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Balcobalancing numbers and balcobalancers

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ABSTRACT. In this work, we determine the general terms of balcobalancing numbers, balcobalancers and also Lucas–balcobalancing numbers in terms of balancing numbers. Further we formulate the sums of these numbers and derive some relations associated with Pell, Pell–Lucas and square triangular numbers.

1. INTRODUCTION

A positive integer n is called a balancing number ([2]) if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1.1)

$$r=\frac{-2n-1+\sqrt{8n^2+1}}{2}. \tag{1.2}$$

From (1.2) we note that n is a balancing number if and only if $8n^2+1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (1.2) we note that $8(0)^2+1=1$ and $8(1)^2+1=3^2$ are perfect squares. So we accept 0 and 1 to be balancing numbers. Let B_n denote the n^{th} balancing number. Then $B_0=0, B_1=1, B_2=6$ and $B_{n+1}=6B_n-B_{n-1}$ for $n\geq 2$.

Later Panda and Ray ([12]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{1.3}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (1.3)

$$r=\frac{-2n-1+\sqrt{8n^2+8n+1}}{2}. \tag{1.4}$$

From (1.4) we note that n is a cobalancing number if and only if $8n^2+8n+1$ is a perfect square. Since $8(0)^2+8(0)+1=1$ is a perfect square, we accept 0 to be a cobalancing number, just like Behera and Panda accepted $0,1$ to be balancing numbers. Cobalancing number is denoted by b_n, and $b_0=b_1=0, b_2=2$ and $b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$.

It is clear from (1.1) and (1.3) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n=r_{n+1}$ and $R_n=b_n$ for $n\geq 1$, where R_n is

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the n^{th} the balancer and r_n is the n^{th} cobalancer. Since $R_n = b_n$, we get from (1.1) that

$$b_n = \frac{-2B_n - 1 + \sqrt{8B_n^2 + 1}}{2} \text{ and } B_n = \frac{2b_n + 1 + \sqrt{8b_n^2 + 8b_n + 1}}{2}. \tag{1.5}$$

Thus from (1.5), we see that B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. So

$$C_n = \sqrt{8B_n^2 + 1} \text{ and } c_n = \sqrt{8b_n^2 + 8b_n + 1} \tag{1.6}$$

are integers which are called the Lucas–balancing number and Lucas–cobalancing number, respectively.

Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ be the roots of the characteristic equation for Pell and Pell–Lucas numbers which are the numbers defined by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ and $Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. Ray ([16]) derived some nice results on balancing numbers and Pell numbers his Phd thesis. Since x is a balancing number if and only if $8x^2+1$ is a perfect square, he set $8x^2+1 = y^2$ for some integer $y \geq 1$. Then $y^2 - 8x^2 = 1$ which is a Pell equation ([1, 3, 9]). The fundamental solution is $(y_1, x_1) = (3, 1)$. So $y_n + x_n\sqrt{8} = (3+\sqrt{8})^n$ for $n \geq 1$ and similarly $y_n - x_n\sqrt{8} = (3-\sqrt{8})^n$. Let $\gamma = 3+\sqrt{8}$ and $\delta = 3 - \sqrt{8}$. Then he get $x_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$ which is the Binet formula for balancing numbers, that is, $B_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$. Since $\alpha^2 = \gamma$ and $\beta^2 = \delta$, he conclude that the Binet formula for balancing numbers is $B_n = \frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}$. Similarly he get $b_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n} + \beta^{2n}}{2}$ and $c_n = \frac{\alpha^{2n-1} + \beta^{2n-1}}{2}$ for $n \geq 1$ (see also [10, 11, 15]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [7], Liptai proved that there is no Fibonacci balancing number except 1 and in [8] he proved that there is no Lucas–balancing number. In [19], Szalay considered the same problem and obtained some nice results by a different method. In [5], Kovács, Liptai and Olajos extended the concept of balancing numbers to the $(a, b)-$balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a + b) + \cdots + (a(n-1) + b) = (a(n+1) + b) + \cdots + (a(n+r) + b)$$

for some positive integers n and r, then $an+b$ is an $(a, b)-$balancing number. The sequence of $(a, b)-$balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [6], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ with $y \geq 4$. Then a positive integer x with $x \leq y - 2$ is called a $(k, l)-$power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for $(k, l)-$power numerical centers. For positive integers k, x, let

$$\Pi_k(x) = x(x+1)\ldots(x+k-1).$$

Then it was proved in [5] that the equation

$$B_m = \Pi_k(x)$$

for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [23] Tengely, considered the case $k = 5$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [14], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas–balancing numbers and in [17], Ray considered the sums of balancing and Lucas–balancing numbers by matrix methods. In [13], Panda and Panda defined the almost balancing number and its balancer. In [21], the first author considered amost balancing numbers, triangular numbers and square triangular numbers and in [22], he considered the sums and spectral norms of all almost balancing numbers.

2. Results.

In this work, we set three new integer sequences called balcobalancing number, balcobalancer and Lucas–balcobalancing number and try to determine the general terms of them in terms of balancing numbers. We also want to derive some relations with Pell, Pell–Lucas and square triangular numbers.

If we sum both sides of (1.1) and (1.3), then we get the Diophantine equation

$$1+2+\cdots+(n-1)+1+2+\cdots+(n-1)+n=2[(n+1)+(n+2)+\cdots+(n+r)]. \quad (2.7)$$

Thus a positive integer n is called a balcobalancing number if the Diophantine equation in (2.7) verified for some positive integer r which is called balcobalancer. For example, $10, 348, 11830, \cdots$ are balcobalancing numbers with balcobalancers $4, 144, 4900, \cdots$. (Here we want to use name "balcobalancing" since it comes from balancing and cobalancing numbers).

From (2.7), we get

$$r=\frac{-2n-1+\sqrt{8n^2+4n+1}}{2}. \quad (2.8)$$

Let B_n^{bc} denote the balcobalancing number and let R_n^{bc} denote the balcobalancer. Then from (2.8), B_n^{bc} is a balcobalancing number if and only if $8(B_n^{bc})^2+4B_n^{bc}+1$ is a perfect square. Thus

$$C_n^{bc}=\sqrt{8(B_n^{bc})^2+4B_n^{bc}+1} \quad (2.9)$$

is an integer which are called the Lucas–balcobancing number. (Here we notice that balcobalancing numbers should be greater that 0. But in (2.9), $8(0)^2+4(0)+1=1$ is a perfect square, so we assume that 0 is a balcobalancing number, that is, $B_0^{bc}=0$. In this case, $R_0^{bc}=0$ and $C_0^{bc}=1$).

In order to determine the general terms of balcobalancing numbers, balcobalancers and Lucas–balcobalancing numbers we have to determine the set of all (positive) integer solutions of the Pell equation

$$x^2-2y^2=-1. \quad (2.10)$$

We see from (2.8) that B_n^{bc} is a balcobalancing number if and only if $8(B_n^{bc})^2+4B_n^{bc}+1$ is a perfect square. So we set $8(B_n^{bc})^2+4B_n^{bc}+1=y^2$ for some integer $y\geq 1$. If we multiply both sides of the last equation by 2, then we get $16(B_n^{bc})^2+8B_n^{bc}+2=2y^2$ and hence $(4B_n^{bc}+1)^2+1=2y^2$. Taking $x=4B_n^{bc}+1$, we get the Pell equation in (2.10).

