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Weighted composition operators from Bloch-type into Bers-type spaces

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ABSTRACT. In this paper we consider the weighted composition operator uC_φ from Bloch-type space B^α into Bers-type space H_β^∞, in three cases, $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$. We give the necessary and sufficient conditions for boundedness and compactness of the above operator.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. An analytic function f on $\mathbb{D}$ is said to belong to the Bloch-type space $B^\alpha(0 < \alpha < \infty)$, if

$$||f||_\alpha = \sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha |f^{'}(z)| < \infty.$$

The expression $\|.\|_\alpha$ defines a seminorm, while the natural norm is given by $\|f\| = |f(0)| + \|f\|_\alpha$. This norm makes Bloch-type space B^α into a Banach space.

Let u be an analytic function on $\mathbb{D}$ and φ a nonconstant analytic self-map of $\mathbb{D}$. We define a linear operator uC_φ on $H(\mathbb{D})$ by

$$uC_\varphi f = u(fo\varphi).$$

This operator is called weighted composition operator. The operator uC_φ can be regarded as a generalization of a multiplication operator and a composition operator. In case $u \equiv 1$, uC_φ reduses to the composition operator C_φ and when $\varphi(z) = z$, uC_φ will be the multiplication operator M_u. For general back ground on composition operators, we refer [2, 9] and references therein.

Boundedness and compactness of composition operator on the Bers-type space were described by He Weixiang and Jiang Lijian in [12]. Zengjuan Lou in [6] characterized the boundedness and compactness of the composition operators between Bloch-type spaces. Several characterizations for the boundedness and compactness of the weighted composition operators from Bloch-type spaces to nth weighted-type spaces, also, some estimates for their essential norms are given by Li, Abbasi and Vaezi in [5]. Vaezi and Houdfar in [11], characterized the boundedness and compactness of composition and weighted composition operators from Bloch-type to Besov-type spaces.

The weighted composition operators acting on various spaces of analytic functions has been studied by many authors. For example, uC_φ was studied by Ohno, Stroethoff and Zhao in [7], where the boundedness and compactness of uC_φ between Bloch-type spaces are investigated. Collona and Li in [1] characterized the bounded and the compact weighted composition operators from the Besove space into Bloch space and Kumar in [4] characterized the boundedness and compactness of uC_φ between Drichlet-type spaces. In

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[8], adjoints of rationally induced weighted composition operators on the Hardy, Bergman and Dirichlet spaces was studied by Salaryan and Vaezi. Boundedness and compactness of this operator on weak vector-valued Bergman spaces and Hardy spaces are investigated by Hassanlou, Vaezi and Wang in [3]. In this paper, we study the operator uC_φ from the Bloch-type space into the Bers-type space. We characterize boundedness and compactness of $uC_\varphi : B^\alpha \to H^\infty_\beta$ in three case, for $\alpha > 1$ in section 2, for $\alpha = 1$ in section 3 and for $0 < \alpha < 1$ in section 4. We need the following lemma (see [10]).

Lemma 1.1. *Let $f \in B^\alpha$, $0 < \alpha < \infty$. Then*

$$|f(z)| \leq C \begin{cases} ||f||_\alpha & \alpha \in (0,1), \\ ||f||_\alpha Ln\frac{e}{1-|z|^2} & \alpha = 1, \\ ||f||_\alpha \frac{1}{(1-|z|^2)^{\alpha-1}} & \alpha > 1. \end{cases}$$

where C is a constant.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Boundedness and compactness of $uC_\varphi : B^\alpha \to H^\infty_\beta$ for $\alpha > 1$

In this section, we characterize the boundedness and compactness of $uC_\varphi : B^\alpha \to H^\infty_\beta$, when $\alpha > 1$.

