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Existence of positive solutions for $3n^{\text{th}}$ order boundary value problems involving p-Laplacian

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ABSTRACT. This paper establishes the existence of positive solutions for $3n^{\text{th}}$ order differential equations with p-Laplacian operator

$$(-1)^n[\phi_p(v^{(3n-3)}(t))]''' = g(t, v(t)),\ t \in [0, 1],$$

satisfying the three-point boundary conditions

$$\left.\begin{array}{c} v^{(3i)}(0) = 0,\ v^{(3i+1)}(0) = 0,\ v^{(3i+1)}(1) = \alpha_{i+1} v^{(3i+1)}(\eta),\ \text{for } 0 \le i \le n-2, \\ [\phi_p(v^{(3n-3)}(t))]_{\text{at } t=0} = 0,\ [\phi_p(v^{(3n-3)}(t))]'_{\text{at } t=0} = 0, \\ [\phi_p(v^{(3n-3)}(t))]'_{\text{at } t=1} = \alpha_n [\phi_p(v^{(3n-3)}(t))]'_{\text{at } t=\eta}, \end{array}\right\}$$

where $n \ge 2$, $\eta \in (0, 1)$, $\alpha_j \in (0, \frac{1}{\eta})$ is a constant for $1 \le j \le n$, by an application of Guo–Krasnosel'skii fixed point theorem.

1. INTRODUCTION

The theory of differential equations has been used in the modeling of physical, biological and medical sciences aspects as well as economics to determine the optimal investment strategies. The boundary value problem involving p-Laplacian operator arises in various real life applications such as biophysics, plasma physics, image processing, rheology, glaciology, turbulent filtration in porous media, radiation of heat etc. Due to the wide applicability in most areas, the researchers have concentrated on establishing the existence of positive solutions to p-Laplacian problems, see [1, 2, 6, 8, 11, 17, 12, 28]. For applications and recent developments, we refer [4, 20, 22, 23, 24].

We establish the existence of positive solutions for $3n^{\text{th}}$ order three-point boundary value problems involving p-Laplacian

$$(-1)^n[\phi_p(v^{(3n-3)}(t))]''' = g(t, v(t)),\ t \in [0, 1], \tag{1.1}$$

$$\left.\begin{array}{c} v^{(3i)}(0) = 0, v^{(3i+1)}(0) = 0,\ v^{(3i+1)}(1) = \alpha_{i+1} v^{(3i+1)}(\eta),\ \text{for } 0 \le i \le n-2, \\ [\phi_p(v^{(3n-3)}(t))]_{\text{at } t=0} = 0,\ [\phi_p(v^{(3n-3)}(t))]'_{\text{at } t=0} = 0, \\ [\phi_p(v^{(3n-3)}(t))]'_{\text{at } t=1} = \alpha_n [\phi_p(v^{(3n-3)}(t))]'_{\text{at } t=\eta}, \end{array}\right\} \tag{1.2}$$

where $n \ge 2$, $\eta \in (0, 1)$, $\alpha_j \in (0, \frac{1}{\eta})$ is a constant for $1 \le j \le n$, and the function $g : [0, 1] \times R^+ \to R^+$ is continuous. The important and significant operator is one-dimensional p-Laplacian operator and is defined by $\phi_p(\tau) = |\tau|^{p-2}\tau$, where $p > 1$, $\phi_p^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. By taking $n = 1$ and $p = 2$ in (1.1) and (1.2), reduces to third order three-point boundary value problem and studied the existence of positive solutions based on various methods by many researchers, see [7, 13, 14, 15, 16, 18, 19, 27, 29, 31]. However, as per our

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knowledge, very few works have been found in the literature on the existence of positive solutions of higher order boundary value problems with p-Laplacian, see [5, 21, 25, 26, 30]. Motivated by above papers, we extend the results to the problem (1.1)-(1.2).

For establishing the new results, throughout this paper we assume the following conditions are fulfilled:

(C1) α_j is a constant such that $0 < \eta\alpha_j < 1$ for $1 \leq j \leq n$, where $\eta \in (0,1)$.
(C2) the function $g(t,v)$ is a non-decreasing for the second variable v, and
(C3) $0 < \int_0^1 G_n(t,s)ds < \infty$.

The remaining part of the paper is organized as follows. The solution of the problem (1.1)-(1.2) is expressed into an equivalent integral equation in terms of Green functions and then certain inequalities are established for the Green functions in Section 2. The existence of positive solutions to the problem (1.1)-(1.2) is established in Section 3. At the end, the established results are demonstrated with examples.

