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On extensions of pseudo-valuations on BCK algebras

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ABSTRACT. In this paper we define a *pseudo-valuation* on a BCK algebra $(A, \rightarrow, 1)$ as a real-valued function $v : A \rightarrow \mathbf{R}$ satisfying $v(1) = 0$ and $v(x \rightarrow y) \geq v(y) - v(x)$, for every $x, y \in A$; v is called a *valuation* if $x = 1$ whenever $v(x) = 0$. We prove that every pseudo-valuation (valuation) v induces a pseudo-metric (metric) on A defined by $d_v(x, y) = v(x \rightarrow y) + v(y \rightarrow x)$ for every $x, y \in A$, where $\rightarrow$ is uniformly continuous in both variables. The aim of this paper is to provide several theorems on extensions of pseudo-valuations (valuations) on BCK algebras.

1. INTRODUCTION AND BASIC RESULTS

BCK algebras are an important class of logical algebras investigated by many researchers (see [2], [3], [6], [8], [9], [10], [13]). BCK algebras were originally introduced by Isèki in [9]. Further properties of them and their connections with other fuzzy structures were established by Iorgulescu in [8].

In [1], Busneag defined pseudo-valuations on Hilbert algebras and proved that every pseudo-valuation induced a pseudo-metric. Using this model, in [4], [5], [7], [11], [12], [14], [15], [16] is introduced the notion of pseudo-valuation on BCK, BCI, BCC algebras and several properties are discussed.

The main goal of this paper is to introduce the notions of pseudo-valuation and valuation on BCK algebras and to prove theorems on extensions of pseudo-valuations (valuations) on BCK algebras.

The paper is organized as follows: In Section 1 we review some relevant concepts relative to BCK algebras. In Section 2 we introduce the notions of pseudo-valuation and valuation on BCK algebras and we induce a pseudo-metric by using pseudo-valuations on BCK algebras (Theorem 2.1). Also, we show that the binary operation $\rightarrow$ is uniformly continuous, see Corollary 2.1. Finally, we prove some theorems (2.2 and 2.3) on extensions of pseudo-valuations (valuations) on BCK algebras. Section 3 contains results about pseudo-valuations on the dual BCK algebra, see Theorem 3.4 and the final section contains conclusions, open problems and future work about the presented topics.

A BCK algebra is an algebra $(A, \rightarrow, 1)$ of type (2,0) satisfying:

(a_1) $x \rightarrow x = 1$;
(a_2) If $x \rightarrow y = y \rightarrow x = 1$, then $x = y$;
(B) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$;
(C) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
(K) $x \leq y \rightarrow x$.

Example 1.1. ([8]) We give an example of a finite bounded BCK algebra. Let $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but a, b are incomparable. A becomes a BCK algebra relative to the

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following operation:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

.

If A is a BCK algebra, then the relation $\leq$ defined by $x \leq y$ iff $x \rightarrow y = 1$ is a partial order on A (called the *natural order*); with respect to this order 1 is the largest element of A. A *bounded BCK algebra* is a BCK algebra with a smallest element 0 relative to the natural order. For a BCK algebra A, two elements $x, y \in A$ and a natural number $n \geq 1$ we denote $x \rightarrow_n y = x \rightarrow (x \rightarrow ...(x \rightarrow y)...)$, where n indicates the number of occurrences of x.

In BCK algebras we have the following rules of calculus (see [2] and [10]):

(c_1) $x \rightarrow 1 = 1, 1 \rightarrow x = x, x \leq (x \rightarrow y) \rightarrow y, x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$;

(c_2) If $x \leq y$, then for every $z \in A$, $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

For a BCK algebra A and $x_1, ..., x_n, x \in A$ $(n \geq 1)$ we define $(x_1, ..., x_n; x) = x_1 \rightarrow (x_2 \rightarrow ...(x_n \rightarrow x)...)$. If σ is a permutation of $\{1, ..., n-1\}$, $n \geq 2$, then:

(c_3) $(x_{\sigma(1)}, ..., x_{\sigma(n-1)}; x_n) = (x_1, ..., x_{n-1}; x_n)$;

(c_4) $x \rightarrow (x_1, ..., x_{n-1}; x_n) = (x, x_1, ..., x_{n-1}; x_n)$;

Let A be a BCK algebra. A subset D of A is called a *deductive system* of A if $1 \in D$ and $x, x \rightarrow y \in D$ implies $y \in D$.