Theorem 2.1. *Let u be an analytic function on $\mathbb{D}$, φ an analytic self-map of $\mathbb{D}$, α and β positive real numbers and $\alpha > 1$. Then $uC_\varphi : B^\alpha \to H^\infty_\beta$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty.$$

Proof. First we obtain sufficiency. For a function $f \in B^\alpha$, we have

$$\begin{aligned} \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|uC_\varphi f(z)| &= \sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^\beta|f(\varphi(z))| \\ &\leq \sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^\beta\frac{C||f||_\alpha}{(1-|\varphi(z)|^2)^{\alpha-1}} \\ &= C\sup_{z\in\mathbb{D}}\frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}||f||_\alpha \\ &= C||f||_\alpha. \end{aligned}$$

In the above inequality we use the Lemma 1.1 for $\alpha > 1$. Thus uC_φ maps B^α boundedly into H^∞_β.

Now, suppose that $uC_\varphi : B^\alpha \to H^\infty_\beta$ is bounded. For fixed $z_0 \in \mathbb{D}$, consider the function f_0 defined by

$$f_0(z) = \frac{1}{(1-z\overline{\varphi(z_0)})^{\alpha-1}}, \tag{2.1}$$

for $z \in \mathbb{D}$.

It is easy to check that $f_0 \in B^\alpha$ and then

$$\begin{aligned} |u(z)|(1-|z|^2)^\beta\frac{1}{|1-\varphi(z)\overline{\varphi(z_0)}|^{\alpha-1}} &= |u(z)|(1-|z|^2)^\beta|f_0(\varphi(z))| \\ &= (1-|z|^2)^\beta|uC_\varphi f_0(z)| \\ &\leq ||uC_\varphi f_0||_\beta \leq C||f_0||_\alpha < \infty. \end{aligned}$$

So for $z_0 \in \mathbb{D}$,

$$\frac{|u(z_0)|(1-|z_0|^2)^\beta}{(1-|\varphi(z_0)|^2)^{\alpha-1}} < \infty.$$

Since z_0 is arbitrary, hence

$$\sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty.$$

□

Theorem 2.2. *Let u be an analytic function on $\mathbb{D}$, φ an analytic self-map of $\mathbb{D}$, α and β positive real numbers and $\alpha > 1$. Then $uC_\varphi : B^\alpha \to H_\beta^\infty$ is compact if and only if $u \in H_\beta^\infty$ and*

$$\lim_{|\varphi(z)|\to 1^-} \frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0.$$

Proof. Suppose that

$$\lim_{|\varphi(z)|\to 1} \frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0. \tag{2.2}$$

By the assumption, for every $\varepsilon > 0$, There exist a $\delta \in (0,1)$, such that

$$\frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} < \varepsilon, \tag{2.3}$$

whenever $\delta < |\varphi(z)| < 1$. To prove the compactness of uC_φ, assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in B^α such that $||f_k||_\alpha \le 1$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. We show that $||uC_\varphi f_k||_\beta \to 0$.

if $|\varphi(z)| > \delta$, then by (2.3),

$$\begin{aligned} ||uC_\varphi f_k||_\beta &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |uC_\varphi f_k(z)| \\ &= \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} ||f_k||_\alpha \\ &< \varepsilon ||f_k||_\alpha \le \varepsilon. \end{aligned}$$

Now consider $|\varphi(z)| \le \delta$. We have

$$||uC_\varphi f_k||_\beta = \sup_{z\in\mathbb{D}} |u(z)|(1-|z|^2)^\beta |f_k(\varphi(z))|.$$

Since $u \in H_\beta^\infty$, so $||uC_\varphi f_k||_\beta \to 0$.
Conversely, note that $u = uC_\varphi 1 \in H_\beta^\infty$. Now we are going to prove that (2.2) is also necessary condition for compactness of uC_φ. Suppose that $(z_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$.
Consider the functions f_k defined by

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-z\overline{\varphi(z_k)})^\alpha} \qquad \text{for} \quad z \in \mathbb{D}.$$