Let Ω denotes the set of all integer solutions of (2.10), that is, $\Omega=\{(x,y):x^2-2y^2=-1\}$. Then we can give the following theorem.

Theorem 2.1. *The set of all integer solutions of (2.10) is* $\Omega=\{(c_n, 2b_n+1):n\geq 1\}$.

Proof. For the Pell equation $x^2-2y^2=-1$, the set of representatives $\text{Rep}=\{[\pm 1 \quad 1]\}$ and $M=\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$. In this case $[-1 \quad 1]M^n$ generates all integer solutions (x_n, y_n) for $n\geq 1$. It can be easily seen that the n^{th} power of M is

$$M^n=\begin{bmatrix} C_n & 2B_n \\ 4B_n & C_n \end{bmatrix}$$

for $n \geq 1$. So

$$[x_n \quad y_n] = [-1 \quad 1] \begin{bmatrix} C_n & 2B_n \\ 4B_n & C_n \end{bmatrix} = [-C_n + 4B_n \quad -2B_n + C_n].$$

Thus the set of all integer solutions is $\Omega = \{(-C_n + 4B_n, -2B_n + C_n) : n \geq 1\}$. But it can be easily seen that $-C_n + 4B_n = c_n$ and $-2B_n + C_n = 2b_n + 1$. So we conclude that the set of all integer solutions of (2.10) is $\Omega = \{(c_n, 2b_n + 1) : n \geq 1\}$. □

From Theorem 2.1, we can give the following result.

Theorem 2.2. *The general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers are*

$$B_n^{bc} = \frac{c_{2n+1} - 1}{4}, \ C_n^{bc} = 2b_{2n+1} + 1 \ \textit{and} \ R_n^{bc} = \frac{4b_{2n+1} - c_{2n+1} + 1}{4}$$

for $n \geq 1$.

Proof. We proved in Theorem 2.1 that $\Omega = \{(c_n, 2b_n + 1) : n \geq 1\}$. Since $x = 4B_n^{bc} + 1$, we get

$$B_n^{bc} = \frac{x_{2n+1} - 1}{4} = \frac{c_{2n+1} - 1}{4}$$

for $n \geq 1$. Thus from (2.9), we obtain

$$\begin{aligned} C_n^{bc} &= \sqrt{8(B_n^{bc})^2 + 4B_n^{bc} + 1} \\ &= \sqrt{8(\frac{c_{2n+1} - 1}{4})^2 + 4(\frac{c_{2n+1} - 1}{4}) + 1} \\ &= \sqrt{\frac{c_{2n+1}^2 + 1}{2}} \\ &= \sqrt{\frac{(\frac{\alpha^{4n+1} + \beta^{4n+1}}{2})^2 + 1}{2}} \\ &= \sqrt{\left[2(\frac{\alpha^{4n+1} - \beta^{4n+1}}{2\sqrt{2}} - \frac{1}{2}) + 1\right]^2} \\ &= 2b_{2n+1} + 1. \end{aligned}$$

Finally from (2.8), we deduce that

$$R_n^{bc} = \frac{4b_{2n+1} - c_{2n+1} + 1}{4}$$

as we wanted. □

We can also give the general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers in terms of balancing and cobalancing numbers as follows.

Theorem 2.3. *The general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers are*

$$B_n^{bc} = \frac{B_{2n} + b_{2n+1}}{2}, C_n^{bc} = 2b_{2n+1} + 1 \ \textit{and} \ R_n^{bc} = \frac{-B_{2n} + b_{2n+1}}{2}$$

for $n \geq 1$.

Proof. We proved in Theorem 2.2 that $B_n^{bc} = \frac{c_{2n+1}-1}{4}$. So we easily deduce that

$$\begin{aligned}
B_n^{bc} &= \frac{c_{2n+1}-1}{4} \\
&= \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4} \\
&= \frac{\alpha^{4n+1}(\frac{\alpha^{-1}+1}{4\sqrt{2}}) + \beta^{4n+1}(\frac{-\beta^{-1}-1}{4\sqrt{2}})}{2} - \frac{1}{4} \\
&= \frac{\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2}}{2} \\
&= \frac{B_{2n}+b_{2n+1}}{2}.
\end{aligned}$$

$C_n^{bc} = 2b_{2n+1}+1$ is already proved in Theorem 2.2. Similarly it can be proved that $R_n^{bc} = \frac{-B_{2n}+b_{2n+1}}{2}$. □

As in Theorem 2.3, we can give the general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers in terms of only balancing numbers or only Lucas–balancing numbers as follows.

Theorem 2.4. *The general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers are*

$$B_n^{bc} = 2B_n(B_{n+1}-B_n),\ C_n^{bc} = B_{2n+1}-B_{2n},\ R_n^{bc} = 4B_n^2$$

or

$$B_n^{bc} = \frac{C_{2n+1}-C_{2n}-2}{8},\ C_n^{bc} = \frac{C_{2n+1}+C_{2n}}{4},\ R_n^{bc} = \frac{C_{2n}-1}{4}$$

for $n \geq 1$.

Proof. From Theorem 2.3, we get

$$\begin{aligned}
B_n^{bc} &= \frac{B_{2n}+b_{2n+1}}{2} \\
&= \frac{\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2}}{2} \\
&= \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4} \\
&= \frac{\alpha^{4n+1}-\alpha^{2n}\beta^{2n+1}-\beta^{2n}\alpha^{2n+1}+\beta^{4n+1}}{8} \\
&= 2\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{2\sqrt{2}}\right) \\
&= 2\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)\left(\frac{\alpha^{2n}(\alpha^2-1)-\beta^{2n}(\beta^2-1)}{4\sqrt{2}}\right) \\
&= 2\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)\left(\frac{\alpha^{2n+2}-\beta^{2n+2}}{4\sqrt{2}} - \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right) \\
&= 2B_n(B_{n+1}-B_n).
\end{aligned}$$

The others can be proved similarly. □

In Theorems 2.2, 2.3 and 2.4, we can give the general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers in terms of balancing, cobalancing, Lucas–balancing and Lucas–cobalancing numbers. Conversely, we can give the general terms of balancing, cobalancing, Lucas–balancing and Lucas–cobalancing numbers in terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers as follows.