2. Green's function and its bounds

The present section contains some preparatory results that are necessary for establishing the main results. For this, we first build a Green's function $G_i(t,s)$ $(1 \leq i \leq n)$ for the following third order three-point problem

$$-v'''(t) = 0,\ t \in [0,1], \tag{2.3}$$

$$v(0) = 0,\ v'(0) = 0,\ v'(1) = \alpha_i v'(\eta). \tag{2.4}$$

Using $G_i(t,s)$ $(1 \leq i \leq n-1)$, we obtain Green's function $H_{n-1}(t,s)$ recursively for the following problem of $(3n-3)^{\text{th}}$ order with three-point boundary conditions

$$(-1)^{n-1}v^{(3n-3)}(t) = 0,\ t \in [0,1], \tag{2.5}$$

$$v^{(3i)}(0) = 0,\ v^{(3i+1)}(0) = 0,\ v^{(3i+1)}(1) = \alpha_{i+1}v^{(3i+1)}(\eta), \tag{2.6}$$

for $0 \leq i \leq n-2$, where $n \geq 3$.

Lemma 2.1. *If the assumption (C1) is fulfilled, then the Green's function $G_i(t,s)$ $(1 \leq i \leq n)$ of the problem* (2.3)-(2.4) *is*

$$G_i(t,s) = \begin{cases} G_{i1}(t,s), & 0 \leq t \leq s \leq \eta \leq 1, \\ G_{i2}(t,s), & 0 \leq s \leq \min\{t,\eta\} \leq 1, \\ G_{i3}(t,s), & 0 \leq \max\{t,\eta\} \leq s \leq 1, \\ G_{i4}(t,s), & 0 \leq \eta \leq s \leq t \leq 1, \end{cases} \tag{2.7}$$

where

$$G_{i1}(t,s) = \frac{t^2}{2}(1-s) + \frac{\alpha_i t^2}{2(1-\eta\alpha_i)}s(1-\eta),$$

$$G_{i2}(t,s) = \frac{1}{2}[t^2(1-s) - (t-s)^2] + \frac{\alpha_i t^2}{2(1-\eta\alpha_i)}s(1-\eta),$$

$$G_{i3}(t,s) = \frac{t^2}{2}(1-s) + \frac{\alpha_i t^2}{2(1-\eta\alpha_i)}\eta(1-s),$$

$$G_{i4}(t,s) = \frac{1}{2}[t^2(1-s) - (t-s)^2] + \frac{\alpha_i t^2}{2(1-\eta\alpha_i)}\eta(1-s).$$

Proof. The result can be proved as in [27]. □

Lemma 2.2. *Suppose the assumption (C1) is fulfilled. If we denote $G_1(t,s) = H_1(t,s)$ and define*

$$H_i(t,s) = \int_0^1 H_{i-1}(t,r)G_i(r,s)dr, \text{ for } 2 \le i \le n, \tag{2.8}$$

recursively, then the Green's function for $3n^{th}$ order problem

$$(-1)^n v^{(3n)}(t) = 0,\ t \in [0,1],$$
$$v^{(3i)}(0) = 0,\, v^{(3i+1)}(0) = 0,\ v^{(3i+1)}(1) = \alpha_{i+1} v^{(3i+1)}(\eta),$$

for $0 \le i \le n-1$ and $n \ge 2$, is given by $H_n(t,s)$.

Proof. One can establish the result in a recursive manner. □

By using the Lemmas 2.1 and 2.2, the solution of the problem (1.1)-(1.2) is

$$v(t) = \int_0^1 H_{n-1}(t,s)\phi_q\left[\int_0^1 G_n(s,r)g(r,v(r))dr\right]ds. \tag{2.9}$$

Lemma 2.3. *If the assumption (C1) is fulfilled, then $G_i(t,s)$ $(1 \le i \le n)$ fulfills the following conditions:*

(i) $G_i(t,s) \ge 0$, *for all $t,s \in [0,1]$,*
(ii) $G_i(t,s) \le G_i(1,s)$, *for all $t,s \in [0,1]$,*
(iii) $\min_{t\in I} G_i(t,s) \ge \eta^2 G_i(1,s)$, *for all $s \in [0,1]$, where $I = [\eta, 1]$.*

Proof. We can establish the result by simple algebraic computations. □

Lemma 2.4. *If the assumption (C1) is fulfilled and if we define $\mathcal{K}_n = \prod_{i=1}^{n-1} K_i$, $\mathcal{L}_n = \prod_{i=1}^{n-1} L_i$, then $H_n(t,s)$ fulfills the following conditions:*

(i) $0 \le H_n(t,s) \le \mathcal{K}_n G_n(1,s)$, *for all $t,s \in [0,1]$,*
(ii) $H_n(t,s) \ge \eta^{2n}\mathcal{L}_n G_n(1,s)$, *for all $t \in I$ and $s \in [0,1]$,*

where $K_i = \int_0^1 G_i(1,r)dr$ and $L_i = \int_{r\in I} G_i(1,r)dr$, for $1 \le i \le n$.