Clearly $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, and

$$\begin{aligned}|f_k^{'}(z)| &= \frac{\alpha(1-|\varphi(z_k)|^2)|\overline{\varphi(z_k)}|}{|1-z\overline{\varphi(z_k)}|^{\alpha+1}}\\ &\leq \frac{\alpha(1-|\varphi(z_k)|^2)}{(1-|z||\varphi(z_k)|)^{\alpha+1}}\\ &= \frac{\alpha(1-|\varphi(z_k)|^2)}{(1-|z||\varphi(z_k)|)^{\alpha}(1-|z||\varphi(z_k)|)}\\ &\leq \frac{\alpha(1+|\varphi(z_k)|)}{(1-|z|)^{\alpha}}\\ &\leq \frac{2\alpha}{(1-|z|)^{\alpha}} < \infty.\end{aligned}$$

So,

$$\begin{aligned}||f_k||_\alpha &= \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}|f_k^{'}(z)|\\ &\leq \sup_{z\in\mathbb{D}} \frac{2\alpha(1-|z|^2)^{\alpha}}{(1-|z|)^{\alpha}}\\ &\leq 2\alpha\cdot 2^{\alpha}\\ &= \alpha 2^{\alpha+1} < \infty.\end{aligned}$$

Hence, $(||f_k||_\alpha)_{k\in\mathbb{N}}$ is uniformly bounded.

Note that

$$f_k(\varphi(z_k)) = \frac{1-|\varphi(z_k)|^2}{(1-\varphi(z_k)\overline{\varphi(z_k)})^{\alpha}} = \frac{1}{(1-|\varphi(z_k)|^2)^{\alpha-1}}.$$

Thus

$$\begin{aligned}\frac{|u(z_k)|(1-|z_k|^2)^{\beta}}{(1-|\varphi(z_k)|^2)^{\alpha-1}} &= (1-|z_k|^2)^{\beta}|uC_\varphi f_k(z_k)|\\ &\leq ||uC_\varphi f_k||_\beta.\end{aligned}$$

Since $uC_\varphi : B^\alpha \to H_\beta^\infty$ is compact, it follow from the proof of the Weak Convergence Theorem in [9] that $||uC_\varphi f_k||_\beta \to 0$.Therefore

$$\frac{|u(z_k)|(1-|z_k|^2)^{\beta}}{(1-|\varphi(z_k)|^2)^{\alpha-1}} \to 0$$

as $k \to \infty$. So if uC_φ is compact, then (2.2) holds. $\square$

3. Boundedness and compactness of $uC_\varphi : B^\alpha \to H_\beta^\infty$ for $\alpha = 1$

In this section, we characterize the boundedness and compactness of $uC_\varphi : B^\alpha \to H_\beta^\infty$, when $\alpha = 1$.

Theorem 3.3. *Let u be an analytic function on $\mathbb{D}$, φ an analytic self-map of $\mathbb{D}$, $\alpha = 1$ and β a positive real number. Then $uC_\varphi : B^\alpha \to H_\beta^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^{\beta}Ln\frac{e}{1-|\varphi(z)|^2} < \infty.$$

Proof. First we obtain sufficiency. For a function $f \in B^\alpha$, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|uC_\varphi f(z)| &= \sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^\beta|f(\varphi(z))|\\ &\le \sup_{z\in\mathbb{D}}|u(z)(1-|z|^2)^\beta C||f||_\alpha Ln\frac{e}{1-|\varphi(z)|^2}\\ &= C||f||_\alpha.\end{aligned}$$

In the above inequality we use the Lemma 1.1 for $\alpha = 1$. Thus uC_φ maps B^α boundedly into H^∞_β.
Now, suppose that $uC_\varphi : B^\alpha \to H^\infty_\beta$ is bounded. For fixed $z_0 \in \mathbb{D}$, consider the function f_0 defined by

$$f_0(z) = Ln\frac{e}{1-z\overline{\varphi(z_0)}}$$

for $z \in \mathbb{D}$.
It is easy to check that $f_0 \in B$, and then

$$\begin{aligned}|u(z)|(1-|z|^2)^\beta\left|Ln\frac{e}{1-\varphi(z)\overline{\varphi(z_0)}}\right| &= |u(z)|(1-|z|^2)^\beta|f_0(\varphi(z))|\\ &= (1-|z|^2)^\beta|uC_\varphi f_0(z)|\\ &\le ||uC_\varphi f_0||_\beta\\ &\le C||f_0||_\alpha < \infty.\end{aligned}$$

