

Application of subordination for estimating the Hankel determinant for subclass of univalent functions

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ABSTRACT. By using subordination structure, a new subclass of convex functions is introduced. The estimate of the third Hankel determinants is also investigated.

1. INTRODUCTION

The coefficient estimate of univalent functions is one of the most important subjects which has been addressed in many recent articles. See [5], [6] and [7]. The well-known Fekete-Szegő inequality and the bounds of second and third Hankel determinants are investigated. See [1], [3], [4], [8] and [9].

Let \mathcal{A} denote the class of univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Further, by \mathcal{S} we shall denote the class of all functions in \mathcal{A} which are normalized univalent in \mathbb{U} .

Let f and g be analytic in \mathbb{U} . Then f is said to be subordinate to g , written $f \prec g$ or $f(z) \prec g(z)$, if there exists a function ω analytic in \mathbb{U} , with $\omega(0) = 0$, $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$.

If g is univalent, the $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Suppose that \mathcal{P} denote the class of analytic functions p of the type

$$p(z) = 1 + \sum_{n=2}^{\infty} a_n z^n \quad (1.2)$$

such that $\operatorname{Re} p(z) > 0$. Also

$$SL^C = \left\{ f \in \mathcal{A} : \left| \left(1 + \frac{zf''(z)}{f'(z)} \right)^2 - 1 \right| < 1, \quad z \in \mathbb{U} \right\}.$$

Thus the values of $1 + \frac{zf''(z)}{f'(z)}$ where $f \in SL^C$ lies in the region which bounded by the right half of the lemniscate of Bernoulli given by $|\omega^2 - 1| < 1$.

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It is easy to see that $f \in SL^C$ if it satisfies the condition

$$1 + \frac{zf''(z)}{f'(z)} \prec \sqrt{1+z}, \quad z \in \mathbb{U}. \tag{1.3}$$

The determinants of the q^{th} Hankel matrix denoted by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix} \quad q \in \mathbb{N} \setminus 1, \quad n \in \mathbb{N}.$$

and is called the q^{th} Hankel determinant. In the particular cases $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$.

the Hankel determinant simplifies respectively to

$$H_2(1) = |a_3 - a_2^2| \quad \text{and} \quad H_2(2) = |a_2a_4 - a_3^2|.$$

In the case $q = 3$ and $n = 1$,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

$H_2(2)$ and $H_3(1)$ are respectively called second and third Hankel determinant

For $f \in \mathcal{S}, a_1 = 1$ so that,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_1a_4 - a_2a_3) + a_5(a_1a_3 - a_2^2).$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

In this paper, we introduce a new subclass of convex functions associated with differential subordination and extremum point.

Lemma 1.1. [7] *If $p \in \mathcal{P}$ be of the form 1.2, Then*

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

for some $x, |x| \leq 1$, and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some $z, |z| \leq 1$.

2. MAIN RESULTS

In this section we obtain some coefficient estimates and bounds of third Hankel determinant.

Lemma 2.2. *If $p \in \mathcal{P}$, then $|p_k| \leq 2$ for each k . [2]*

Theorem 2.1. *Let f of the form 1.1 be in the class SL^C , Then*

$$|a_3 - a_2^2| \leq \frac{5}{24}.$$

Proof. If $f \in SL^C$, then it follows from 1.3 that

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \tag{2.4}$$

where $\phi(z) = \sqrt{1+z}$. Define a function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$$

It is clear that $p \in \mathcal{P}$. This implies that

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1}.$$

From 2.4, we have

$$1 + \frac{zf''(z)}{f'(z)} = \phi(\omega(z)).$$

with

$$\phi(\omega(z)) = \left(\frac{2p(z)}{p(z) + 1} \right)^{\frac{1}{2}}.$$

Now

$$\left(\frac{2p(z)}{p(z) + 1} \right)^{\frac{1}{2}} = 1 + \frac{1}{4}p_1z + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2 \right)z^2 + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3 \right)z^3 \dots$$

Similarly,

$$\frac{1 + zf''(z)}{f'(z)} = 1 + 2a_2z + (6a_3 - 4a_2^2)z^2 + (12a_4 - 18a_2a_3 + 8a_2^3)z^3 \dots$$

Now, equating the coefficients we get

$$2a_2 = \frac{1}{4}p_1,$$

then

$$a_2 = \frac{1}{8}p_1. \tag{2.5}$$

and

$$6a_3 - 4a_2^2 = \frac{1}{4}p_2 - \frac{5}{32}p_1^2,$$

$$a_3 = \frac{1}{24}p_2 - \frac{1}{64}p_1^2. \tag{2.6}$$

and

$$12a_4 - 18a_2a_3 + 8a_2^3 = \frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3$$

$$a_4 = \frac{1}{48}p_3 - \frac{7}{384}p_1p_2 + \frac{13}{3072}p_1^3. \tag{2.7}$$

Applying Lemma 2.2 for the coefficients p_1 and p_2 , we get:

$$|a_3 - a_2^2| \leq \frac{5}{24}.$$

□

Theorem 2.2. Let f of the form 1.1 be in the class SL^C , Then

$$|a_2a_4 - a_3^2| \leq \frac{1}{72}.$$

Proof. From 2.5, 2.6 and 2.7, we obtain

$$\begin{aligned} a_2 a_4 - a_3^2 &= \frac{1}{384} \left(p_1 p_3 - \frac{7}{8} p_1^2 p_2 + \frac{13}{64} p_1^4 \right) - \left(\frac{1}{24} p_2 - \frac{1}{64} p_1^2 \right)^2 \\ &= \frac{1}{384} p_1 p_3 - \frac{7}{3072} p_1^2 p_2 + \frac{13}{24576} p_1^4 - \frac{1}{576} p_2^2 \\ &\quad + \frac{1}{768} p_1^2 p_2 - \frac{1}{4096} p_1^4 \\ &= \frac{1}{24576} \left(64 p_1 p_3 + 24 p_1^2 p_2 - \frac{128}{3} p_2^2 + 7 p_1^4 \right). \end{aligned}$$

