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Semiprime rings with multiplicative (generalized)-derivations involving left multipliers

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ABSTRACT. Let R be a semiprime ring, I a non zero ideal of R. A mapping $F : R \longrightarrow R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation of R if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where d is any mapping on R. A map $H : R \longrightarrow R$ (not necessarily additive) is called a multiplicative left multiplier if

$$H(xy) = H(x)y, \text{ holds for all } x, y \in R.$$

The main objective of this article is to study the following situations:
(i) $F(xoy) \pm H(xoy) = 0$,
(ii) $F(xoy) \pm H[x, y] = 0$,
(iii) $F[x, y] \pm [x, H(y)] = 0$,
(iv) $F(xoy) \pm [x, H(y)] = 0$,
(v) $F(xy) \pm [x, H(y)] \in Z(R)$,
(vi) $F(xy) \pm [H(x), H(y)] \in Z(R)$,
for all x, y in some appropriate subsets of R.

1. INTRODUCTION

Let R denote an associative ring with center $Z(R)$. A ring R is called a prime ring if for any $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$ and is called a semiprime ring if $aRa = 0$ implies that $a = 0$. For any $x, y \in R$, we shall denote the commutator and anti-commutator by the symbols

$$[x, y] = xy - yx$$

and

$$(xoy) = xy + yx,$$

respectively.

An additive map $d : R \longrightarrow R$ is called a derivation of R if

$$d(xy) = d(x)y + xd(y)$$

holds for all $x, y \in R$.

An additive mapping $F : R \longrightarrow R$ associated with a derivation $d : R \longrightarrow R$ is called a generalized derivation of R if

$$F(xy) = F(x)y + xd(y),$$

holds for all $x, y \in R$.

In [6], Bresar introduced the notion of generalized derivation. Obviously, every derivation is a generalized derivation of R. Thus, generalized derivation covers both the concept of derivation and the concept of left multipliers. Let S be a non-empty subset of R.

A map $f : S \longrightarrow R$ is called a centralizing(commuting) map on S if

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$[f(x),x]\in Z(R)($ or $[f(x),x]=0)$, for all $x\in S$.

The concept of multiplicative derivations appears for the first time in the work of Daif [9] and it was motivated by the work of Martindale [18]. According to Daif [9]: A map $d:R\longrightarrow R$ is called a multiplicative derivation of R if $d\left(xy\right)=d\left(x\right)y+xd\left(y\right)$ holds for all $x,y\in R$. Further, the complete description of those maps were given by Goldmann and semrl in [13]. The notion of multiplicative derivation was extended to multiplicative generalized derivation by Daif and Tammam-El-Sayiad [11] as follows: a map $F:R\longrightarrow R$ is called a multiplicative generalized derivation if there exists a derivation d such that

$$F\left(xy\right)=F\left(x\right)y+xd\left(y\right)$$

for all $x,y\in R$.

Recently, Dhara and Ali [12] gave a definition of multiplicative(generalized)-derivation as follows: a mapping $F:R\longrightarrow R$ (not necessarily additive) is said to be multiplicative (generalized)-derivation if

$$F\left(xy\right)=F\left(x\right)y+xd\left(y\right)$$

holds for all $x,y\in R$, where d is any map on R (not necessarily a derivation nor additive). Hence the concept of multiplicative (generalized)-derivation covers the concept of multiplicative derivation. A mapping $H:R\longrightarrow R$ (not necessarily additive) is said to be a multiplicative left multiplier if

$$H\left(xy\right)=H\left(x\right)y$$

holds for all $x,y\in R$ ([19]).

Moreover, multiplicative(generalized)-derivation with $d=0$ covers the concept of multiplicative left multipliers. Many papers in literature have investigated the commutativity of prime and semiprime rings satisfying certain functional identitites involving multiplicative generalized derivations or multiplicative(generalized)-derivations ([3], [4], [5], [7], [15], [16], [17], [20], [21] and [22]).