Proof. We can prove these inequalities by using induction on n. □

The fixed point theorem of Guo–Krasnosel'skii stated below is used as the fundamental tool to establish the existence of positive solutions of the problem (1.1)-(1.2).

Theorem 2.1. [3, 9, 10] *Let B be a Banach Space and the set $\kappa \subseteq \mathsf{B}$ be a cone. Assume the sets Ω_1 and Ω_2 are any two open subsets of B such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Further, suppose that the operator $\mathsf{T} : \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \kappa$ is a completely continuous such that, either*

(i) $\|\mathsf{T}v\| \le \|v\|,\ v \in \kappa \cap \partial\Omega_1$ *and* $\|\mathsf{T}v\| \ge \|v\|,\ v \in \kappa \cap \partial\Omega_2$, *or*
(ii) $\|\mathsf{T}v\| \ge \|v\|,\ v \in \kappa \cap \partial\Omega_1$ *and* $\|\mathsf{T}v\| \le \|v\|,\ v \in \kappa \cap \partial\Omega_2$ *holds.*

Then the operator T has a fixed point in $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence of positive solutions

This section presents the existence of positive solutions to the problem (1.1)-(1.2). For our construction, let $\mathsf{B} = \{v : v \in C[0,1]\}$ be a Banach space with norm, $\|v\| = \max_{t\in[0,1]} |v(t)|$. Let $\mathcal{M} = \frac{\eta^{2n-2}\mathcal{L}_{n-1}}{\mathcal{K}_{n-1}}$. We now consider the set

$$\kappa = \{v \in \mathsf{B} : v(t) \ge 0 \text{ on } t \in [0,1] \text{ and } \min_{t\in I} v(t) \ge \mathcal{M}\|v\|\}.$$

Then the set κ is a cone in B. Define an operator $\mathsf{T}:\kappa\to\mathsf{B}$ as

$$\mathsf{T}v(t)=\int_0^1 H_{n-1}(t,s)\phi_q\left[\int_0^1 G_n(s,r)g(r,v(r))dr\right]ds. \tag{3.10}$$

The following non-negative extended real numbers g_0, g^0, g_∞ and g^∞ are defined as

$$g_0=\lim_{v\to 0^+}\min_{t\in[0,1]}\frac{g(t,v)}{\phi_p(v)},\ g^0=\lim_{v\to 0^+}\max_{t\in[0,1]}\frac{g(t,v)}{\phi_p(v)},$$

$$g_\infty=\lim_{v\to\infty}\min_{t\in[0,1]}\frac{g(t,v)}{\phi_p(v)}\text{ and } g^\infty=\lim_{v\to\infty}\max_{t\in[0,1]}\frac{g(t,v)}{\phi_p(v)},$$

and also assume that they will exist. The case $g^0=0$ and $g_\infty=\infty$ is called superlinear and the case $g_0=\infty$ and $g^\infty=0$ is called the sublinear.

Lemma 3.5. *If the operator $\mathsf{T}:\kappa\to\mathsf{B}$ is defined by* (3.10)*, then T is a self map on the cone κ.*

Proof. By $(C3)$ and the non-negative of $G_n(t,s)$, $H_{n-1}(t,s)$ in Lemmas 2.3, 2.4 that $\mathsf{T}v(t)\geq 0$ for $v\in\kappa$ and $t\in[0,1]$. Then, by Lemma 2.4 and for $v\in\kappa$, we get

$$\begin{aligned}\mathsf{T}v(t)&=\int_0^1 H_{n-1}(t,s)\phi_q\bigg(\int_0^1 G_n(s,r)g\big(r,v(r)\big)dr\bigg)ds\\&\leq\mathcal{K}_{n-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(s,r)g\big(r,v(r)\big)dr\bigg)ds\end{aligned}$$

so that

$$\|\mathsf{T}v\|\leq\mathcal{K}_{n-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(s,r)g\big(r,v(r)\big)dr\bigg)ds. \tag{3.11}$$