Theorem 2.5. *The general terms of balancing, cobalancing, Lucas–balancing and Lucas–cobalancing numbers are*

$$B_n = \begin{cases} B^{bc}_{\frac{n}{2}} - R^{bc}_{\frac{n}{2}} & n \geq 2 \text{ even} \\ (B^{bc}_{\frac{n+1}{2}} + B^{bc}_{\frac{n-1}{2}} - 2R^{bc}_{\frac{n+1}{2}})/2 & n \geq 1 \text{ odd} \end{cases}$$

$$b_n = \begin{cases} -B^{bc}_{\frac{n}{2}} + 3R^{bc}_{\frac{n}{2}} & n \geq 2 \text{ even} \\ B^{bc}_{\frac{n-1}{2}} + R^{bc}_{\frac{n-1}{2}} & n \geq 1 \text{ odd} \end{cases}$$

$$C_n = \begin{cases} -4B^{bc}_{\frac{n}{2}} + 2C^{bc}_{\frac{n}{2}} - 1 & n \geq 2 \text{ even} \\ 4B^{bc}_{\frac{n-1}{2}} + 2C^{bc}_{\frac{n-1}{2}} + 1 & n \geq 1 \text{ odd} \end{cases}$$

$$c_n = \begin{cases} 12B^{bc}_{\frac{n}{2}} - 4C^{bc}_{\frac{n}{2}} + 3 & n \geq 2 \text{ even} \\ 4B^{bc}_{\frac{n-1}{2}} + 1 & n \geq 1 \text{ odd.} \end{cases}$$

Proof. From Theorem 2.3, we get $B^{bc}_n = \frac{B_{2n}+b_{2n+1}}{2}$ and $R^{bc}_n = \frac{-B_{2n}+b_{2n+1}}{2}$. Thus we get $B_{2n} = B^{bc}_n - R^{bc}_n$ and hence

$$B_n = B^{bc}_{\frac{n}{2}} - R^{bc}_{\frac{n}{2}}$$

for even $n \geq 2$. The others can be proved similarly. □

Thus we construct one-to-one correspondence between all balcobalancing numbers and all balancing numbers.

3. Binet Formulas, Recurrence Relations and Companion Matrix.

Theorem 3.6. *Binet formulas for balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers are*

$$B^{bc}_n = \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}, \ C^{bc}_n = \frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}} \ \text{ and } \ R^{bc}_n = \frac{\alpha^{4n}+\beta^{4n}}{8} - \frac{1}{4}$$

for $n \geq 1$.

Proof. Since $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$ and $b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$, we get from Theorem 2.3 that

$$\begin{aligned} B^{bc}_n &= \frac{B_{2n}+b_{2n+1}}{2} \\ &= \frac{\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2}}{2} \\ &= \frac{\alpha^{4n}(\frac{1+\alpha}{4\sqrt{2}}) + \beta^{4n}(\frac{-1-\beta}{4\sqrt{2}})}{2} - \frac{1}{4} \\ &= \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}. \end{aligned}$$

The others can be proved similarly. □

Recall that balancing numbers satisfy recurrence relation $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$. Similarly we can give the following result.

Theorem 3.7. *B_n^{bc}, C_n^{bc} and R_n^{bc} satisfy the recurrence relations*

$$B_n^{bc} = 35(B_{n-1}^{bc} - B_{n-2}^{bc}) + B_{n-3}^{bc}$$
$$R_n^{bc} = 35(R_{n-1}^{bc} - R_{n-2}^{bc}) + R_{n-3}^{bc}$$

for $n \geq 3$ and

$$C_n^{bc} = 34C_{n-1}^{bc} - C_{n-2}^{bc}$$

for $n \geq 2$.

Proof. Recall that $B_n^{bc} = \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}$ by Theorem 3.6. Since $35\alpha^{-3} - 35\alpha^{-7} + \alpha^{-11} = \alpha$ and $35\beta^{-3} - 35\beta^{-7} + \beta^{-11} = \beta$, we get

$$\begin{aligned}
&35(B_{n-1}^{bc} - B_{n-2}^{bc}) + B_{n-3}^{bc} \\
&= 35\left[(\frac{\alpha^{4n-3}+\beta^{4n-3}}{8} - \frac{1}{4}) - (\frac{\alpha^{4n-7}+\beta^{4n-7}}{8} - \frac{1}{4})\right] \\
&\quad + \frac{\alpha^{4n-11}+\beta^{4n-11}}{8} - \frac{1}{4} \\
&= \frac{\alpha^{4n}(35\alpha^{-3} - 35\alpha^{-7} + \alpha^{-11}) + \beta^{4n}(35\beta^{-3} - 35\beta^{-7} + \beta^{-11})}{8} - \frac{1}{4} \\
&= \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4} \\
&= B_n^{bc}
\end{aligned}$$

The others can be proved similarly. □

Recall that the companion matrix for balancing numbers is

$$M = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}.$$

It can be easily seen that the n^{th} power of M is

$$M^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix} \tag{3.11}$$

for $n \geq 1$. Since $B_n^{bc} = 35(B_{n-1}^{bc} - B_{n-2}^{bc}) + B_{n-3}^{bc}$ and $R_n^{bc} = 35(R_{n-1}^{bc} - R_{n-2}^{bc}) + R_{n-3}^{bc}$ by Theorem 3.7, the companion matrix for balcobalancing numbers and balcobalancers are same and is

$$M^{bc} = \begin{bmatrix} 35 & -35 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and since $C_n^{bc} = 34C_{n-1}^{bc} - C_{n-2}^{bc}$, the companion matrix for Lucas–balcobalancing numbers is

$$N^{bc} = \begin{bmatrix} 34 & -1 \\ 1 & 0 \end{bmatrix}.$$

As in (3.11), we can give the following theorem.

Theorem 3.8. *The n^{th} power of M^{bc} is*

$$(M^{bc})^n = \begin{bmatrix} \sum_{i=0}^{\frac{n}{2}} B_{4i+1} & -\sum_{i=1}^{n} B_{2i+1} & \sum_{i=0}^{\frac{n-2}{2}} B_{4i+3} \\ \sum_{i=0}^{\frac{n-2}{2}} B_{4i+3} & -\sum_{i=1}^{n-1} B_{2i+1} & \sum_{i=0}^{\frac{n-2}{2}} B_{4i+1} \\ \sum_{i=0}^{\frac{n-2}{2}} B_{4i+1} & -\sum_{i=1}^{n-2} B_{2i+1} & \sum_{i=0}^{\frac{n-4}{2}} B_{4i+3} \end{bmatrix}$$

for even $n \geq 4$ or

$$(M^{bc})^n = \begin{bmatrix} \sum_{i=0}^{\frac{n-1}{2}} B_{4i+3} & -\sum_{i=1}^{n} B_{2i+1} & \sum_{i=0}^{\frac{n-1}{2}} B_{4i+1} \\ \sum_{i=0}^{\frac{n-1}{2}} B_{4i+1} & -\sum_{i=1}^{n-1} B_{2i+1} & \sum_{i=0}^{\frac{n-3}{2}} B_{4i+3} \\ \sum_{i=0}^{\frac{n-3}{2}} B_{4i+3} & -\sum_{i=1}^{n-2} B_{2i+1} & \sum_{i=0}^{\frac{n-3}{2}} B_{4i+1} \end{bmatrix}$$

for odd $n \geq 3$, and the n^{th} power of N^{bc} is

$$(N^{bc})^n = (-1)^n \begin{bmatrix} \sum_{i=1}^{n+1} (-1)^{i+1} B_{2i-1} & \sum_{i=1}^{n} (-1)^{i+1} B_{2i-1} \\ -\sum_{i=1}^{n} (-1)^{i+1} B_{2i-1} & -\sum_{i=1}^{n-1} (-1)^{i+1} B_{2i-1} \end{bmatrix}$$

for every $n \geq 1$.