Clearly, if D is a deductive system of A and $x \leq y$ with $x \in D$, then $y \in D$.

We denote by $Ds(A)$ the set of all deductive systems of a BCK algebra A.

For a non-empty subset $X \subseteq A$, we denote by $< X >= \cap\{D \in Ds(A) : X \subseteq D\}$; $< X >$ is called the *deductive system of A generated by X*. If $X = \{x_1, ..., x_n\}$ we denote $< \{x_1, ..., x_n\} >$ by $< x_1, ..., x_n >$; also, we denote by $< a >$ the deductive system generated by $\{a\}$. It is easy to prove (see [2]) that $< a >= \{x \in A : a \rightarrow_n x = 1$, for some natural number $n \geq 1\}$ ($< a >$ is called *principal*).

Let A be a bounded BCK algebra. An element $x \in A$ is called *boolean* (see [6]) if $< x > \cap < x^* >= \{1\}$. Let $B(A)$ the set of all boolean elements of A.

In [2] it is proved that if A is a BCK algebra and $X \subseteq A$ then

(c_5) $< X > = \{x \in A : (x_1, ..., x_n; x) = 1$, for some $x_1, ..., x_n \in X$ and $n \geq 1\}$;

(c_6) If $D_1, D_2 \in Ds(A)$ and we define $D_1 \vee D_2 =< D_1 \cup D_2 >$, then $D_1 \vee D_2 = \{x \in A : d_1 \rightarrow (d_2 \rightarrow x) = 1$, for some $d_1 \in D_1$ and $d_2 \in D_2\}$.

Remark 1.1. If $D \in Ds(A)$, then D is a BCK subalgebra of A (since $1 \in D$ and if $x, y \in D$ from $y \leq x \rightarrow y$ we deduce that $x \rightarrow y \in D$).

2. Pseudo-valuations (valuations) on BCK algebras

Using the model of Hilbert algebras (see [1]), in this section we introduce the notions of pseudo-valuations and valuations on BCK algebras and we prove some theorems of extension for these.

Let A be a BCK algebra. A real-valued function $v : A \rightarrow \mathbf{R}$ is called a *pseudo-valuation* on A if $v(1) = 0$ and $(^*)$: $v(x \rightarrow y) \geq v(y) - v(x)$, for every $x, y \in A$. The pseudo-valuation v is called *valuation* if $v(x) = 0$ implies $x = 1$.

If we interpret A as an implicational calculus, $x \rightarrow y$ as the proposition $x \Rightarrow y$ and 1 as truth, a pseudo-valuation on A can be interpreted as a "falsity-valuation".

Example 2.2. $v : A \rightarrow \mathbf{R}$, $v(x) = 0$ for every $x \in A$ is a pseudo-valuation on A.

Example 2.3. If $D \in Ds(A)$ and $0 \leq r \in \mathbf{R}$, then $v_D : A \to \mathbf{R}$, $v_D(x) = 0$, if $x \in D$ and r otherwise, is a pseudo-valuation on A. Indeed, $v_D(1) = 0$ since $1 \in D$. Let $x, y \in A$. If $x, y \in D$, since $y \leq x \to y$ we obtain $x \to y \in D$. So, $v_D(x \to y) = v_D(x) = v_D(y) = 0$ and $v_D(x \to y) = v_D(y) - v_D(x)$. If $x, y \notin D$, then $v_D(y) - v_D(x) = r - r = 0 \leq v_D(x \to y)$. If $x \notin D$ and $y \in D$ we deduce that $x \to y \in D$, so $0 = v_D(x \to y) \geq v_D(y) - v_D(x) = 0 - r = -r$. If $y \notin D$ and $x \in D$ then $x \to y \notin D$. We obtain $r = v_D(x \to y) = v_D(y) - v_D(x) = r - 0 = r$.

