Application of subordination for estimating the Hankel determinant for subclass of univalent functions

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ABSTRACT. By using subordination structure, a new subclass of convex functions is introduced. The estimate of the third Hankel determinants is also investigated.

1. INTRODUCTION

The coefficient estimate of univalent functions is one of the most important subjects which has been addressed in many recent articles. See [5], [6] and [7]. The well-known Fekete-Szegö inequality and the bounds of second and third Hankel determinants are investigated. See [1], [3], [4], [8] and [9].

Let $\mathcal{A}$ denote the class of univalent functions of the form
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \] (1.1)
which are analytic in the open unit disk
\[ \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}. \]
Further, by $S$ we shall denote the class of all functions in $\mathcal{A}$ which are normalized univalent in $\mathbb{U}$.

Let $f$ and $g$ be analytic in $\mathbb{U}$. Then $f$ is said to be subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function $\omega$ analytic in $\mathbb{U}$, with $\omega(0) = 0$, $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$.

If $g$ is univalent, the $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Suppose that $\mathcal{P}$ denote the class of analytic functions $p$ of the type
\[ p(z) = 1 + \sum_{n=2}^{\infty} a_n z^n \] (1.2)
such that $Re p(z) > 0$. Also
\[ SL^C = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} \right|^2 - 1 < 1, \quad z \in \mathbb{U} \right\}. \]

Thus the values of $1 + \frac{zf''(z)}{f'(z)}$ where $f \in SL^C$ lies in the region which bounded by the right half of the lemniscate of Bernoulli given by $|\omega^2 - 1| < 1$.
It is easy to see that \( f \in SL^C \) if it satisfies the condition
\[
1 + \frac{zf''(z)}{f'(z)} < \sqrt{1 + z}, \quad z \in \mathbb{U}. \tag{1.3}
\]

The determinants of the \( q^{th} \) Hankel matrix denoted by
\[
H_q(n) = \begin{vmatrix}
  a_n & a_{n+1} & \ldots & a_{n+q-1} \\
  a_{n+1} & a_{n+2} & \ldots & a_{n+q-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+q-1} & a_{n+q-2} & \ldots & a_{n+2q-2}
\end{vmatrix} \quad q \in \mathbb{N} \setminus 1, \quad n \in \mathbb{N}.
\]
and is called the \( q^{th} \) Hankel determinant. In the particular cases
\( q = 2, n = 1, a_1 = 1 \) and \( q = 2, n = 2 \),
the Hankel determinant simplifies respectively to
\( H_2(1) = |a_3 - a_2^2| \) and \( H_2(2) = |a_2a_4 - a_3^2| \).
In the case \( q = 3 \) and \( n = 1 \),
\[
H_3(1) = \begin{vmatrix}
  a_1 & a_2 & a_3 \\
  a_2 & a_3 & a_4 \\
  a_3 & a_4 & a_5
\end{vmatrix}
\]
\( H_2(2) \) and \( H_3(1) \) are respectively called second and third Hankel determinant
For \( f \in S, a_1 = 1 \) so that,
\[
H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_1a_4 - a_2a_3) + a_5(a_1a_3 - a_2^2).
\]
and by using the triangle inequality, we have
\[
|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.
\]

In this paper, we introduce a new subclass of convex functions associated with differential subordination and extremum point.

**Lemma 1.1.** [7] If \( p \in \mathcal{P} \) be of the form 1.2, Then
\[
2p_2 = p_1^2 + x(4 - p_1^2)
\]
for some \( x, |x| \leq 1, \) and
\[
4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z
\]
for some \( z, |z| \leq 1. \)

2. Main Results

In this section we obtain some coefficient estimates and bounds of third Hankel determinant.