Proof. It can be proved by induction on n. □

We can rewrite the n^{th} power of M^{bc} and N^{bc} in terms of balancing and Lucas–balancing numbers instead of sums of balancing numbers. For this purpose, we set two integer sequences k_n and l_n to be

$$k_n = \frac{-8B_{2n} + 3C_{2n} - 3}{96} \text{ and } l_n = \frac{-288B_{2n} - 102C_{2n} + 102}{96}$$

for $n \geq 0$. Then we can give the following theorem.

Theorem 3.9. *The n^{th} power of M^{bc} is*

$$(M^{bc})^n = \begin{bmatrix} k_{n+2} & l_n & k_{n+1} \\ k_{n+1} & l_{n-1} & k_n \\ k_n & l_{n-2} & k_{n-1} \end{bmatrix}$$

for every $n \geq 2$, and the n^{th} power of N^{bc} is

$$(N^{bc})^n=(-1)^n\begin{cases}\begin{bmatrix}k_{n+2}-k_{n+1} & k_n-k_{n+1}\\ -k_n+k_{n+1} & -k_n+k_{n-1}\end{bmatrix} & \text{for even } n\geq 2\\[2ex] \begin{bmatrix}k_{n+1}-k_{n+2} & k_{n+1}-k_n\\ -k_{n+1}+k_n & -k_{n-1}+k_n\end{bmatrix} & \text{for odd } n\geq 1.\end{cases}$$

Proof. It can be proved by induction on n. □

4. Relationship with Pell and Pell–Lucas Numbers.

Recall that general terms of all balancing numbers can be given in terms of Pell numbers

$$B_n=\frac{P_{2n}}{2},\ b_n=\frac{P_{2n-1}-1}{2},\ C_n=P_{2n}+P_{2n-1} \text{ and } c_n=P_{2n-1}+P_{2n-2}$$

and also in terms of Pell–Lucas numbers

$$B_n=\frac{Q_{2n}+Q_{2n-1}}{8},\ b_n=\frac{Q_{2n}-Q_{2n-1}-4}{8},\ C_n=\frac{Q_{2n}}{2} \text{ and } c_n=\frac{Q_{2n-1}}{2}.$$

Similarly we can give the following theorem.

Theorem 4.10. *The general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers are*

$$B_n^{bc}=P_{2n+1}P_{2n},\ C_n^{bc}=P_{2n+1}^2+P_{2n}^2,\ R_n^{bc}=P_{2n}^2$$

or

$$B_n^{bc}=\frac{Q_{2n+1}Q_{2n}-4}{8},\ C_n^{bc}=\frac{Q_{2n+1}^2+Q_{2n}^2}{8},\ R_n^{bc}=(\frac{Q_{2n+1}-Q_{2n}}{4})^2$$

for $n\geq 1$.

Proof. We deduce from Theorem 2.3 that

$$\begin{aligned}B_n^{bc}&=\frac{B_{2n}+b_{2n+1}}{2}\\ &=\frac{\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}}+\frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}}-\frac{1}{2}}{2}\\ &=\frac{\alpha^{4n+1}(\alpha^{-1}+1)+\beta^{4n+1}(-1-\beta^{-1})}{8\sqrt{2}}-\frac{1}{4}\\ &=\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}-\frac{1}{4}\\ &=\frac{\alpha^{4n+1}+\beta^{4n+1}-(\alpha\beta)^{2n}(\alpha+\beta)}{8}\\ &=\frac{\alpha^{4n+1}-\alpha^{2n+1}\beta^{2n}-\beta^{2n+1}\alpha^{2n}+\beta^{4n+1}}{8}\\ &=\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{2\sqrt{2}}\right)\left(\frac{\alpha^{2n}-\beta^{2n}}{2\sqrt{2}}\right)\\ &=P_{2n+1}P_{2n}.\end{aligned}$$

The others can be proved similarly. □

Conversely, we can give the general terms of Pell and Pell–Lucas numbers in terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers as follows.

Theorem 4.11. *The general terms of Pell and Pell–Lucas numbers are*

$$P_n = \begin{cases} 2(B^{bc}_{\frac{n}{4}} - R^{bc}_{\frac{n}{4}}) & n \equiv 0(mod4) \\ C^{bc}_{\frac{n-1}{4}} & n \equiv 1(mod4) \\ 4B^{bc}_{\frac{n-2}{4}} + C^{bc}_{\frac{n-2}{4}} + 1 & n \equiv 2(mod4) \\ 8B^{bc}_{\frac{n-3}{4}} + 3C^{bc}_{\frac{n-3}{4}} + 2 & n \equiv 3(mod4) \end{cases}$$

and

$$Q_n = \begin{cases} 8R^{bc}_{\frac{n}{4}} + 2 & n \equiv 0(mod4) \\ 8B^{bc}_{\frac{n-1}{4}} + 2 & n \equiv 1(mod4) \\ 8B^{bc}_{\frac{n-2}{4}} + 4C^{bc}_{\frac{n-2}{4}} + 2 & n \equiv 2(mod4) \\ (C^{bc}_{\frac{n+1}{4}} - C^{bc}_{\frac{n-3}{4}})/2 & n \equiv 3(mod4). \end{cases}$$

Proof. It can be proved as in the same way that Theorem 2.3 was proved. □

Thus we construct one-to-one correspondence between all balcobalancing numbers and Pell and Pell–Lucas numbers.

5. Relationship with Triangular and Square Triangular Numbers.

Recall that triangular numbers denoted by T_n are the numbers of the form

$$T_n = \frac{n(n+1)}{2}.$$

It is known that there is a correspondence between balancing (and also cobalancing) numbers and triangular numbers. Indeed from (1.1), we note that n is a balancing number if and only if n^2 is a triangular number since

$$\frac{(n+r)(n+r+1)}{2} = n^2.$$

So

$$T_{B_n+R_n} = B_n^2, \tag{5.12}$$

Similarly from (1.3), n is a cobalancing number if and only if n^2+n is a triangular number since

$$\frac{(n+r)(n+r+1)}{2} = n^2+n.$$

So

$$T_{b_n+r_n} = b_n^2 + b_n.$$

As in (5.12), we can give the following theorem.

Theorem 5.12. *B_n^{bc} is a balcobalancing number if and only if $(B_n^{bc})^2 + \frac{B_n^{bc}}{2}$ is a triangular number, that is,*

$$T_{B_n^{bc}+R_n^{bc}} = (B_n^{bc})^2 + \frac{B_n^{bc}}{2}.$$

Proof. From (2.7), we get $n^2 = 2nr + r^2 + r$ and hence

$$\frac{(n+r)(n+r+1)}{2} = n^2 + \frac{n}{2}.$$

So the result is obvious. □

There are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and is denoted by S_n. Notice that

$$S_n = s_n^2 = \frac{t_n(t_n+1)}{2},$$

where s_n and t_n are the sides of the corresponding square and triangle. We can give the general terms of S_n, s_n and t_n in terms of balancing and cobalancing numbers, namely, $S_n = B_n^2, s_n = B_n$ and $t_n = B_n + b_n$. Their Binet formulas are

$$S_n = \frac{\alpha^{4n}+\beta^{4n}-2}{32}, \ s_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}} \text{ and } t_n = \frac{\alpha^{2n}+\beta^{2n}-2}{4} \tag{5.13}$$

for $n \geq 1$. We can give the general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers in terms of s_n and t_n as follows.