Remark 2.2. Let A be a non trivial BCK algebra, $D \in Ds(A)$, $r \geq 0$ and $v_D : A \to \mathbf{R}$ the function given by $v_D(x) = 0$ if $x \in D$ and r otherwise. Then v_D is a valuation if and only if $D = \{1\}$ and $r > 0$.

Example 2.4. Let M be a finite set with n elements and $A = P(M)$ be the power set of M (the set of all subsets of M). Then $(P(M), \cap, \cup, C_M, \varnothing, M)$ is a Boolean algebra (where for $X \subseteq M$, $C_M(X) = M \backslash X$). The function $v : P(M) \to \mathbf{R}$, defined by $v(X) = n - n(X)$ is a valuation on A, where $n(X)$ is the number of elements of X. Indeed, $v(M) = 0$. Let $X, Y \subseteq M$. We have $v(X \to Y) = v(C_M X \cup Y) = n - n(C_M X \cup Y) = n - n(C_M X) - n(Y) + n(C_M X \cap Y) = n(X) - n(Y) + n(C_M X \cap Y) \geq n(X) - n(Y) = v(Y) - v(X)$. Obviously, $v(X) = 0$ iff $X = M$.

Lemma 2.1. *If $v : A \to \mathbf{R}$ is a pseudo-valuation on A and $x, x_1, ..., x_n \in A$ such that $(x_1, ..., x_n; x) = 1$ then*

(c_7) $v(x) \leq \sum_{i=1}^{n} v(x_i)$.

Proof. $0 = v(1) = v((x_1, ..., x_n; x)) \geq v(x) - \sum_{i=1}^{n} v(x_i)$, so $v(x) \leq \sum_{i=1}^{n} v(x_i)$. $\square$

A pseudo-valuation $v : A \to \mathbf{R}$ is called *decreasing* if $v(x) \geq v(y)$ for every $x, y \in A$ with $x \leq y$.

Lemma 2.2. *A pseudo-valuation v is a positive decreasing function satisfying*

(c_8) $v(x \to y) + v(y \to z) \geq v(x \to z)$, *for any $x, y, z \in A$.*

Proof. If in $(^*)$ we put $y = 1$ we obtain $v(x \to 1) \geq v(1) - v(x)$, so $v(x) \geq 0$, for every $x \in A$. If $x \leq y$, then $x \to y = 1$, so from $(^*)$ we deduce that $0 = v(1) = v(x \to y) \geq v(y) - v(x)$. We conclude that $v(x) \geq v(y)$ for every $x \leq y$, so, v is a decreasing function. Let now $x, y, z \in A$. Since v is a decreasing function, from (B), we deduce that $v(x \to y) \geq v((y \to z) \to (x \to z)) \geq v(x \to z) - v(y \to z)$. Thus, $v(x \to z) \leq v(x \to y) + v(y \to z)$. $\square$

We recall that by a *pseudo-metric space* we mean an ordered pair (M, d), where M is a non-empty set and $d : M \times M \to \mathbf{R}$ is a positive function satisfying the following properties: $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in M$. If in the pseudo-metric space (M, d), $d(x, y) = 0$ implies $x = y$, then (M, d) is called a *metric space.*

Theorem 2.1. *Let $v : A \to \mathbf{R}$ be a pseudo-valuation on A. If we define $d_v : A \times A \to \mathbf{R}$, $d_v(x, y) = v(x \to y) + v(y \to x)$, for every $(x, y) \in A \times A$, then*