Now, by Lemma 2.4, for $v\in\kappa$ that

$$\begin{aligned}\min_{t\in I}\mathsf{T}v(t)&=\min_{t\in I}\left\{\int_0^1 H_{n-1}(t,s)\phi_q\bigg(\int_0^1 G_n(s,r)g\big(r,v(r)\big)dr\bigg)ds\right\}\\&\geq\eta^{2n-2}\mathcal{L}_{n-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(s,r)g\big(r,v(r)\big)dr\bigg)ds\\&\geq\left(\frac{\eta^{2n-2}\mathcal{L}_{n-1}}{\mathcal{K}_{n-1}}\right)\|Tv\|=\mathcal{M}\|\mathsf{T}v\|.\end{aligned}$$

Therefore, $\mathsf{T}:\kappa\to\kappa$ and hence, it is proved. □

Moreover, the operator T is completely continuous by Arzela–Ascoli theorem. Now, we prove the existence of positive solutions to the problem (1.1)-(1.2) by superlinear case and sublinear case.

Theorem 3.2. *Suppose the assumptions* ($C1$), ($C2$) *and* ($C3$) *are fulfilled. If $g^0=0$ and $g_\infty=\infty$ hold, then the problem* (1.1)-(1.2) *has at least one positive solution in the cone κ.*

Proof. From the definition of $g^0=0$, there exist $\xi_1>0$ and $J_1>0$ such that $g(t,v)\leq\xi_1\phi_p(v)$, for $0<v\leq J_1$, where ξ_1 satisfies

$$(\xi_1)^{q-1}\mathcal{K}_{n-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(1,r)dr\bigg)ds\leq 1. \tag{3.12}$$

Choose $v \in \kappa$ with $\|v\| = J_1$. Then, for $t \in [0,1]$, and by Lemmas 2.3, 2.4, we get

$$\begin{aligned}\mathsf{T}v(t) &= \int_0^1 H_{n-1}(t,s)\phi_q\bigg(\int_0^1 G_n(s,r)g\big(r,v(r)\big)dr\bigg)ds\\ &\leq \mathcal{K}_{n-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(1,r)\xi_1\phi_p(v)dr\bigg)ds\\ &\leq (\xi_1)^{q-1}\mathcal{K}_{n-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(1,r)dr\bigg)ds\|v\| \leq \|v\|.\end{aligned}$$

Hence, $\|\mathsf{T}v\| \leq \|v\|$. Now, if we are setting $\Omega_1 = \{v \in \mathsf{B} : \|v\| < J_1\}$, then

$$\|\mathsf{T}u\| \leq \|v\|, \text{ for } v \in \kappa \cap \partial\Omega_1. \tag{3.13}$$

Further, since $g_\infty = \infty$, there exist $\xi_2 > 0$ and $\bar{J}_2 > 0$ such that $g(t,v(t)) \geq \xi_2\phi_p(v)$, for $v \geq \bar{J}_2$, where ξ_2 satisfies

$$(\xi_2)^{q-1}\bigg(\frac{\eta^{4n-2}\mathcal{L}_n^2}{\mathcal{K}_n L_{n-1}}\bigg)\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\eta^2\int_{r\in I} G_n(1,r)dr\bigg)ds \geq 1. \tag{3.14}$$

Let $J_2 = \max\left\{2J_1, \frac{\bar{J}_2}{\mathcal{M}}\right\}$. Choose $v \in \kappa$ and $\|v\| = J_2$. Then $\min\limits_{t\in I} v(t) \geq \mathcal{M}\|v\| \geq \bar{J}_2$. Using the Lemmas 2.3, 2.4, and for $t \in [0,1]$, we obtain

$$\begin{aligned}\mathsf{T}v(t) &= \int_0^1 H_{n-1}(t,s)\phi_q\bigg(\int_0^1 G_n(s,r)g(r,v(r))dr\bigg)ds\\ &\geq \min_{t\in I}\left\{\int_0^1 H_{n-1}(t,s)\phi_q\bigg(\int_0^1 G_n(s,r)g(r,v(r))dr\bigg)ds\right\}\\ &\geq \eta^{2n-2}\mathcal{L}_{n-1}\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(s,r)g(r,v(r))dr\bigg)ds\\ &\geq \eta^{2n-2}\mathcal{L}_{n-1}\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\eta^2\int_{r\in I} G_n(1,r)\xi_2\phi_p(v)dr\bigg)ds\\ &\geq \eta^{2n-2}\mathcal{L}_{n-1}(\xi_2)^{q-1}\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\eta^2\int_{r\in I} G_n(1,r)dr\bigg)\mathcal{M}\|v\|ds\\ &\geq \bigg(\frac{\eta^{4n-2}\mathcal{L}_n^2}{\mathcal{K}_n L_{n-1}}\bigg)(\xi_2)^{q-1}\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\eta^2\int_{r\in I} G_n(1,r)dr\bigg)\|v\|ds \geq \|v\|.\end{aligned}$$