Theorem 5.13. *The general terms of balcobalancing numbers, Lucas–balcobalancing numbers and balcobalancers are*

$$\begin{aligned}
B_n^{bc} &= \frac{2s_{2n+1}-t_{2n+1}-1}{2} \\
C_n^{bc} &= -2s_{2n+1}+2t_{2n+1}+1 \\
R_n^{bc} &= \frac{-4s_{2n+1}+3t_{2n+1}+1}{2}
\end{aligned}$$

for $n \geq 1$.

Proof. From Theorem 2.3, we get

$$\begin{aligned}
B_n^{bc} &= \frac{B_{2n}+b_{2n+1}}{2} \\
&= \frac{\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}} + \frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2}}{2} \\
&= \frac{\alpha^{4n+2}(\frac{1}{2\sqrt{2}}-\frac{1}{4}) + \beta^{4n+2}(-\frac{1}{2\sqrt{2}}-\frac{1}{4}) - \frac{1}{2}}{2} \\
&= \frac{2(\frac{\alpha^{4n+2}-\beta^{4n+2}}{4\sqrt{2}}) - (\frac{\alpha^{4n+2}+\beta^{4n+2}-2}{4}) - 1}{2} \\
&= \frac{2s_{2n+1}-t_{2n+1}-1}{2}
\end{aligned}$$

by (5.13). The others can be proved similarly. □

Conversely, we can give the following theorem.

Theorem 5.14. *The general terms of* S_n, s_n *and* t_n *are*

$$\begin{aligned}
S_n &= \frac{R_n^{bc}}{4} \\
s_n &= \begin{cases} B_{\frac{n}{2}}^{bc} - R_{\frac{n}{2}}^{bc} & n \geq 2 \text{ even} \\ (4B_{\frac{n-1}{2}}^{bc} + C_{\frac{n-1}{2}}^{bc} + 1)/2 & n \geq 1 \text{ odd} \end{cases} \\
t_n &= \begin{cases} 2R_{\frac{n}{2}}^{bc} & n \geq 2 \text{ even} \\ 2B_{\frac{n-1}{2}}^{bc} + C_{\frac{n-1}{2}}^{bc} & n \geq 1 \text{ odd.} \end{cases}
\end{aligned}$$

Proof. From Theorem 2.3, we get

$$\begin{aligned} R_n^{bc} &= \frac{-B_{2n}+b_{2n+1}}{2} \\ &= \frac{-\frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}}+\frac{\alpha^{4n+1}-\beta^{4n+1}}{4\sqrt{2}}-\frac{1}{2}}{2} \\ &= \frac{\alpha^{4n}(-1+\alpha)+\beta^{4n}(1-\beta)}{8\sqrt{2}}-\frac{1}{4} \\ &= \frac{\alpha^{4n}+\beta^{4n}}{8}-\frac{1}{4}. \end{aligned}$$

So from (5.13), we observe that

$$S_n=\frac{\alpha^{4n}+\beta^{4n}-2}{32}=\frac{\frac{\alpha^{4n}+\beta^{4n}}{8}-\frac{1}{4}}{4}=\frac{R_n^{bc}}{4}$$

as we wanted. The others can be proved similarly. □

Thus we construct one-to-one correspondence between all balcobalancing numbers and square triangular numbers.

Finally, we want to construct a correspondence between triangular and square triangular numbers via balcobalancing numbers, that is, we want to find out that for which balcobalancing numbers m, the equation $T_m=S_n$ holds. The answer is given below.

Theorem 5.15. *For triangular numbers T_n and square triangular numbers S_n, we have*

(1) if $n\geq 1$ is odd, then

$$T_{2B^{bc}_{\frac{n-1}{2}}+C^{bc}_{\frac{n-1}{2}}}=S_n.$$

(2) if $n\geq 2$ is even, then

$$T_{-2B^{bc}_{\frac{n}{2}}+C^{bc}_{\frac{n}{2}}-1}=S_n.$$

Proof. (1) Let $n\geq 1$ be odd. Then

$$\begin{aligned} T_{2B^{bc}_{\frac{n-1}{2}}+C^{bc}_{\frac{n-1}{2}}} &= \frac{(2B^{bc}_{\frac{n-1}{2}}+C^{bc}_{\frac{n-1}{2}})(2B^{bc}_{\frac{n-1}{2}}+C^{bc}_{\frac{n-1}{2}}+1)}{2} \\ &= \left\{\begin{array}{c} \left[2\left(\frac{\alpha^{2n-1}+\beta^{2n-1}}{8}-\frac{1}{4}\right)+\frac{\alpha^{2n-1}-\beta^{2n-1}}{2\sqrt{2}}\right]\times \\ \left[2\left(\frac{\alpha^{2n-1}+\beta^{2n-1}}{8}-\frac{1}{4}\right)+\frac{\alpha^{2n-1}-\beta^{2n-1}}{2\sqrt{2}}+1\right] \end{array}\right\}/2 \\ &= \frac{\left(\alpha^{2n}+\beta^{2n}-2\right)\left(\alpha^{2n}+\beta^{2n}+2\right)}{32} \\ &= \frac{\alpha^{4n}+\beta^{4n}-2}{32} \\ &= S_n \end{aligned}$$

by (5.13). The other case can be proved similarly. □

6. Sums of Balcobalancing Numbers.

Theorem 6.16. *The sum of first $n-$terms of B_n^{bc}, C_n^{bc} and R_n^{bc} is*

$$\sum_{i=1}^{n} B_i^{bc} = \frac{b_{2n+2} - 2n - 2}{8}$$
$$\sum_{i=1}^{n} C_i^{bc} = \frac{c_{2n+2} - 7}{8}$$
$$\sum_{i=1}^{n} R_i^{bc} = \frac{B_{2n+1} - 2n - 1}{8}$$

for $n \geq 1$.