Daif and Bell [10] proved that if a semiprime ring R admits a derivation d such that $d\left[x,y\right]\pm\left[x,y\right]=0$ holds for all x, y in a non-zero ideal I of R, then R is commutative. Hongan [14] generalized these results by taking the same situations in the center of the ring R. Asma Ali et al.[1] investigated the commutativity of a prime ring admitting a generalized derivation satisfying any one of the following identities: (i)$F\left(\left[x,y\right]\right)\pm\left[x,y\right]\in Z\left(R\right)$ (ii)$F\left(xoy\right)\pm\left(xoy\right)\pm Z\left(R\right)$ in some appropriate subset of R. Recently, Ali et al.[2] proved multiplicative(generalized)-derivation and left ideals in semiprime rings. Dedem Camci and Neset Aydin [8] studied the following identities related to multiplicative(generalized)-derivations in semiprime rings:

(i) $F\left(xy\right)\pm H\left(xy\right)=0$,
(ii) $F\left(xy\right)\pm H\left(yx\right)=0$,
(iii) $F\left(x\right)F\left(y\right)\pm H\left(xy\right)=0$,
(iv) $F\left(xy\right)\pm H\left(xy\right)\in Z$,
(v) $F\left(xy\right)\pm H\left(yx\right)\in Z$,
(vi) $F\left(x\right)F\left(y\right)\pm H\left(xy\right)\in Z$,

for all $x,y\in R$.

In this line of investigation, it is more interesting to study the semiprime rings with multiplicative(generalized)-derivations involving left multipliers in some appropriate subsets of R.

Throughout the paper, R will be a semiprime ring, I a non zero ideal of R, F be a multiplicative(generalized)-derivation of R and H be a multiplicative left multiplier of R.

We shall frequently use the following basic commutator and anti-commutator identities in the proofs of our results:

$(i)\ [x, yz] = y[x, z] + [x, y]z,$
$(ii)\ [xy, z] = [x, z]y + x[y, z],$
$(iii)\ xoyz = (xoy)z - y[x, z] = y(xoz) + [x, y]z,$
$(iv)\ xyoz = x(yoz) - [x, z]y = (xoz)y + x[y, z],$
for all $x, y, z \in R$.

2. MAIN RESULTS

We begin with our first theorem:

Theorem 2.1. *Let R be a semiprime ring and I a non-zero ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that $F(xoy) \pm H(xoy) = 0$ for all $x, y \in I$, then $I[x, d(x)] = 0$ for all $x \in I$.*

Proof. By the hypothesis, we have

$$F(xoy) \pm H(xoy) = 0, \text{ for all } x, y \in I. \tag{2.1}$$

Replacing y by yx in (2.1), we obtain

$$F((xoy)x) \pm H((xoy)x) = 0,$$

Using (2.1), it reduces to

$$(xoy)d(x) = 0, \text{ for all } x, y \in I. \tag{2.2}$$

Substituting $d(x)y$ for y and using (2.2), we get

$$[x, d(x)]yd(x) = 0, \text{ for all } x, y \in I. \tag{2.3}$$

Right multiplying (2.3) by x, we get

$$[x, d(x)]yd(x)x = 0, \text{ for all } x, y \in I. \tag{2.4}$$

Replace y by yx in (2.3), we obtain

$$[x, d(x)]yxd(x) = 0, \text{ for all } x, y \in I. \tag{2.5}$$

subtract (2.4) from (2.5), we get

$$[x, d(x)]y[x, d(x)] = 0, \text{ for all } x, y \in I. \tag{2.6}$$

Replacing y by ry, we obtain

$$[x, d(x)]ry[x, d(x)] = 0, \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.7}$$

left multiplying (2.7) by y, we get

$$y[x, d(x)]Ry[x, d(x)] = 0, \text{ for all } x, y \in I.$$

By the semiprimeness of R, we conclude that $y[x, d(x)] = 0$, for all $x, y \in I$, that is, $I[x, d(x)] = 0$, for all $x \in I$. □

Theorem 2.2. *Let R be a semiprime ring and I a non-zero ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that $F(xoy) \pm H[x, y] = 0$ for all $x, y \in I$, then $I[x, d(x)] = 0$ for all $x \in I$.*

Proof. By the hyothesis, we have

$$F(xoy) \pm H[x, y] = 0, \text{ for all } x, y \in I. \tag{2.8}$$

Replacing y by yx in (2.8), we obtain

$$F((xoy)x) \pm H([x, y]x), \text{ for all } x, y \in I, \tag{2.9}$$

Using (2.8), it reduces to

$$(xoy)\, d(x) = 0, \text{ for all } x, y \in I. \tag{2.10}$$

Using the same arguments after (2.2) in the proof of Theorem (2.1), we get the required result. □