**Lemma 2.2.** If \( p \in \mathcal{P} \), then \( |p_k| \leq 2 \) for each \( k \). [2]

**Theorem 2.1.** Let \( f \) of the form 1.1 be in the class \( SL^C \), Then
\[
|a_3 - a_2^2| \leq \frac{5}{24}.
\]

**Proof.** If \( f \in SL^C \), then it follows from 1.3 that
\[
1 + \frac{zf''(z)}{f'(z)} < \phi(z), \tag{2.4}
\]
where $\phi(z) = \sqrt{1 + z}$. Define a function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \ldots.$$ 

It is clear that $p \in \mathcal{P}$. This implies that

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1}.$$ 

From 2.4, we have

$$1 + \frac{zf''(z)}{f'(z)} = \phi(\omega(z)).$$

with

$$\phi(\omega(z)) = \left(\frac{2p(z)}{p(z) + 1}\right)^{\frac{1}{2}}.$$

Now

$$\left(\frac{2p(z)}{p(z) + 1}\right)^{\frac{1}{2}} = 1 + \frac{1}{4}p_1z + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right)z^2 + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3\right)z^3 + \ldots.$$ 

Similarly,

$$\frac{1 + zf''(z)}{f'(z)} = 1 + 2a_2z + (6a_3 - 4a_2^2)z^2 + (12a_4 - 18a_2a_3 + 8a_2^3)z^3 + \ldots.$$ 

Now, equating the coefficients we get

$$2a_2 = \frac{1}{4}p_1,$$

then

$$a_2 = \frac{1}{8}p_1.$$ 

(2.5)

and

$$6a_3 - 4a_2^2 = \frac{1}{4}p_2 - \frac{5}{32}p_1^2,$$

$$a_3 = \frac{1}{24}p_2 - \frac{1}{64}p_1^2.$$ 

(2.6)

and

$$12a_4 - 18a_2a_3 + 8a_2^3 = \frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3,$$

$$a_4 = \frac{1}{48}p_3 - \frac{7}{384}p_1p_2 + \frac{13}{3072}p_1^3.$$ 

(2.7)

Applying Lemma 2.2 for the coefficients $p_1$ and $p_2$, we get:

$$|a_3 - a_2^2| \leq \frac{5}{24}.$$ 

□

**Theorem 2.2.** Let $f$ of the form 1.1 be in the class $SL^C$, Then

$$|a_2a_4 - a_3^2| \leq \frac{1}{72}.$$
Proof. From 2.5, 2.6 and 2.7, we obtain
\[
\begin{align*}
  a_2a_4 - a_3^2 &= \frac{1}{384} \left( p_1^3 - \frac{7}{8} p_2^2 p_3 + \frac{13}{64} p_4^3 \right) - \left( \frac{1}{24} p_2 - \frac{1}{64} p_1^2 \right)^2 \\
  &= \frac{1}{384} p_1^3 - \frac{7}{3072} p_2^2 p_3 + \frac{13}{24576} p_4^3 - \frac{1}{576} p_2^2 \\
  &\quad + \frac{1}{768} p_1^2 p_2 - \frac{1}{4096} p_4^4 \\
  &= \frac{1}{24576} \left( 64p_1p_3 + 24p_2^2 p_3 - \frac{128}{3} p_2^2 + 7p_4 \right).
\end{align*}
\]

Putting the values of \( p_2 \) and \( p_3 \) from Lemma 1.1, letting \( p > 0 \) and taking \( p_2 = p = \in [0, 2] \), we get
\[
|a_2a_4 - a_3^2| = \frac{1}{24576} \left| 16p_1\left( p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 \right) \\
+ 2(4 - p_1^2)(1 - |x|^2)z + 12p_1^2 (p_1^2 + x(4 - p_1^2)) \\
- \frac{32}{3} \left( p_1^2 + x(4 - p_1^2)^2 \right) + 16p_1^2 (p_1^2 + x(4 - p_1^2)) - 6p_4^4 \right|.
\]

After a simple calculation, we get
\[
|a_2a_4 - a_3^2| = \frac{1}{24576} \left| \frac{82}{3} p^4 + \frac{116}{3} (4 - p^2) p^2 x \\
+ 32(4 - p^2)p(1 - |x|^2)z - \frac{16}{3} (4 - p^2)(p^2 + 8)x^2 \right|.
\]

Now, applying the triangle inequality and replacing \(|x|\) by \( \rho \), we obtain
\[
|a_2a_4 - a_3^2| \leq \frac{1}{73728} \left| 82p^4 + 192p(4 - p^2) + 116(4 - p^2)p^2 \rho + 16\rho^2(4 - p^2)(p^2 + 8) \right| = F(p, \rho).
\]