(i) *(A, d_v) is a pseudo-metric space satisfying:*

(c_9) $\max\{d_v(x \to z, y \to z), d_v(z \to x, z \to y)\} \leq d_v(x, y)$, *for every $x, y, z \in A$;*

(ii) *d_v is a metric on A iff v is a valuation on A.*

Proof. (i). Let $x, y, z \in A$. Clearly, $d_v(x, y) = d_v(y, x) \geq 0$ and $d_v(x, x) = v(x \to x) + v(x \to x) = v(1) + v(1) = 0 + 0 = 0$. Also, $d_v(x, y) + d_v(y, z) = [v(x \to y) + v(y \to x)] + [v(y \to z) + v(z \to y)] = [v(x \to y) + v(y \to z)] + [v(z \to y) + v(y \to x)] \overset{(c_8)}{\geq} v(x \to$

$z) + v(z \to x) = d_v(x,z)$, hence d_v is a pseudo-metric on A. Now, we prove (c_9). We have $d_v(x \to z, y \to z) = v((x \to z) \to (y \to z)) + v((y \to z) \to (x \to z))$. Since, from (B), $x \to y \leq (y \to z) \to (x \to z)$ and $y \to x \leq (x \to z) \to (y \to z)$ we deduce that $v(x \to y) \geq v((y \to z) \to (x \to z))$ and $v(y \to x) \geq v((x \to z) \to (y \to z))$, hence, $d_v(x,y) = v(x \to y) + v(y \to x) \geq v((y \to z) \to (x \to z)) + v((x \to z) \to (y \to z)) = d_v(x \to z, y \to z)$. Since, from (c_1), $x \to y \leq (z \to x) \to (z \to y)$ and $y \to x \leq (z \to y) \to (z \to x)$, analogously as above we deduce that $d_v(x,y) \geq d_v(z \to x, z \to y)$. So, $\max\{d_v(x \to z, y \to z), d_v(z \to x, z \to y)\} \leq d_v(x,y)$, for every $x,y,z \in A$.

(ii). First, we suppose that d_v is a metric on A and let $x \in A$ such that $v(x) = 0$. Since $d_v(x,1) = v(x \to 1) + v(1 \to x) = v(1) + v(x) = 0 + 0 = 0$, then $x = 1$, that is, v is a valuation on A. Conversely, if v is a valuation on A, let $x,y \in A$ such that $d_v(x,y) = 0$. We obtain $v(x \to y) = v(y \to x) = 0$. Hence $x \to y = y \to x = 1$, so $x = y$, that is, d_v is a metric on A. $\square$

We shall call d_v *the pseudo-metric (metric) induced* by the pseudo-valuation (valuation) v. If we interpret a pseudo-valuation as a measure, then d_v is well known metric induced by a measure.

Corollary 2.1. *Let $v : A \to \mathbf{R}$ be a valuation. Then the operation $\to : A \times A \to A$ is uniformly continuous.*

