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A note on an integral by Grigorii Mikhailovich Fichtenholz

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ABSTRACT. In this manuscript, the authors derived a definite integral involving the logarithmic function, function of powers and polynomials in terms of the Lerch function. A summary of the results is produced in the form of a table of definite integrals for easy referencing by readers.

1. INTRODUCTION

In 1948 Grigorii Mikhailovich Fichtenholz produced volume II of his three volume collection [5, 8]. In volume II the authors found an integral (3.244.4) in [1] which is of interest because of its closed form solution over the real line. However, upon closer inspection and evaluation of this integral and applying our simultaneous contour integral method we found this integral is not symmetric over the real line when the logarithmic function is introduced into the integrand. This logarithmic term appears after applying our contour integral method to this Fichtenholz integral. This consequence lead us to produce this manuscript which achieves two objectives. The first, is that of producing formal derivations for some definite integrals in Table 3.244 in [1]. The second goal is to produce more definite integrals as an expansion of the current Table 3.244 in [1]. These gaols are achieved by using this integral by Fichtenholz along with our contour integral method to form a closed solution in terms of the Lerch function. The Lerch function being a special function has the fundamental property of analytic continuation, which enables us to widen the range of evaluation for the parameters involved in our definite integral.

The definite integral the authors derived using the integral by Fichtenholz in this manuscript is given by

$$\int_0^\infty \frac{\left(x^{2n}-x^{2m}\right)\log^k\left(ax^2\right)}{x^{2l}-1}dx \tag{1.1}$$

in terms of the Lerch function, where the parameters k, a, m, n and l are general complex numbers where $Re(l) > Re(m)$ and $Re(l) > Re(n)$ in order for the integral to exist. A summary of the results is given in a table of integrals for easy reading. This work is important because the authors were unable to find similar results in current literature. Tables of definite integrals provide a useful summary and reference for readers seeking such integrals for potential use in their research. We use our simultaneous contour integration method to aid in our derivations of the closed forms solutions in terms of the Lerch function, which provides analytic continuation of the results. The derivations follow the method used by us in [2]. The generalized Cauchy's integral formula is given by

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$$\frac{y^k}{k!} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{1.2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant [2] has the same value at the end points of the contour. This method involves using a form of equation (1.2) then multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. A second contour integral is derived by multiplying equation (1.2) by a function and performing some substitutions so that the contour integrals are the same.

2. Definite integral of the contour integral

We use the method in [2]. Here the contour is similar to Figure 2 in [2]. Using a generalization of Cauchy's integral formula equation (1.2) replace y by $\log(ax^2)$ followed by multiplying both sides by $\frac{x^{2m}-x^{2n}}{1-x^{2l}}$ and taking the definite integral over $x \in [0,\infty)$ to get

$$\begin{aligned}
&\frac{1}{k!}\int_0^\infty \frac{\left(x^{2n}-x^{2m}\right)\log^k\left(ax^2\right)}{x^{2l}-1}dx \\
&= \frac{1}{2\pi i}\int_0^\infty\int_C \frac{a^w w^{-k-1}\left(x^{2(m+w)}-x^{2(n+w)}\right)}{1-x^{2l}}dwdx \\
&= \frac{1}{2\pi i}\int_C\int_0^\infty \frac{a^w w^{-k-1}\left(x^{2(m+w)}-x^{2(n+w)}\right)}{1-x^{2l}}dxdw \\
&= \frac{1}{2\pi i}\int_C \frac{\pi a^w w^{-k-1}\left(\cot\left(\frac{\pi(2(m+w)+1)}{2l}\right)-\cot\left(\frac{\pi(2(n+w)+1)}{2l}\right)\right)}{2l}dw
\end{aligned} \tag{2.3}$$

from equation (3.244.4) in [1] where the integral is valid for a, m, l, n, k complex and $-1 < Re(w+m) < 0$ and $-1 < Re(w+n) < 0$. The contour C is defined where the cut lies in the second quadrant going from the origin vertically to infinity and the contour C lies on opposite sides of the cut going round the origin with zero radius. The logarithmic function is defined in equation (4.1.2) in [4]