Differentiating with respect to \( \rho \), we have
\[
\frac{\partial F(p, \rho)}{\partial \rho} = \frac{1}{73728} \left( 116(4 - p^2)p^2 + 32\rho(4 - p^2)(p^2 + 8) \right).
\]

It is clear that \( \frac{\partial F(p, \rho)}{\partial \rho} > 0 \), which shows that \( F(p, \rho) \) is an increasing function on the closed interval \([0, 1]\). This implies that maximum occurs at \( \rho = 1 \). Therefore \( \text{Max } F(p, \rho) = F(p, 1) = G(p) \). Now
\[
G(p) = \frac{1}{73728} \left( -148p^4 + 336p^2 + 1024 \right).
\]

Therefore
\[
G'(p) = \frac{1}{73728} \left( -592p^3 + 672p \right)
\]

and
\[
G''(p) = \frac{1}{73728} \left( -1776p^2 + 672 \right) < 0
\]

for \( p = 0 \). This shows that maximum of \( G(p) \) occurs at \( p = 0 \). Hence, we obtain
\[
|a_2a_4 - a_3^2| \leq \frac{1024}{73728} = \frac{1}{72}.
\]
Theorem 2.3. Let \( f \) of the form 1.1 be in the class \( SL^C \), then

\[
|a_2a_3 - a_4| \leq \frac{1}{12}.
\]

Proof. Since

\[
a_2 = \frac{1}{8}p_1, \quad a_3 = \frac{1}{24}p_2 - \frac{1}{64}p_1^2, \quad a_4 = \frac{1}{48}p_3 - \frac{7}{384}p_1p_2 + \frac{13}{3072}p_1^3,
\]

therefore, by using Lemma 1.2 and replacing \(|x|\) by \( \rho \), we have

\[
|a_2a_3 - a_4| \leq \frac{1}{3072} \left| 39p^3 + 64(4 - p^2) + 4(4 - p^2)p\rho + 16p\rho^2(4 - p^2) \right|.
\]

Let

\[
F_1(p, \rho) = \frac{1}{3072} \left| 39p^3 + 64(4 - p^2) + 4(4 - p^2)p\rho + 16p\rho^2(4 - p^2) \right|.
\] (2.8)

We assume that the upper bound occurs at the interior of the rectangle \([0, 2] \times [0, 1]\). Differentiating 2.8 with respect to \( \rho \), we have

\[
\frac{\partial F_1(p, \rho)}{\partial \rho} = \frac{1}{3072} (4p(4 - p^2) + 64p\rho(4 - p^2)).
\]

For \( 0 < \rho < 1 \), and fixed \( p \in (0, 2) \), it is clear that \( \frac{\partial F_1(p, \rho)}{\partial \rho} < 0 \), which shows that \( F_1(p, \rho) \) is an decreasing function of \( p \), which contradicts our assumption, therefore \( Max F_1(p, \rho) = F_1(p, 0) = G_1(p) \). Now

\[
G_1(p) = \frac{1}{3072} (-4p^3 + 16p).
\]

So

\[
G_1'(p) = \frac{1}{3072} (-12p^2 + 16)
\]

and

\[
G_1''(p) = \frac{1}{3072} (-24p) < 0
\]

for \( p = 0 \). This shows that maximum of \( G_1(p) \) occurs at \( p = 0 \). Hence, we get the required result.

\[
\square
\]

Lemma 2.3. If the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) belongs to the class \( SL^C \), then

\[
|a_2| \leq \frac{1}{4}, \quad |a_3| \leq \frac{1}{12}, \quad |a_4| \leq \frac{1}{24}, \quad |a_5| \leq \frac{1}{40}.
\]

These estimations are sharp.

Theorem 2.4. Let \( f \) of the form 1.1 be in the class \( SL^C \), Then

\[
|H_3(1)| \leq \frac{49}{8640}.
\]
Proof. Since
\[ H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_1a_4 - a_2a_3) + a_5(a_1a_3 - a_2^2), \]
Now, using the triangle inequality, we obtain
\[ |H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \]
Using the fact that \( a_1 = 1 \) with the results of Theorem 2.1, Theorem 2.2, Theorem 2.3 and Lemma 2.3, we obtain
\[ |H_3(1)| \leq \frac{49}{8640}. \]
□

REFERENCES


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