Proof. Let $x, x', y, y' \in A$ and $0 < \varepsilon \in \mathbf{R}$. Then $\overline{d_v} : A \times A \to \mathbf{R}$, $\overline{d_v}((x,y),(x',y')) = \max\{d_v(x,x'), d_v(y,y')\}$, for every $(x,y),(x',y') \in A \times A$ is a metric on $A \times A$. Obviously, by definition, $\overline{d_v}$ is a positive function. Since v is a valuation on A, using Theorem 2.1, we deduce that d_v is a metric on A. Thus, $\overline{d_v}((x,y),(x,y)) = \max\{d_v(x,x), d_v(y,y)\} = 0$ and $\overline{d_v}((x,y),(x',y')) = \max\{d_v(x,x'), d_v(y,y')\} = \max\{d_v(x',x), d_v(y',y)\} = \overline{d_v}((x',y'),(x,y))$, for every $(x,y),(x',y') \in A \times A$. Also, for $(x,y),(x',y'),(x'',y'') \in A \times A$ we have: $\overline{d_v}((x,y),(x'',y'')) = \max\{d_v(x,x''), d_v(y,y'')\} \leq \max\{d_v(x,x') + d_v(x',x''), d_v(y,y') + d_v(y',y'')\} \leq \max\{d_v(x,x'), d_v(y,y')\} + \max\{d_v(x',x''), d_v(y',y'')\} = \overline{d_v}((x,y),(x',y')) + \overline{d_v}((x',y'),(x'',y''))$ and $\overline{d_v}((x,y),(x',y')) = 0$ implies $d_v(x,x') = d_v(y,y') = 0$ so, $x = x'$ and $y = y'$. We conclude that $(x,y) = (x',y')$. Thus, $\overline{d_v}$ is a metric on $A \times A$. If $\overline{d_v}((x,y),(x',y')) < \varepsilon/2$ then $d_v(x,x'), d_v(y,y') < \varepsilon/2$. We have $d_v(x \to y, x' \to y') \leq d_v(x \to y, x' \to y) + d_v(x' \to y, x' \to y') \leq d_v(x,x') + d_v(y,y') \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$, that is, $\to$ is uniformly continuous. $\square$

We have the following theorems of extension:

Theorem 2.2. *Let A and B two BCK algebras such that A is a subalgebra of B and $v : A \to \mathbf{R}$ is a pseudo-valuation on A. Then there exists a pseudo-valuation $v' : B \to \mathbf{R}$ such that $v'_{|A} = v$.*

Proof. For $x \in B$ we define $v'(x) = \inf\{\sum\limits_{i=1}^{n} v(x_i) : x_1,...,x_n \in A$ and $(x_1,...,x_n;x) = 1\}$. Since $1 \in A$ and $1 \to 1 = 1$ we deduce that $v'(1) = v(1) = 0$. For $x,y \in B$, let $x_1,...,x_n,z_1,...,z_m \in A$ such that $(x_1,...,x_n;x) = (z_1,...,z_m;x \to y) = 1$. We deduce that $(x_1,...,x_n,z_1,...,z_m;y) = 1$, hence, by the definition of v' we have $v'(y) \leq \sum\limits_{i=1}^{m} v(z_i) + \sum\limits_{i=1}^{n} v(x_i)$, so, $v'(y) \leq \inf\{\sum\limits_{i=1}^{m} v(z_i) : z_1,...,z_m \in A$ and $(z_1,...,z_m;x \to y) = 1\} + \inf\{\sum\limits_{i=1}^{n} v(x_i)$ $: x_1,...,x_n \in A$ and $(x_1,...,x_n;x) = 1\}$.

Thus, $v'(y) \leq v'(x \to y) + v'(x)$, so, $v'(y) - v'(x) \leq v'(x \to y)$, for every $x,y \in B$. We conclude that v' is a pseudo-valuation on B.

If $x \in A$, since $x \to x = 1$, we deduce that $v'(x) \leq v(x)$. Let $x_1, ..., x_n \in A$ such that $(x_1, ..., x_n; x) = 1$. From Lemma 2.1, $v(x) \leq \sum_{i=1}^{n} v(x_i)$, hence $v(x) \leq \inf\{\sum_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A$ and $(x_1, ..., x_n; x) = 1\} = v'(x)$, that is, $v'_{|A} = v$. $\square$

Remark 2.3. If A and B are two BCK algebras such that A is a subalgebra of B, $v : A \to \mathbf{R}$ is a pseudo-valuation on A and $v' : B \to \mathbf{R}$ is a real-valued function such that $v'_{|A} = v$, then v' is not necessarily a pseudo-valuation on B. Indeed, let $B = \{0, a, b, c, 1\}$ be BCK algebra from Example 1.1. Obviously, $A = \{1\}$ is a sub-BCK-algebra of B and $v : A \to \mathbf{R}$, $v(1) = 0$ is a pseudo-valuation on A, see Example 2.2. Let $v' : B \to \mathbf{R}$ be a real-valued function on B defined by $v' = \begin{pmatrix} 0 & a & b & c & 1 \\ 7 & 2 & 2 & 2 & 0 \end{pmatrix}$. Then $v'_{|A} = v$, but v' is not a pseudo-valuation on B since $v'(b \to 0) = v'(a) = 2 < v'(0) - v'(b) = 7 - 2 = 5$.

