varnothing, X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Put $A = \{a\}$. Then A is not a weakly $\mathcal{I}_{\hat{\tau}}$-open subset of X such that $cl(A) = X$. However, $cl(A)$ is weakly $\mathcal{I}_{\hat{\tau}}$-open.

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