Putting the values of p_2 and p_3 from Lemma 1.1, letting $p > 0$ and taking $p_1 = p \in [0, 2]$, we get

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{1}{24576} \left| 16 p_1 (p_1^3 + 2(4 - p_1^2) p_1 x - (4 - p_1^2) p_1 x^2 \right. \\ &\quad \left. + 2(4 - p_1^2)(1 - |x|^2)z) + 12 p_1^2 (p_1^2 + x(4 - p_1^2)) \right. \\ &\quad \left. - \frac{32}{3} (p_1^2 + x(4 - p_1^2)^2) + 16 p_1^2 (p_1^2 + x(4 - p_1^2)) - 6 p_1^4 \right|. \end{aligned}$$

After a simple calculation, we get

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{1}{24576} \left| \frac{82}{3} p^4 + \frac{116}{3} (4 - p^2) p^2 x \right. \\ &\quad \left. + 32(4 - p^2) p (1 - |x|^2) z - \frac{16}{3} (4 - p^2) (p^2 + 8) x^2 \right|. \end{aligned}$$

Now, applying the triangle inequality and replacing $|x|$ by ρ , we obtain

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq \frac{1}{73728} \left| 82 p^4 + 192 p (4 - p^2) + 116 (4 - p^2) p^2 \rho + 16 \rho^2 (4 - p^2) (p^2 + 8) \right| \\ &= F(p, \rho). \end{aligned}$$

Differentiating with respect to ρ , we have

$$\frac{\partial F(p, \rho)}{\partial \rho} = \frac{1}{73728} (116(4 - p^2) p^2 + 32 \rho (4 - p^2) (p^2 + 8)).$$

It is clear that $\frac{\partial F(p, \rho)}{\partial \rho} > 0$, which shows that $F(p, \rho)$ is an increasing function on the closed interval $[0, 1]$. This implies that maximum occurs at $\rho = 1$. Therefore

$\text{Max } F(p, \rho) = F(p, 1) = G(p)$. Now

$$G(p) = \frac{1}{73728} (-148 p^4 + 336 p^2 + 1024).$$

Therefore

$$G'(p) = \frac{1}{73728} (-592 p^3 + 672 p)$$

and

$$G''(p) = \frac{1}{73728} (-1776 p^2 + 672) < 0$$

for $p = 0$. This shows that maximum of $G(p)$ occurs at $p = 0$. Hence, we obtain

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq \frac{1024}{73728} \\ &= \frac{1}{72}. \end{aligned}$$

□

Theorem 2.3. Let f of the form 1.1 be in the class SL^C , Then

$$|a_2a_3 - a_4| \leq \frac{1}{12}.$$

Proof. Since

$$\begin{aligned} a_2 &= \frac{1}{8}p_1, \\ a_3 &= \frac{1}{24}p_2 - \frac{1}{64}p_1^2, \\ a_4 &= \frac{1}{48}p_3 - \frac{7}{384}p_1p_2 + \frac{13}{3072}p_1^3, \end{aligned}$$

therefore, by using Lemma 1.2 and replacing $|x|$ by ρ , we have

$$|a_2a_3 - a_4| \leq \frac{1}{3072} \left| 39p^3 + 64(4 - p^2) + 4(4 - p^2)p\rho + 16p\rho^2(4 - p^2) \right|.$$

Let

$$F_1(p, \rho) = \frac{1}{3072} \left| 39p^3 + 64(4 - p^2) + 4(4 - p^2)p\rho + 16p\rho^2(4 - p^2) \right|. \quad (2.8)$$

We assume that the upper bound occurs at the interior of the rectangle $[0, 2] \times [0, 1]$. Differentiating 2.8 with respect to ρ , we have

$$\frac{\partial F_1(p, \rho)}{\partial \rho} = \frac{1}{3072} (4p(4 - p^2) + 64p\rho(4 - p^2)).$$

For $0 < \rho < 1$, and fixed $p \in (0, 2)$, it is clear that $\frac{\partial F_1(p, \rho)}{\partial \rho} < 0$, which shows that $F_1(p, \rho)$ is an decreasing function of p , which contradicts our assumption, therefore $Max F_1(p, \rho) = F_1(p, 0) = G_1(p)$. Now

$$G_1(p) = \frac{1}{3072} (-4p^3 + 16p).$$

So

$$G'_1(p) = \frac{1}{3072} (-12p^2 + 16)$$

and

$$G''_1(p) = \frac{1}{3072} (-24p) < 0$$

for $p = 0$. This shows that maximum of $G_1(p)$ occurs at $p = 0$. Hence, we get the required result. □

Lemma 2.3. If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class SL^C , then

$$|a_2| \leq \frac{1}{4}, \quad |a_3| \leq \frac{1}{12}, \quad |a_4| \leq \frac{1}{24}, \quad |a_5| \leq \frac{1}{40}.$$

These estimations are sharp.

Theorem 2.4. Let f of the form 1.1 be in the class SL^C , Then

$$|H_3(1)| \leq \frac{49}{8640}.$$

Proof. Since

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_1a_4 - a_2a_3) + a_5(a_1a_3 - a_2^2),$$

Now, using the triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

Using the fact that $a_1 = 1$ with the results of Theorem 2.1, Theorem 2.2, Theorem 2.3 and Lemma 2.3, we obtain

$$|H_3(1)| \leq \frac{49}{8640}.$$

□

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