We consider $D \in Ds(A)$ and the relation δ_D on A defined by $(x, y) \in \delta_D$ iff $x \to y \in D$ and $y \to x \in D$. Hence δ_D is a congruence on A, see [3] and [13]. For $x \in A$ we denote by x/D the congruence class of x modulo δ_D and by $A/D = \{x/D : x \in A\}$ the quotient algebra. Then A/D is a BCK algebra, where for $x, y \in A$, $x/D \to y/D = (x \to y)/D$. Also, we denote by $p_D : A \to A/D$ the canonical surjective morphism of BCK algebras, $p_D(x) = x/D$, for every $x \in A$. For $x \in D$, we have $x/D = 1/D = \mathbf{1}$.

Theorem 2.3. *If $D \in Ds(A)$ and $v : A \to \mathbf{R}$ is a pseudo-valuation (valuation) on A, then the following assertions are equivalent:*

(i) *There exists a pseudo-valuation (valuation) $v' : A/D \to \mathbf{R}$ such that $v' \circ p_D = v$;*

(ii) *$v(a) = 0$, for every $a \in D$.*

Proof. $(i) \Rightarrow (ii)$. Let $v' : A/D \to \mathbf{R}$ be a pseudo-valuation on A/D such that $v' \circ p_D = v$ and let $a \in D$. Then $v(a) = (v' \circ p_D)(a) = v'(p_D(a)) = v'(\mathbf{1}) = 0$.

$(ii) \Rightarrow (i)$. For $x \in A$ we define $v'(x/D) = v(x)$. Let $x, y \in A$ such that $x/D = y/D$. Then $x \to y \in D$ and $y \to x \in D$. We obtain $0 = v(x \to y) \geq v(y) - v(x)$ and $0 = v(y \to x) \geq v(x) - v(y)$, so, $v(x) = v(y)$, hence v' is correctly defined. Also, we have $v'(1/D) = v(1) = 0$ and for $x, y \in A$, $v'(x/D \to y/D) = v'((x \to y)/D) = v(x \to y) \geq v(y) - v(x) = v'(y/D) - v'(x/D)$, hence v' is a pseudo-valuation on A. Clearly, $v' \circ p_D = v$. If v is a valuation on A and $x \in A$ such that $v'(x/D) = 0$, then $v(x) = 0$, hence $x = 1$. Thus, $x/D = 1/D = \mathbf{1}$. We conclude that v' is a valuation on A/D such that $v' \circ p_D = v$. $\square$

3. THE DUAL BCK ALGEBRA

In this section we introduce the notion of dual BCK algebra and taking as guide line [1], we obtain results for BCK algebras.

Let $A \in Ds(A)$, $D_1 \wedge D_2 = D_1 \cap D_2$, $D_1 \vee D_2 =< D_1 \cup D_2 >$ and $D_1 \to D_2 = \vee\{D \in Ds(A) : D_1 \cap D \subseteq D_2\} = \{a \in A : D_1 \cap < a > \subseteq D_2\}$.

Definition 3.1. The dual BCK algebra of A, denoted by A°, is the Heyting algebra $Ds(A)$ with the order $D_1 \leq D_2$ iff $D_2 \subseteq D_1$.

In $(A^\circ, \leq)$, $\mathbf{0} = A, \mathbf{1} = \{1\}$ and for $D_1, D_2 \in A^\circ$, $D_1 \sqcap D_2 =< D_1 \cup D_2 >= D_1 \vee D_2$, $D_1 \sqcup D_2 = D_1 \cap D_2$ and $D_1 \to D_2 = \sqcup\{D \in A^\circ : D_1 \sqcap D \leq D_2\} = \cap\{D \in A^\circ : D_2 \subseteq D_1 \vee D\}$.