3. The Lerch function

The Lerch function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{3.4}$$

where $|z| < 1, v \neq 0,-1,..$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \tag{3.5}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4. Infinite sum of the contour integral

4.1. **Derivation of the first contour integral.** In this section we will derive the contour integral given by

$$\frac{1}{2\pi i}\int_C \frac{\pi a^w w^{-k-1}\cot\left(\frac{\pi(2(m+w)+1)}{2l}\right)}{2l}dw \tag{4.6}$$

Again, using the method in [2] and equation (1.2), we replace y by $\log(a)+\frac{2i\pi(y+1)}{l}$ multiply both sides by $-\frac{2i\pi}{l}e^{\frac{i\pi(2m+1)(y+1)}{l}}$ and take the infinite sum of both sides over $y\in[0,\infty)$ simplifying in terms the Lerch function to get

$$\begin{aligned}
&-\frac{i(2\pi)^{k+1}\left(\frac{i}{l}\right)^k e^{\frac{i(2\pi m+\pi)}{l}}\Phi\left(e^{\frac{i(2m+1)\pi}{l}},-k,1-\frac{il\log(a)}{2\pi}\right)}{lk!}\\
&=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\frac{2i\pi a^w w^{-k-1}e^{\frac{i\pi(y+1)(2m+2w+1)}{l}}}{l}dw\\
&=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\frac{2i\pi a^w w^{-k-1}e^{\frac{i\pi(y+1)(2m+2w+1)}{l}}}{l}dw\\
&=\frac{1}{2\pi i}\int_C\left(\frac{\pi a^w w^{-k-1}\cot\left(\frac{\pi(2m+2w+1)}{2l}\right)}{l}+\frac{i\pi a^w w^{-k-1}}{l}\right)dw
\end{aligned}\tag{4.7}$$

similar to equation (1.232.1) in [1] where

$$\cot(x)=-2i\sum_{y=0}^{\infty}e^{2xi(y+1)}-i\tag{4.8}$$

where $Im(x)>0$.

4.2. **Derivation of the second contour integral.** In this section we will derive the contour integral given by

$$-\frac{1}{2\pi i}\int_C\frac{\pi a^w w^{-k-1}\cot\left(\frac{\pi(2n+2w+1)}{2l}\right)}{l}dw\tag{4.9}$$

Again, using the method in [2] and equation (1.2), we replace y by $\log(a)+\frac{2i\pi(y+1)}{l}$ multiply both sides by $-\frac{2i\pi}{l}e^{\frac{i\pi(2n+1)(y+1)}{l}}$ and take the infinite sum of both sides over $y\in[0,\infty)$ simplifying in terms the Lerch function to get

$$\begin{aligned}
&\frac{i(2\pi)^{k+1}\left(\frac{i}{l}\right)^k e^{\frac{i(2\pi n+\pi)}{l}}\Phi\left(e^{\frac{i(2n+1)\pi}{l}},-k,1-\frac{il\log(a)}{2\pi}\right)}{lk!}\\
&=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\frac{2i\pi a^w w^{-k-1}e^{\frac{i\pi(y+1)(2n+2w+1)}{l}}}{l}dw\\
&=\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\frac{2i\pi a^w w^{-k-1}e^{\frac{i\pi(y+1)(2n+2w+1)}{l}}}{l}dw\\
&=-\frac{1}{2\pi i}\int_C\left(\frac{\pi a^w w^{-k-1}\cot\left(\frac{\pi(2n+2w+1)}{2l}\right)}{l}-\frac{i\pi a^w w^{-k-1}}{l}\right)dw
\end{aligned}\tag{4.10}$$

similar to equation (1.232.1) in [1] where

$$\cot(x)=-2i\sum_{y=0}^{\infty}e^{2xi(y+1)}-i\tag{4.11}$$

where $Im(x)>0$.