Proof. Recall that $B_n^{bc} = \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}$ by Theorem 3.6. Since

$$\sum_{i=1}^{n}\alpha^{4i+1} = \frac{-\alpha^3(1-\alpha^{4n})}{4\sqrt{2}} \quad \text{and} \quad \sum_{i=1}^{n}\beta^{4i+1} = \frac{\beta^3(1-\beta^{4n})}{4\sqrt{2}},$$

we get

$$\begin{aligned}
\sum_{i=1}^{n} B_i^{bc} &= \sum_{i=1}^{n}\left(\frac{\alpha^{4i+1}+\beta^{4i+1}}{8} - \frac{1}{4}\right) \\
&= \frac{\frac{-\alpha^3(1-\alpha^{4n})}{4\sqrt{2}} + \frac{\beta^3(1-\beta^{4n})}{4\sqrt{2}}}{8} - \frac{n}{4} \\
&= \frac{\alpha^{4n+3} - \beta^{4n+3} - \alpha^3 + \beta^3}{32\sqrt{2}} - \frac{n}{4} \\
&= \frac{\alpha^{4n+3} - \beta^{4n+3} - 10\sqrt{2}}{32\sqrt{2}} - \frac{n}{4} \\
&= \frac{\alpha^{4n+3} - \beta^{4n+3}}{32\sqrt{2}} - \frac{5}{16} - \frac{n}{4} \\
&= \frac{\frac{\alpha^{4n+3}-\beta^{4n+3}}{4\sqrt{2}} - \frac{1}{2} + \frac{1}{2}}{8} - \frac{5}{16} - \frac{n}{4} \\
&= \frac{\frac{\alpha^{4n+3}-\beta^{4n+3}}{4\sqrt{2}} - \frac{1}{2}}{8} - \frac{n+1}{4} \\
&= \frac{b_{2n+2} - 2n - 2}{8}.
\end{aligned}$$

The others can be proved similarly. □

We can give the sums of first $n-$terms of balcobalancing numbers in terms of balancing numbers, sums of first $n-$terms of Lucas–balcobalancing numbers in terms of Lucas–balancing numbers and sums of first $n-$terms of balcobalancers in terms of balancers as follows.

Theorem 6.17. *The sum of first $n-$terms of B_n^{bc}, C_n^{bc} and R_n^{bc} is*

$$\begin{aligned}\sum_{i=1}^{n} B_i^{bc} &= \frac{B_{2n+2}-B_{2n+1}-4n-5}{16}\\ \sum_{i=1}^{n} C_i^{bc} &= \frac{5C_{2n+1}-C_{2n}-14}{16}\\ \sum_{i=1}^{n} R_i^{bc} &= \frac{R_{2n+2}-R_{2n+1}-4n-2}{16}\end{aligned}$$

for $n \geq 1$.

Proof. It can be easily seen that $B_{2n+2} - B_{2n+1} = 2b_{2n+2} + 1$. So from Theorem 6.16, we get

$$\sum_{i=1}^{n} B_i^{bc} = \frac{b_{2n+2}-2n-2}{8} = \frac{\frac{B_{2n+2}-B_{2n+1}-1}{2}-2n-2}{8} = \frac{B_{2n+2}-B_{2n+1}-4n-5}{16}.$$

The others can be proved similarly. □

Recall that the sum of first $n-$terms of all balancing numbers can be given in terms of same balancing numbers, that is,

$$\begin{aligned}\sum_{i=1}^{n} B_i &= \frac{5B_n-B_{n-1}-1}{4}, & \sum_{i=1}^{n} b_i &= \frac{5b_n-b_{n-1}+2-2n}{4}\\ \sum_{i=1}^{n} C_i &= \frac{5C_n-C_{n-1}-2}{4}, & \sum_{i=1}^{n} c_i &= \frac{5c_n-c_{n-1}-2}{4}.\end{aligned}$$

Similarly we can give the sums of first $n-$terms of balcobalancing numbers in terms of balcobalancing numbers, sums of first $n-$terms of Lucas–balcobalancing numbers in terms of Lucas–balcobalancing numbers and sums of first $n-$terms of balcobalancers in terms of balcobalancers as follows.

Theorem 6.18. *The sum of first $n-$terms of B_n^{bc}, C_n^{bc} and R_n^{bc} is*

$$\begin{aligned}\sum_{i=1}^{n} B_i^{bc} &= \frac{33B_n^{bc}-B_{n-1}^{bc}-8n-2}{32}\\ \sum_{i=1}^{n} C_i^{bc} &= \frac{33C_n^{bc}-C_{n-1}^{bc}-28}{32}\\ \sum_{i=1}^{n} R_i^{bc} &= \frac{33R_n^{bc}-R_{n-1}^{bc}-8n+4}{32}\end{aligned}$$

for $n \geq 1$.

Proof. It can be proved similarly. □

We also note that

$$\sum_{i=1}^{n}(-1)^i B_i = \begin{cases} 2B_{\frac{n}{2}}^2 + B_{\frac{n}{2}} C_{\frac{n}{2}} & n \geq 2 \text{ even} \\ -2B_{\frac{n+1}{2}}(b_{\frac{n+1}{2}} + \frac{1}{2}) & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i b_i = \begin{cases} 2B_{\frac{n}{2}}^2 & n \geq 2 \text{ even} \\ -2b_{\frac{n+1}{2}}^2 - 2b_{\frac{n+1}{2}} & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i C_i = \begin{cases} B_n + 8B_{\frac{n}{2}}^2 & n \geq 2 \text{ even} \\ -B_n - 8(b_{\frac{n+1}{2}} + \frac{1}{2})^2 & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i c_i = \begin{cases} B_n & n \geq 2 \text{ even} \\ -B_n & n \geq 1 \text{ odd.} \end{cases}$$

Similarly we can give the following theorem.

Theorem 6.19. *For B_n^{bc}, C_n^{bc} and R_n^{bc}, we get*

$$\sum_{i=1}^{n}(-1)^i B_i^{bc} = \begin{cases} (35B_n^{bc} - B_{n-1}^{bc} - 2)/36 & n \geq 2 \textit{ even} \\ (-35B_n^{bc} + B_{n-1}^{bc} - 10)/36 & n \geq 1 \textit{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i C_i^{bc} = \begin{cases} (35C_n^{bc} - C_{n-1}^{bc} - 30)/36 & n \geq 2 \textit{ even} \\ (-35C_n^{bc} + C_{n-1}^{bc} - 30)/36 & n \geq 1 \textit{ odd} \end{cases}$$

$$\sum_{i=1}^{n}(-1)^i R_i^{bc} = \begin{cases} (35R_n^{bc} - R_{n-1}^{bc} + 4)/36 & n \geq 2 \textit{ even} \\ (-35R_n^{bc} + R_{n-1}^{bc} - 4)/36 & n \geq 1 \textit{ odd.} \end{cases}$$

Proof. It can be proved similarly. □

In [20], Tekcan and Tayat set two integer sequences

$$X_n = \frac{\alpha^{n+1} + \beta^{n+1}}{2} \text{ and } Y_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{2}}$$

for $n \geq 0$ and proved that

$$\sum_{i=1}^{n} B_i C_i = \frac{X_n X_{n-1} Y_n Y_{n-1}}{8}.$$

It can be easily seen that

$$\sum_{i=1}^{n} B_i C_i = \frac{C_{2n+1} - 3}{32}. \tag{6.14}$$

As in (6.14), we can give the following theorem.