Theorem 2.3. *Let R be a semiprime ring and I a non-zero ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that $F[x, y] \pm [x, H(y)] = 0$ for all $x, y \in I$, then $I[x, d(x)] = 0$ for all $x \in I$.*

Proof. By the hypothesis, we have

$$F[x, y] \pm [x, H(y)] = 0 for\ all\ x, y \in I. \tag{2.11}$$

Replacing y by yx in (2.11), we obtain

$$F([x, y]\, x) \pm [x, H(y)\, x] = 0 for\ all\ x, y \in I, \tag{2.12}$$

Using (2.11), it reduces to

$$[x, y]\, d(x) = 0 for\ all\ x, y \in I. \tag{2.13}$$

Substituting $d(x)\, y$ for y and using (2.13), we get

$$[x, d(x)]\, yd(x) = 0 for\ all\ x, y \in I. \tag{2.14}$$

Right multiplying (2.14) by x, we obtain

$$[x, d(x)]\, yd(x)\, x = 0 for\ all\ x, y \in I. \tag{2.15}$$

Replacing y by yx in (2.14), we get

$$[x, d(x)]\, yxd(x) = 0 for\ all\ x, y \in I. \tag{2.16}$$

Subtracting (2.15) from (2.16), we get

$$[x, d(x)]\, y\, [x, d(x)] = 0 for\ all\ x, y \in I. \tag{2.17}$$

Replacing y by ry, we obtain

$$[x, d(x)]\, ry\, [x, d(x)] = 0 for\ all\ x, y \in I\ and\ r \in R. \tag{2.18}$$

left multiplying (2.17) by y, we get

$$y\, [x, d(x)]\, Ry\, [x, d(x)] = 0 for\ all\ x, y \in I.$$

The semiprimeness of R yields that $y\,[x, d(x)] = 0$ for all $x, y \in I$. Therefore $I\,[x, d(x)] = 0$ for all $x \in I$. □

Theorem 2.4. *Let R be a semiprime ring and I a non-zero ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that $F[x, y] \pm [x, H(y)] = 0$ for all $x, y \in I$, then $I[x, d(x)] = 0$ for all $x \in I$.*

Proof. By the hypothesis, we have

$$F(xoy) \pm [x, H(y)] = 0 for\ all\ x, y \in I. \tag{2.19}$$

Replacing y by yx in (2.19), we get

$$F((xoy)\, x) \pm [x, H(y)\, x] = 0 for\ all\ x, y \in I, \tag{2.20}$$

Using (2.19), it reduces to

$$(xoy)\, d(x) = 0 for\ all\ x, y \in I. \tag{2.21}$$

Then by the same argument as in the proof of Theorem(2.1), we get $I\,[x, d(x)] = 0$ for all $x \in I$. □

Theorem 2.5. *Let R be a semiprime ring and I a non-zero ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that $F(xy) \pm [x, H(y)] \in Z(R)$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. By the hypothesis, we have

$$F(xy) + [x, H(y)] \in Z(R) \, for \; all \; x, y \in I. \tag{2.22}$$

Replacing y by yz in (2.22), we get

$$F(xy)z + xyd(z) + H(y)[x, z] + [x, H(y)]z \in Z(R) \, for \; all \; x, y, z \in I,$$