5. Main results

5.1. Definite integral in terms of the Lerch function.

Theorem 5.1. *For all $a, k \in \mathbb{C}$, $Re(m) < Re(l)$ and $Re(n) < Re(l)$,*

$$\int_0^\infty \frac{\left(x^{2n}-x^{2m}\right)\log^k\left(ax^2\right)}{x^{2l}-1}dx = \frac{i2^k\pi^{k+1}\left(\frac{i}{l}\right)^k e^{\frac{i(2\pi n+\pi)}{l}}\Phi\left(e^{\frac{i(2n+1)\pi}{l}},-k,1-\frac{il\log(a)}{2\pi}\right)}{l} - \frac{i2^k\pi^{k+1}\left(\frac{i}{l}\right)^k e^{\frac{i(2\pi m+\pi)}{l}}\Phi\left(e^{\frac{i(2m+1)\pi}{l}},-k,1-\frac{il\log(a)}{2\pi}\right)}{l} \tag{5.12}$$

Proof. Since the right-hand side of equation (2.3) is equal to the sum of equations (4.7) and (4.10) we can equate the left-hand sides and simplify the factorials. □

5.2. Derivation of a logarithmic integral.

Theorem 5.2. *For $Re(m) < Re(l)$ and $Re(n) < Re(l)$,*

$$\int_0^\infty \frac{\log\left(x^2\right)\left(x^{2n}-x^{2m}\right)}{x^{2l}-1}dx = \frac{\pi^2\left(\csc^2\left(\frac{2\pi n+\pi}{2l}\right)-\csc^2\left(\frac{2\pi m+\pi}{2l}\right)\right)}{2l^2} \tag{5.13}$$

Proof. Use equation (5.12) and set $k = a = 1$ simplify using entry (2) in Table below (64:12:7) in [3]. □

5.3. Derivation of an integral by Fichtenholz entry 3.244.4 in [1] and 640 in [8] volume II.

Theorem 5.3. *For $Re(m) < Re(l)$ and $Re(n) < Re(l)$,*

$$\int_0^\infty \frac{x^{2n}-x^{2m}}{x^{2l}-1}dx = \frac{\pi\left(\cot\left(\frac{2\pi m+\pi}{2l}\right)-\cot\left(\frac{2\pi n+\pi}{2l}\right)\right)}{2l} \tag{5.14}$$

Proof. Use equation (5.13) and set $k = 0$ simplify using entry (2) in Table below (64:12:7) in [3]. □

5.4. Derivation of entry 4.235.1 in [1].

Lemma 5.1. *For $Re(n) > 1$,*

$$\int_0^\infty \frac{(x-1)x^{n-2}\log(x)}{x^{2n}-1}dx = -\frac{\pi^2\tan^2\left(\frac{\pi}{2n}\right)}{4n^2} \tag{5.15}$$

Proof. Use equation (5.13) set $a = k = 1$ and simplify using entry (3) in Table below (64:12:7) in [3]. Next replace n by $\frac{n-2}{2}$, m by $\frac{n-1}{2}$ and l by n and simplify. □

5.5. Derivation of entry 4.235.2 in [1].

Theorem 5.4. *For $Re(n) > Re(m) > 0$,*

$$\int_0^\infty \frac{\left(x^2-1\right)x^{m-1}\log(x)}{x^{2n}-1}dx = -\frac{\pi^2\left(\csc^2\left(\frac{\pi m}{2n}\right)-\csc^2\left(\frac{\pi(m+2)}{2n}\right)\right)}{4n^2} \tag{5.16}$$

Proof. Use equations (5.13) set $a = k = 1$ and simplify using entry (3) in Table below (64:12:7) in [3]. Next replace n by $\frac{m-1}{2}$, m by $\frac{m+1}{2}$ and l by n simplify using equation (64:10:2) in [3]. □

5.6. **Derivation of entry 4.235.3 in** [1].