Theorem 6.20. *For B_n^{bc} and C_n^{bc}, we get*

$$\sum_{i=1}^{n} B_i^{bc} C_i^{bc} = \frac{(3B_n^{bc} + C_n^{bc})^2 - 1}{12}.$$

Proof. From Theorem 3.6, we find that

$$\sum_{i=1}^{n} B_i^{bc} C_i^{bc} = B_1^{bc} C_1^{bc} + B_2^{bc} C_2^{bc} + \cdots + B_n^{bc} C_n^{bc}$$

$$
\begin{aligned}
&= \left(\frac{\alpha^5+\beta^5}{8}-\frac{1}{4}\right)\left(\frac{\alpha^5-\beta^5}{2\sqrt{2}}\right)+\left(\frac{\alpha^9+\beta^9}{8}-\frac{1}{4}\right)\left(\frac{\alpha^9-\beta^9}{2\sqrt{2}}\right) \\
&\quad +\cdots+\left(\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}-\frac{1}{4}\right)\left(\frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}}\right) \\
&= \frac{(\alpha^{10}+\alpha^{18}+\cdots+\alpha^{8n+2})-(\beta^{10}+\beta^{18}+\cdots+\beta^{8n+2})}{16\sqrt{2}} \\
&\quad -\frac{(\alpha^5+\alpha^9+\cdots+\alpha^{4n+1})-(\beta^5+\beta^9+\cdots+\beta^{4n+1})}{8\sqrt{2}} \\
&= \frac{1}{16\sqrt{2}}\left[\frac{\alpha^{10}(\alpha^{8n}-1)}{\alpha^8-1}-\frac{\beta^{10}(\beta^{8n}-1)}{\beta^8-1}\right]-\frac{1}{8\sqrt{2}}\left[\frac{\alpha^5(\alpha^{4n}-1)}{\alpha^4-1}-\frac{\beta^5(\beta^{4n}-1)}{\beta^4-1}\right] \\
&= \frac{1}{32}\left[\frac{\alpha^{8n+6}+\beta^{8n+6}-198}{24}\right]-\frac{1}{16}\left[\frac{\alpha^{4n+3}+\beta^{4n+3}-14}{4}\right] \\
&= \frac{1}{32.24}\left[(\alpha^{4n+3}+\beta^{4n+3}-6)^2-64\right] \\
&= \frac{1}{12}\left[3\left(\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}-\frac{1}{4}\right)+\left(\frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}}\right)\right]^2-\frac{1}{12} \\
&= \frac{(3B_n^{bc}+C_n^{bc})^2-1}{12}.
\end{aligned}
$$

This completes the proof. □

In [18], Santana and Diaz–Barrero proved that

$$P_{2n+1}\left|\sum_{i=0}^{2n}P_{2i+1}\right. \quad \text{and} \quad P_{2n}\left|\sum_{i=1}^{2n}P_{2i-1}\right. .$$

Similarly we can give the following theorem.

Theorem 6.21. $C_n^{bc}\left|\sum_{i=0}^{4n}P_{2i+1}\right.$.

Proof. As in Theorem 6.16, we find that

$$\sum_{i=0}^{4n}P_{2i+1}=C_n^{bc}(4B_n^{bc}+1).$$

So the result is obvious. □

7. Sums of Pell and Balancing Numbers.

Panda and Ray proved in [11] that the sum of first $2n-1$ Pell numbers is equal to the sum of n^{th} balancing number and its balancer, that is,

$$\sum_{i=1}^{2n-1}P_i=B_n+b_n. \tag{7.15}$$

Later Gözeri, Özkoç and Tekcan proved in [4] that the sum of Pell–Lucas numbers from 0 to $2n-1$ is equal to the the sum of n^{th} Lucas–balancing and Lucas–cobalancing number, that is,

$$\sum_{i=0}^{2n-1}Q_i=C_n+c_n.$$

Since $R_n = b_n$, (7.15) becomes

$$\sum_{i=1}^{2n-1} P_i = B_n + R_n. \tag{7.16}$$

As in (7.16), we can give the following result.

Theorem 7.22. *The sum of even ordered Pell numbers from 1 to $2n$ is equal to the sum of the n^{th} balcobalancing number and its balcobalancer, that is,*

$$\sum_{i=1}^{2n} P_{2i} = B_n^{bc} + R_n^{bc}.$$

Proof. Since $\sum\limits_{i=1}^{2n} \alpha^{2i} = \frac{-\alpha(1-\alpha^{4n})}{2}$ and $\sum\limits_{i=1}^{2n} \beta^{2i} = \frac{-\beta(1-\beta^{4n})}{2}$, we deduce that

$$\begin{aligned}
\sum_{i=1}^{2n} P_{2i} &= \sum_{i=1}^{2n} (\frac{\alpha^{2i} - \beta^{2i}}{2\sqrt{2}}) \\
&= \frac{\frac{-\alpha(1-\alpha^{4n})}{2} - \frac{-\beta(1-\beta^{4n})}{2}}{2\sqrt{2}} \\
&= \frac{\alpha^{4n+1} - \beta^{4n+1}}{4\sqrt{2}} - \frac{1}{2} \\
&= \frac{\alpha^{4n+1}(1+\alpha^{-1}) + \beta^{4n+1}(1+\beta^{-1})}{8} - \frac{1}{2} \\
&= \left(\frac{\alpha^{4n+1} + \beta^{4n+1}}{8} - \frac{1}{4}\right) + \left(\frac{\alpha^{4n} + \beta^{4n}}{8} - \frac{1}{4}\right) \\
&= B_n^{bc} + R_n^{bc}
\end{aligned}$$

by Theorem 3.6. □

Similarly we can give the following theorem which can be proved similarly.

Theorem 7.23. *For the sums of Pell, Pell–Lucas and balancing numbers, we have*

(1) the sum of odd ordered Pell numbers from 1 to $2n$ is equal to the difference of the n^{th} balcobalancing number and its balcobalancer, that is,

$$\sum_{i=1}^{2n} P_{2i-1} = B_n^{bc} - R_n^{bc}.$$

(2) the half of the sum of Pell numbers from 1 to $4n$ is equal to the n^{th} balcobalancing number, that is,

$$\frac{\sum\limits_{i=1}^{4n} P_i}{2} = B_n^{bc}.$$

(3) the sum of Pell–Lucas numbers from 0 to $4n+1$ is equal to the sum of the twelve times of the n^{th} balcobalancing number, four times of the its balcobalancer plus 4, that is,

$$\sum_{i=0}^{4n+1} Q_i = 12B_n^{bc} + 4R_n^{bc} + 4.$$

(4) *the sum of Pell–Lucas numbers from* 1 *to* $4n$ *is equal to the two times of the* n^{th} *Lucas–balcobalancing number minus* 1*, that is,*

$$\sum_{i=1}^{4n} Q_i = 2(C_n^{bc} - 1).$$

(5) *the sum of balancing numbers from* 1 *to* $4n+1$ *is equal to the product of the sum of three times of the* n^{th} *balcobalancing number, its balcobalancer plus* 1 *and the four times of the* n^{th} *balcobalancing number plus* 1*, that is,*

$$\sum_{i=1}^{4n+1} B_i = (3B_n^{bc} + R_n^{bc} + 1)(4B_n^{bc} + 1).$$

In [18], Santana and Diaz–Barrero proved that the sum of first nonzero $4n+1$ terms of Pell numbers is a perfect square, that is,

$$\sum_{i=1}^{4n+1} P_i = \left[\sum_{i=0}^{n} \begin{pmatrix} 2n+1 \\ 2i \end{pmatrix} 2^i\right]^2.$$

In fact this sum is equals to c_{n+1}^2, that is,

$$\sum_{i=1}^{4n+1} P_i = c_{n+1}^2.$$

Similarly we can give the following result.