Therefore, $\|\mathsf{T}v\| \geq \|v\|$. So, if we take $\Omega_2 = \{v \in \mathsf{B} : \|v\| < J_2\}$, then

$$\|\mathsf{T}v\| \geq \|v\| \text{ for } v \in \kappa \cap \partial\Omega_2. \tag{3.15}$$

By an application of Theorem 2.1 to the equations (3.13) and (3.15), the operator T has a fixed point $v \in \kappa \cap (\Omega_2 \setminus \bar{\Omega}_1)$ and that v is a positive solution to the problem (1.1)-(1.2). □

Theorem 3.3. *Suppose the assumptions* (*C1*), (*C2*) *and* (*C3*) *are fulfilled. If* $g_0 = \infty$ *and* $g^\infty = 0$ *hold, then the problem* (1.1)-(1.2) *has at least one positive solution in the cone* κ.

Proof. From the definition of $g_0 = \infty$, there exist $\xi_3 > 0$ and $J_3 > 0$ such that $g(t,v) \geq \xi_3\phi_p(v)$, for $0 < v \leq J_3$, where $\xi_3 \geq \xi_2$ and ξ_2 is given in (3.14). Let $v \in \kappa$ and $\|v\| = J_3$.

Then, for $t \in [0,1]$ and by Lemmas 2.3, 2.4, we get

$$
\begin{aligned}
\mathsf{T}v(t) &= \int_0^1 H_{n-1}(t,s)\phi_q\bigg(\int_0^1 G_n(s,r)g(r,v(r))dr\bigg)ds \\
&\geq \min_{t\in I}\left\{\int_0^1 H_{n-1}(t,s)\phi_q\bigg(\int_0^1 G_n(s,r)g(r,v(r))dr\bigg)ds\right\} \\
&\geq \eta^{2n-2}\mathcal{L}_{n-1}\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(s,r)g(r,v(r))dr\bigg)ds \\
&\geq \eta^{2n-2}\mathcal{L}_{n-1}\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\eta^2\int_{r\in I} G_n(1,r)\xi_3\phi_p(v)dr\bigg)ds \\
&\geq \eta^{2n-2}\mathcal{L}_{n-1}(\xi_3)^{q-1}\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\eta^2\int_{r\in I} G_n(1,r)dr\bigg)\mathcal{M}\|v\|ds \\
&\geq \left(\frac{\eta^{4n-2}\mathcal{L}_n^2}{\mathcal{K}_n L_{n-1}}\right)(\xi_3)^{q-1}\int_{s\in I} G_{n-1}(1,s)\phi_q\bigg(\eta^2\int_{r\in I} G_n(1,r)dr\bigg)\|v\|ds \geq \|v\|.
\end{aligned}
$$

Therefore, $\|\mathsf{T}v\| \geq \|v\|$. Now, if we are setting $\Omega_3 = \{v \in \mathsf{B} : \|v\| < J_3\}$, then

$$
\|\mathsf{T}v\| \geq \|v\|, \text{ for } v \in \kappa \cap \partial\Omega_3. \tag{3.16}
$$

Next, since $g^\infty = 0$, there exist $\xi_4 > 0$ and $\bar{J}_4 > 0$ such that $g(t,v(t)) \leq \xi_4\phi_p(v),$ for $v \geq \bar{J}_4$, where $\xi_4 \leq \xi_1$ and ξ_1 is given in (3.12). Set $g^*(t,v) = \sup_{0\leq s\leq v} g(t,s)$. Then, it is obvious that g^* is a non-decreasing real-valued function, $g \leq g^*$ and

$$
\lim_{v\to\infty}\frac{g^*(t,v)}{v} = 0.
$$

It follows that there exists $J_4 > \max\{2J_3, \bar{J}_4\}$ such that $g^*(t,v) \leq g^*(t,J_4),$ for $0 < v \leq J_4$. Choose $v \in \kappa$ with $\|v\| = J_4$. Then