So for $z_0 \in \mathbb{D}$, we have

$$|u(z_0)|(1-|z_0|^2)^\beta Ln\frac{e}{1-|\varphi(z_0)|^2} < \infty.$$

Since z_0 is arbitrary, hence

$$\sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^\beta Ln\frac{e}{1-|\varphi(z)|^2} < \infty.$$

□

Theorem 3.4. *Let u be an analytic function on $\mathbb{D}$, φ an analytic self-map of $\mathbb{D}$, $\alpha = 1$ and β a positive real number. Then $uC_\varphi : B^\alpha \to H^\infty_\beta$ is compact if and only if $u \in H^\infty_\beta$ and*

$$\lim_{|\varphi(z)|\to 1}|u(z)|(1-|z|^2)^\beta Ln\frac{e}{1-|\varphi(z)|^2} = 0.$$

Proof. Suppose that

$$\lim_{|\varphi(z)|\to 1}|u(z)|(1-|z|^2)^\beta Ln\frac{e}{1-|\varphi(z)|^2} = 0. \tag{3.4}$$

By the assumption, for every $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that

$$|u(z)|(1-|z|^2)^\beta Ln\frac{e}{1-|\varphi(z)|^2} < \varepsilon \tag{3.5}$$

whenever $\delta < |\varphi(z)| < 1$. To prove the compactness of uC_φ, assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in B^α such that $||f_k||_\alpha \le 1$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. We show that $||uC_\varphi f_k||_\beta \to 0$.
if $|\varphi(z)| > \delta$, then by (3.5),

$$\begin{aligned}||uC_\varphi f_k||_\beta &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|uC_\varphi f_k(z)|\\ &= \sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^\beta||f_k||_\alpha Ln\frac{e}{1-|\varphi(z)|^2}\\ &< \varepsilon||f_k||_\alpha \le \varepsilon.\end{aligned}$$

Now consider $|\varphi(z)| \leq \delta$. In this case we have

$$||uC_\varphi f_k||_\beta = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |uC_\varphi f_k(z)|.$$

Since $u \in H_\beta^\infty$, therefore $||uC_\varphi f_k||_\beta \to 0$.
We are going to prove that (3.4) is also necessary condition for compactness of uC_φ. Suppose that $(z_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$.
Consider the functions f_k defined by

$$f_k(z) = \frac{1}{Ln\frac{e}{1-|\varphi(z_k)|^2}}\left(Ln\frac{e}{1-z\overline{\varphi(z_k)}}\right)^2 \qquad \text{for} \quad z \in \mathbb{D}.$$

Clearly $f_k \to 0$ uniformly on compact subset of $\mathbb{D}$, and

$$|f_k^{'}(z)| = \frac{2}{Ln\frac{e}{1-|\varphi(z_n)|^2}}\frac{|\overline{\varphi(z_n)}|}{|1-z\overline{\varphi(z_n)}|}\cdot Ln\frac{e}{|1-z\overline{\varphi(z_n)}} < \infty.$$

Therefore, $f_k \in B$ and $(||f_k||_\alpha)_{k\in\mathbb{N}}$ is uniformly bounded.
Not that

$$f_k(\varphi(z_n)) = Ln\frac{e}{1-|\varphi(z_n)|^2}.$$

Thus

$$\begin{aligned} |u(z_k)|(1-|z_k|^2)^\beta Ln\frac{e}{1-|\varphi(z_n)|^2} &= (1-|z_k|^2)^\beta |uC_\varphi f_k(z_k)| \\ &\leq ||uC_\varphi f_k||_\beta. \end{aligned}$$

Since $uC_\varphi : B \to H_\beta^\infty$ is compact, it follows from the proof of the Weak Convergence Theorem in [9] that, $||uC_\varphi f_k||_\beta \to 0$. Therefore

$$|u(z_n)|(1-|z_k|^2)^\beta Ln\frac{e}{1-|\varphi(z_n)|^2} \to 0$$

as $k \to \infty$. So, if uC_φ is compact, then (3.4) holds. □

4. Boundedness and compactness of $uC_\varphi : B^\alpha \to H_\beta^\infty$ for $0 < \alpha < 1$

In this section, we characterize the boundedness and compactness of $uC_\varphi : B^\alpha \to H_\beta^\infty$ when $0 < \alpha < 1$.