$$(F(xy) + [x, H(y)])z + xyd(z) + H(y)[x, z] \in Z(R). \tag{2.23}$$

Combining (2.21) and (2.22), we obtain

$$[xyd(z), z] + [H(y)[x, z], z] = 0. \tag{2.24}$$

Replacing x by xz in (2.23), we get

$$[xzyd(z), z] + [H(y)[xz, z], z] = 0,$$

$$[xzyd(z), z] + [H(y)[x, z]z, z] = 0. \tag{2.25}$$

Right multiplying (2.23) by z and subtracting from (2.24), we get

$$[x[yd(z), z], z] = 0 for \; all \; x, y, z \in I. \tag{2.26}$$

Replacing x by wx in (2.25) and using (2.25), we obtain

$$[w, z]x[yd(z), z] = 0 for \; all \; x, y, z, w \in I. \tag{2.27}$$

Replacing w by $yd(z)$ and using semiprimeness of R, we get

$$[yd(z), z] = 0 for \; all \; y, z \in I. \tag{2.28}$$

Substituting $d(z)y$ instead of y in (2.27) and using (2.27), we obtain

$$[d(z), z]yd(z) = 0 for \; all \; y, z \in I.$$

Replacing z by x, we get

$$[d(x), x]yd(x) = 0 for \; all \; x, y \in I. \tag{2.29}$$

Replacing y by yx in (2.28), we get

$$[d(x), x]yxd(x) = 0 for \; all \; x, y \in I. \tag{2.30}$$

Right multiplying (2.28) by x, we get

$$[d(x), x]yd(x)x = 0 for \; all \; x, y \in I. \tag{2.31}$$

Subtracting (2.29) from (2.30), we get

$$[d(x), x]y[d(x), x] = 0 for \; all \; x, y \in I. \tag{2.32}$$

Replacing y by ry in (2.31), we obtain

$$[d(x), x]ry[d(x), x] = 0 for \; all \; x, y \in I \; and \; r \in R. \tag{2.33}$$

Left multiplying (2.32) by y, we get

$$y[d(x), x]Ry[d(x), x] = 0 for \; all \; x, y \in I. \tag{2.34}$$

By semiprimeness of R, we conclude that $y[x, d(x)] = 0$, for all $x, y \in I$, then $I[x, d(x)] = 0$, for all $x \in I$.

In the same manner the conclusion can be obtained when $F(xy) - [x, H(y)] \in Z(R)$ for all $x, y \in I$. □

Theorem 2.6. *Let R be a semiprime ring and I a non-zero ideal of R. If $F : R \longrightarrow R$ is a multiplicative(generalized)-derivation associated with a map $d : R \longrightarrow R$ such that*

$$F(xy) \pm [H(x), H(y)] \in Z(R),$$

for all $x, y \in I$, then

$$I[d(x), x] = 0,$$

for all $x \in I$.

Proof. By the hypothesis, we have

$$F(xy) + [H(x), H(y)] \in Z(R), \text{ for all } x, y \in I. \tag{2.35}$$

Replacing y by yz in (2.35), we get

$$F(xy)z + xyd(z) + H(y)[x, z] + [H(x), H(y)]z \in Z(R). \tag{2.36}$$

Combining (2.34) and (2.35), we obtain

$$[xyd(z), z] + [H(y)[x, z], z] = 0, \text{ for all } x, y, z \in I. \tag{2.37}$$

Replacing x by xz in (2.37), we find that

$$[xzyd(z), z] + [H(y)[xz, z], z] = 0,$$

$$[xzyd(z), z] + [H(y)[x, z]z, z] = 0, \text{ for all } x, y, z \in I. \tag{2.38}$$

Then by the same argument as in the proof of Theorem(2.5), we get $I[d(x), x] = 0$, for all $x \in I$.
In the same manner the conclusion can be obtained when $F(xy) - [H(x), H(y)] \in Z(R)$ for all $x, y \in I$. □

Corollary 2.1. *Let R be a semiprime ring admitting a multiplicative(generalized)-derivation $F : R \longrightarrow R$ associated with a map $d : R \longrightarrow R$ and $H : R \longrightarrow R$ be a multiplicative left multiplier. If R satisfies any one of the following identities:*
(i) $F(xoy) \pm H(xoy) = 0$,
(ii) $F(xoy) \pm H[x, y] = 0$,
(iii) $F[x, y] \pm [x, H(y)] = 0$,
(iv) $F(xoy) \pm [x, H(y)] = 0$,
(v) $F(xy) \pm [x, H(y)] \in Z(R)$,
(vi) $F(xy) \pm [H(x), H(y)] \in Z(R)$ holds for all $x, y \in R$, then the map d is a commuting map on R.

Example 2.1. Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} / \text{ a, b, c} \in \text{Z} \right\}$, where Z is set of integers. We define the maps $F, d, H : R \longrightarrow R$ by F$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

d$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

respectively.

It is verified that F is a multiplicative (generalized)-derivation associated with a map d respectively and

$$H(xy) = H(x)y$$

holds for all $x, y \in R$.

It is easy to see that the identity

$$F\left(xoy\right) \pm H\left(xoy\right) = 0,$$

for all $x, y \in R$.

Here R is not semiprime because $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$.

Hence, the condition of semiprimeness in Corollary 2.7 cannot be removed.

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