$$
\begin{aligned}
\mathsf{T}v(t) &= \int_0^1 H_{n-1}(t,s)\phi_q\bigg(\int_0^1 G_n(s,r)g(r,v(r))dr\bigg)ds \\
&\leq \mathcal{K}_{n-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(s,r)g(r,J_4)dr\bigg)ds \\
&\leq \mathcal{K}_{n-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(s,r)\xi_4\phi_p(J_4)dr\bigg)ds \\
&\leq \mathcal{K}_{n-1}(\xi_4)^{q-1}\int_0^1 G_{n-1}(1,s)\phi_q\bigg(\int_0^1 G_n(1,r)dr\bigg)dsJ_4 \\
&\leq J_4 = \|v\|.
\end{aligned}
$$

Hence, $\|\mathsf{T}v\| \leq \|v\|$. So, if we are setting $\Omega_4 = \{v \in \mathsf{B} : \|v\| < J_4\}$, then

$$
\|\mathsf{T}v\| \leq \|v\|, \text{ for } v \in \kappa \cap \partial\Omega_4. \tag{3.17}
$$

Using Theorem 2.1, the equations (3.16) and (3.17) yields that the operator T has a fixed point $v \in \kappa \cap (\Omega_4 \setminus \bar{\Omega}_3)$ and that v is a positive solution to the problem (1.1)-(1.2). $\square$

Let us consider the examples to demonstrate our results.

Example 3.1. Let $n = 1$, $p = 2$, $\eta = \frac{1}{2}$, $\alpha_1 = \frac{1}{3}$. Consider the p-Laplacian problem

$$
-v'''(t) = g(t,v(t)),\ t \in [0,1], \tag{3.18}
$$

$$
v(0) = 0,\ v'(0) = 0,\ v'(1) = \frac{1}{3}v'\left(\frac{1}{2}\right). \tag{3.19}
$$

(a) If $g(t,v(t)) = e^{t(1-t)}v^{3/2}$, then all the conditions of the Theorem 3.2 are satisfied. Therefore, the problem (3.18)-(3.19) has at least one positive solution.

(b) If $g(t,v(t)) = (1+t^2)e^{-v}$, then all the conditions of the Theorem 3.3 are satisfied. Therefore, the problem (3.18)-(3.19) has at least one positive solution.

Example 3.2. Let $n = 3$, $\eta = \frac{1}{3}$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{3}{2}$, $\alpha_3 = 2$. Consider the p-Laplacian problem

$$(-1)^3[\phi_p(v^{(6)}(t))]''' = g(t,v(t)),\ t \in [0,1], \tag{3.20}$$

$$\left.\begin{aligned} &v(0)=0,\ v'(0)=0,\ v'(1)=\frac{1}{2}v'\left(\frac{1}{3}\right),\ v'''(0)=0,\ v^{(4)}(0)=0,\\ &v^{(4)}(1)=\frac{3}{2}v^{(4)}\left(\frac{1}{3}\right),[\phi_p(v^{(6)}(0))]=0,\ [\phi_p(v^{(6)}(t))]'_{\text{at } t=0}=0,\\ &[\phi_p(v^{(6)}(t))]'_{\text{at } t=1}=2\left[\phi_p\left(v^{(6)}\left(t\right)\right)\right]'_{\text{at } t=\eta=\frac{1}{3}}. \end{aligned}\right\} \tag{3.21}$$

By setting $p = 2$ and some algebraic calculations, we obtain $K_1 = 0.133333$, $K_2 = 0.33333$, $L_1 = 0.08395$, $L_2 = 0.17284$, $\mathcal{K}_3 = 0.04444$, $\mathcal{L}_3 = 0.01451$ and $\mathcal{M} = 0.00777$.

(a) If $g(t,v(t)) = (1+e^{t(1-2t)})v^2$, then all the conditions of the Theorem 3.2 are satisfied. Therefore, the problem (3.20)-(3.21) has at least one positive solution.

(b) If $g(t,v(t)) = (t^3+1)^{2/3}v^{3/4}$, then all the conditions of the Theorem 3.3 are satisfied. Therefore, the problem (3.20)-(3.21) has at least one positive solution.

4. Conclusions

In this paper, we proved the existence of at least one positive solution to $3n^{\text{th}}$ order boundary value problem with p-Laplacian by an application of Guo–Krasnosel'skii fixed point theorem. It will be interesting to obtain multiple positive solutions for the problem involving more general nonlinear terms by applying various fixed point theorems.

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