Example 3.5. Let A be the BCK algebra from Example 1.1. It is imediate to prove that

$$Ds(A) = \{\{1\}, \{1, c\}, \{1, a, c\}, \{1, b, c\}, A\}$$

and A° is the Heyting algebra $Ds(A)$ with the order $D_1 \leq D_2$ iff $D_2 \subseteq D_1$, for $D_1, D_2 \in Ds(A)$.

Also, we define $j_A : A \to A^\circ$, $j_A(a) =< a >$, for every $a \in A$. Hence $j_A(1) = \mathbf{1}$ and $j_A(x \to y) \supseteq j_A(x) \to j_A(y)$, for every $x, y \in A$. Indeed, $j_A(1) =< 1 >= \{1\} = \mathbf{1}$. Also, $j_A(x) \to j_A(y) =< x >\to< y > = \cap\{D \in Ds(A) :< y >\subseteq< x > \vee D\}$. Since from (c_1), $x \to ((x \to y) \to y) = 1$, we deduce (using (c_6)) that $y \in< x > \vee < x \to y >$, so $< y >\subseteq< x > \vee < x \to y >$. Thus, $j_A(x \to y) =< x \to y >\supseteq j_A(x) \to j_A(y)$, for every $x, y \in A$.

Lemma 3.3. *For every $x, y \in A$, there is a natural number $m \geq 1$ such that*

$$j_A(x \to_m y) \subseteq j_A(x) \to j_A(y).$$

Proof. We have that $j_A(x) \to j_A(y) = \cap\{D \in Ds(A) :< y >\subseteq< x > \vee D\}$. So let $D \in Ds(A)$ such that $< y >\subseteq< x > \vee D$. Since $< x > \vee D = \{z \in A : t \to (d \to z) = 1$, for some $d \in D$ and $t \in< x >\}$ and $y \in< y > \subseteq < x > \vee D$, we deduce that $t \to (d \to y) = 1$, for some $d \in D$ and $t \in< x >$. But $t \to (d \to y) = d \to (t \to y)$, so, $d \to (t \to y) = 1$ and $d \leq t \to y$. We deduce that $t \to y \in D$ for some $t \in< x >$ (i.e., $x \to_n t = 1$, for some $n \geq 1$).

Finally, $x \to_n y \in D$, for some $n \geq 1$. Hence, there is a natural number $m \geq 1$ such that $x \to_m y \in D$, for every $D \in Ds(A)$. We conclude that there is $m \geq 1$ such that $j_A(x \to_m y) \subseteq j_A(x) \to j_A(y)$, for every $x, y \in A$. □

We recall that if A and B are two BCK algebras, a function $f : A \to B$ is a morphism of BCK algebras if $f(x \to y) = f(x) \to f(y)$ for every $x, y \in A$.

Lemma 3.4. *$j_{B(A)}$ is an injective morphism of BCK algebras.*

Proof. We recall that (see [6]) if $a \in B(A)$ then $a \to (a \to x) = a \to x$ for every $x \in A$, so, $< a >= \{x \in A : a \leq x\}$. Using Lemma 3.3, if we consider $x, y \in B(A)$ we deduce that $j_A(x) \to j_A(y) = j_A(x \to y)$. Also, if $j_A(x) = j_A(y)$, then $< x >=< y >$ so, $x \leq y$ and $y \leq x$. Thus, $x = y$. We conclude that $j_{B(A)}$ is an injective morphism of BCK algebras. □

Definition 3.2. We say that a BCK algebra A has property $\mathcal{F}$ if for every $D \in A^\circ$ there exist $x_1, ..., x_n \in A$ such that $D \subseteq< x_1, ..., x_n >$.