Theorem 4.5. *Let u be an analytic function on $\mathbb{D}$, φ analytic self-map of D, β a positive real number and $0 < \alpha < 1$. Then $uC_\varphi : B^\alpha \to H_\beta^\infty$ is bounded if and only if $u \in H_\beta^\infty$.*

Proof. Suppose that $u \in H_\beta^\infty$. For a function $f \in B^\alpha$, we have

$$\begin{aligned} \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |uC_\varphi f(z)| &= \sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^\beta |f(\varphi(z))| \\ &\leq \sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^\beta C||f||_\alpha \\ &= C||f||_\alpha. \end{aligned}$$

In the above inequality we use the Lemma 1.1 for $\alpha \in (0,1)$. Thus uC_φ maps B^α boundedly into H_β^∞.
Now, suppose that uC_φ maps B^α boundedly into H_β^∞. Then $u = uC_\varphi 1 \in H_\beta^\infty$. This completes the proof of theorem. □

Theorem 4.6. *Let u be an analytic function on $\mathbb{D}$, φ analytic self-map of $\mathbb{D}$, β a positive number and $0 < \alpha < 1$. Then $uC_\varphi : B^\alpha \to H_\beta^\infty$ is compact if and only if $u \in H_\beta^\infty$.*

Before proving the above theorem, we need the following lemma, see [7].

Lemma 4.2. *Let $0 < \alpha < 1$ and T be a bounded linear operator from B^α into normed linear space Y. Then T is compact if and only if $||Tf_k||_Y \to 0$ whenever $(f_k)_{k\in\mathbb{N}}$ is a norm-bounded sequence in B^α that converges to 0 uniformly on $\bar{\mathbb{D}}$.*

Proof of Theorem 4.6. We have already shown that $u \in H^\infty_\beta$ is necessary for the weighted composition operator $uC_\varphi : B^\alpha \to H^\infty_\beta$ to be bounded. Suppose that $f_k \in B^\alpha$ with $||f_k||_\alpha \leq 1$ for all $k = 1, 2, \ldots$ and $f_k \to 0$ uniformly on $\overline{\mathbb{D}}$.

Then

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|uC_\varphi f_k(z)| &= \sup_{z\in\mathbb{D}}|u(z)|(1-|z|^2)^\beta|f_k(\varphi(z))| \\ &\leq ||u||_\beta \sup_{|w|\leq 1}|f_k(w)| \to 0\end{aligned}$$

as $k \to \infty$. So, $||uC_\varphi f_k||_\beta \to 0$ as $k \to \infty$. It follows from lemma 4.2 that the operator uC_φ maps B^α compactly into H^∞_β.

Now, suppose that uC_φ maps B^α compactly into H^∞_β. Then $u = uC_\varphi 1 \in H^\infty_\beta$. This completes the proof of theorem (4.6). □

5. Conclusions

Boundedness and compactness of (weighted) composition operators on different spaces of analytic functions are studied by many authors. This concepts for composition operators on Bers-type and between Bloch-type spaces were studied in [12] and [6]. We have studied the boundedness and compactness of weighted composition operators from Bloch-type to nth weighted-type spaces in [5]. Also, in [11], by using the hyperbolic analytic Besov-type classes, we investigate the boundedness and compactness of composition and weighted composition operators from Bloch-type to Besov-type spaces.

In this paper we give the necessary and sufficient conditions for boundedness and compactness of weighted composition operators from Bloch-type space B^α into Berse-type spaces in three cases, $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$. So, the methods and results of this paper are essentially different from [11].

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