Theorem 7.24. *The sum of Pell numbers from* 1 *to* $8n+1$ *is a perfect square and is*

$$\sum_{i=1}^{8n+1} P_i = (4B_n^{bc} + 1)^2.$$

Proof. Since $\sum\limits_{i=1}^{n} P_i = \frac{P_{n+1}+P_n-1}{2}$, we get

$$\begin{aligned}
\sum_{i=1}^{8n+1} P_i &= \frac{P_{8n+2} + P_{8n+1} - 1}{2} \\
&= \frac{\frac{\alpha^{8n+2}-\beta^{8n+2}}{2\sqrt{2}} + \frac{\alpha^{8n+1}-\beta^{8n+1}}{2\sqrt{2}} - 1}{2} \\
&= \frac{\frac{\alpha^{8n+2}(1+\alpha^{-1})+\beta^{8n+2}(-1-\beta^{-1})}{2\sqrt{2}}}{2} - \frac{1}{2} \\
&= \frac{\alpha^{8n+2} + \beta^{8n+2}}{4} - \frac{1}{2} \\
&= \frac{\alpha^{8n+2} + 2\alpha^{4n+1}\beta^{4n+1} + \beta^{8n+2}}{4}
\end{aligned}$$

$$= 16\left[\left(\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}\right)^2 - 2\left(\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}\right)(\frac{1}{4}) + \frac{1}{16}\right]$$
$$+ 8\left(\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}\right) - 2 + 1$$
$$= 16\left[\frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}\right]^2 + 8\left[\frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}\right] + 1$$
$$= 16(B_n^{bc})^2 + 8B_n^{bc} + 1$$
$$= (4B_n^{bc} + 1)^2$$

by Theorem 3.6. □

Apart from Theorem 7.24, we can give the following theorem which can be proved similarly.

Theorem 7.25. *For the sums of Pell, Pell–Lucas, balancing and Lucas–cobalancing numbers, we have*

(1) the sum of Pell numbers from 1 *to* $8n+3$ *plus* 1 *is a perfect square and is*

$$1 + \sum_{i=1}^{8n+3} P_i = (4B_n^{bc} + 2C_n^{bc} + 1)^2.$$

(2) the sum of odd ordered Pell–Lucas numbers from 1 *to* $4n+2$ *is a perfect square and is*

$$\sum_{i=1}^{4n+2} Q_{2i-1} = (8B_n^{bc} + 2C_n^{bc} + 2)^2.$$

(3) the half of the sum of odd ordered Pell–Lucas numbers from 0 *to* $4n$ *is a perfect square and is*

$$\frac{\sum\limits_{i=0}^{4n} Q_{2i+1}}{2} = (4B_n^{bc} + 1)^2.$$

(4) the sum of odd ordered balancing numbers from 1 *to* $2n+1$ *is a perfect square and is*

$$\sum_{i=1}^{2n+1} B_{2i-1} = (3B_n^{bc} + R_n^{bc} + 1)^2.$$

(5) the sum of Lucas–cobalancing numbers from 1 *to* $4n+2$ *plus* 1 *is a perfect square and is*

$$1 + \sum_{i=1}^{4n+2} c_i = (8B_n^{bc} + 4R_n^{bc} + 3)^2.$$

REFERENCES

[1] Barbeau, E. J. Pell's Equation. *Springer–Verlag New York*, 2003.

[2] Behera, A.; Panda, G. K. On the Square Roots of Triangular Numbers. *The Fibonacci Quart.* **37** (1999), no. 2, 98–105.

[3] Flath, D. E. Introduction to number theory. With errata. Reprint of the 1989 *AMS Chelsea Publishing, Providence, RI*, 2018.

[4] Gözeri, G. K.; Özkoç, A.; Tekcan, A. Some Algebraic Relations on Balancing Numbers. *Utilitas Mathematica* **103** (2017), 217–236.

[5] Kovacs, T; Liptai, K.; Olajos, P. On (a, b)-Balancing Numbers. *Publ. Math. Deb.* **77** (2010), no. (3-4) 485–498.

[6] Liptai, K.; Luca, F.; Pinter, A.; Szalay, L. Generalized Balancing Numbers. *Indag. Mathem. N. S.* **20** (2009), no. 1, 87–100.

[7] Liptai, K. Fibonacci Balancing Numbers. *The Fibonacci Quarterly* **42** (2004), no. 4, 330–340.

[8] Liptai, K. Lucas Balancing Numbers. *Acta Math. Univ. Ostrav.* **14** (2006), 43–47.

[9] Mollin, R. A. *Quadratics.* CRS Press, Boca Raton, New York, London, Tokyo, 1996.

[10] Olajos, P., Properties of Balancing, Cobalancing and Generalized Balancing Numbers. *Annales Mathematicae et Informaticae* **37** (2010), 125–138.

[11] Panda, G. K.; Ray, P. K. Some Links of Balancing and Cobalancing Numbers with Pell and Associated Pell Numbers. *Bul. of Inst. of Math. Acad. Sinica* **6** (2011), no. 1, 41–72.

[12] Panda, G. K.; Ray, P. K. Cobalancing Numbers and Cobalancers. *Int. J. Math. Math. Sci.* **8** (2005), 1189–1200.

[13] Panda, G. K.; Panda, A. K. Almost Balancing Numbers. *Jour. of the Indian Math. Soc.* **82** (2015), no. (3-4), 147–156.

[14] Panda, G. K.; Komatsu, T.; Davala, R. K. Reciprocal Sums of Sequences Involving Balancing and Lucas-balancing Numbers. *Math. Reports* **20 (70)** (2018), no. 2, 201–214.

[15] Panda, A. K. *Some Variants of the Balancing Sequences.* Ph.D. dissertation, National Institute of Technology Rourkela, India, 2017.

[16] Ray, P. K. *Balancing and Cobalancing Numbers*. Ph.D. dissertation, Department of Mathematics, National Institute of Technology, Rourkela, India, 2009.

[17] Ray, P. K. Balancing and Lucas-balancing Sums by Matrix Methods. *Math. Reports* **17 (67)**(2015), no. 2, 225–233.

[18] Santana, S. F.; Diaz–Barrero, J. L. Some Properties of Sums Involving Pell Numbers. *Missouri Journal of Mathematical Science* **18** (2006), no. 1, 33–40.

[19] Szalay, L. On the Resolution of Simultaneous Pell Equations. *Ann. Math. Inform.* **34** (2007), 77–87.

[20] Tekcan, A.; Tayat, M. Generalized Pell numbers, balancing numbers and binary quadratic forms. *Creat. Math. Inform.* **23** (2014), no. 1, 115–122.

[21] Tekcan, A. Almost Balancing, Triangular and Square Triangular Numbers. *Notes on Number Theory and Discrete Math.* **25** (2019), no. 1, 108–121.

[22] Tekcan, A. Sums and Spectral Norms of all Almost Balancing Numbers. *Creat. Math. Inform.* **28** (2019), no. 2, 203–214.

[23] Tengely, S. Balancing Numbers which are Products of Consecutive Integers. *Publ. Math. Deb.* **83** (2013), no. (1-2), 197–205.

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