Example 3.6. If we consider BCK algebra from Example 1.1, then A has property $\mathcal{F}$ since $A^\circ = Ds(A) = \{\{1\}, \{1, c\}, \{1, a, c\}, \{1, b, c\}, A\}$ and $\{1\} =< 1 >$, $\{1, c\} =< 1, c >$, $\{1, a, c\} =< 1, a, c >$, $\{1, b, c\} =< 1, b, c >$ and $A =< 0 >$.

Remark 3.4. Examples of BCK algebras with property $\mathcal{F}$ are bounded BCK algebras (since $A =< 0 >$) and finite BCK algebras.

Theorem 3.4. *Let A be a BCK algebra with property $\mathcal{F}$ and $v : A \to \mathbf{R}$ a pseudo-valuation on A. Then there exists a pseudo-valuation on A°, $v' : A^\circ \to \mathbf{R}$ such that $v' \circ j_A = v$.*

Proof. For $D \in A^\circ$ we define $v'(D) = \inf\{\sum\limits_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A$ and $D \subseteq < x_1, ..., x_n >\}$. First, we prove that $v\prime$ is a pseudo-valuation on A°.

Clearly, $v'(\mathbf{1}) = \inf\{\sum\limits_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A$ and $\{1\} \subseteq< x_1, ..., x_n >\} = v(1) = 0$. To verify (*), let $D_1, D_2 \in A^\circ$ and $x_1, ..., x_n, z_1, ..., z_m \in A$ such that $D_1 \subseteq < x_1, ..., x_n >$ and $D_1 \to D_2 \subseteq< z_1, ..., z_m >$. Then $D_2 \subseteq D_1 \vee (D_1 \to D_2) \subseteq < x_1, ..., x_n > \vee < z_1, ..., z_m > \subseteq < x_1, ..., x_n, z_1, ..., z_m >$. Thus $v'(D_2) \leq \sum\limits_{i=1}^{n} v(x_i) + \sum\limits_{j=1}^{m} v(z_j)$, so, $v'(D_2) \leq \inf\{\sum\limits_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A$ and $D_1 \subseteq < x_1, ..., x_n >\} + \inf\{\sum\limits_{j=1}^{m} v(z_j) : z_1, ..., z_m \in A$ and $D_1 \to D_2 \subseteq< z_1, ..., z_m >\} =$

$v'(D_1) + v'(D_1 \to D_2)$. We obtain that $v'(D_1 \to D_2) \geq v'(D_2) - v'(D_1)$.

If $a, x_1, ..., x_n \in A$ such that $< a > \subseteq < x_1, ..., x_n >$ then $(x_{i_1}, ..., x_{i_k}; a) = 1$, for some $x_{i_1}, ..., x_{i_k} \in \{x_1, ..., x_n\}$. From Lemma 2.1 we deduce that $v(a) \leq \sum_{i=1}^{n} v(x_i)$, so $v(a) \leq \inf\{\sum_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A$ and $< a > \subseteq < x_1, ..., x_n >\} = v'(< a >)$. Since $< a > \subseteq < \{a\} >$ it follows that $v'(< a >) = v(a)$. We conclude that $v' \circ j_A = v$. □

4. Conclusions and future work

In [1], is defined a pseudo-valuation on a Hilbert algebra. In this paper, we generalize this concept for BCK algebras and we prove theorems on extensions of pseudo-valuations (valuations) on BCK algebras. Since the power set of a non-empty set is a BCK algebra, using of pseudo-valuations can be useful in the study of theory of sets. As another direction of research one could define and study the concept of free Hilbert algebra with infimum over a BCK algebra. Specifically, the questions are the following: *Are there the free Hilbert algebras with infimum over BCK algebras?* and if the answer is positive, *Which is the relation of these algebras with pseudo-valuations?* It is interesting to note that, if the BCK algebra is a Hilbert algebra, then an explicit construction of the free semilattice extension of a Hilbert algebra is not immediate. Also, [1] contains results about pseudo-valuations on free Hertz algebra over a Hilbert algebra.

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