Theorem 5.5. *For $Re(n) > 2$ and $Im(n) > 2$,*

$$\int_0^\infty \frac{(x^2-1)\, x^{n-3}\log(x)}{x^{2n}-1}dx = -\frac{\pi^2\tan^2\left(\frac{\pi}{n}\right)}{4n^2} \tag{5.17}$$

Proof. Use equations (5.13) set $a = k = 1$ and simplify using entry (3) in Table below (64:12:7) in [3]. Next replace n by $\frac{n-3}{2}$, m by $\frac{n-1}{2}$ and l by n simplify using equation (64:4:2) in [3]. □

5.7. **Definite integral in terms of the Polylogarithm function.** Using equation (5.12) and setting $a = 1$ simplifying to get

$$\int_0^\infty \frac{\log^k(x)\left(x^{2n}-x^{2m}\right)}{x^{2l}-1}dx = \frac{\pi^{k+1}\left(\frac{i}{l}\right)^{k-1}\left(\text{Li}_{-k}\left(e^{\frac{i(2m+1)\pi}{l}}\right)-\text{Li}_{-k}\left(e^{\frac{i(2n+1)\pi}{l}}\right)\right)}{l^2} \tag{5.18}$$

from equation (64:12:2) in [3].

5.8. **Definite integral in terms of the logarithm of trigonometric functions.**

Theorem 5.6. *For all $m, n, p, q, l \in \mathbb{C}$*

$$\int_0^\infty \frac{x^{2m}-x^{2n}-x^{2p}+x^{2q}}{(x^{2l}-1)\log(x)}dx = \log\left(\frac{\left(\cos\left(\frac{\pi(n-p)}{l}\right)-\cos\left(\frac{\pi(n+p+1)}{l}\right)\right)e^{-\frac{i\pi(m-n-p+q)}{l}}}{\cos\left(\frac{\pi(m-q)}{l}\right)-\cos\left(\frac{\pi(m+q+1)}{l}\right)}\right) \tag{5.19}$$

Proof. Form a second equation by using equation (5.12) and replacing m by p and n by q. Then we take the difference between these two equations and setting $k = -1, a = 1$ simplify using entry (1) in Table below (64:12:7) in [3]. □

5.9. **Evaluation of a Definite integral of a nested logarithmic function.**

Proposition 5.1.

$$\begin{aligned}&\int_0^\infty \frac{\left(x-x^{2/3}\right)\log(\log(x))}{x^4-1}dx\\ &= \frac{\pi}{8\left(\sqrt{3}+(2-i)\right)}\left(4\left((1+2i)+i\sqrt{3}\right)\text{Li}_0'\left((-1)^{5/6}\right)+\left(-\sqrt{3}\right.\right.\\ &\left.\left.+(2+i)\right)\pi+\left(4+4i\sqrt{3}\right)\log\left(\frac{\pi}{2}\right)\right)\end{aligned} \tag{5.20}$$

Proof. Use equation (5.12) and set $n = 1/2, l = 2, m = 1/3, a = 1$ and simplify using entry (2) in Table below (64:7), equations (64:12:1) and (64:12:2) in [3]. Then take the first partial derivative with respect to k and then set $k = 0$ simplify. □

5.10. **Definite integral in terms of the trigonometric functions.**

Theorem 5.7. *For all $m, n, l \in \mathbb{C}$*

$$\int_0^\infty \frac{\log^2(x)\left(x^{2n}-x^{2m}\right)}{x^{2l}-1}dx$$
$$= -\frac{1}{32l^3}\left(\pi^3\csc^3\left(\frac{2\pi m+\pi}{2l}\right)\csc^3\left(\frac{2\pi n+\pi}{2l}\right)\left(6\sin\left(\frac{\pi(m-n)}{l}\right)\right.\right.$$
$$-\sin\left(\frac{\pi(3m-n+1)}{l}\right)-\sin\left(\frac{\pi(3m+n+2)}{l}\right)+\sin\left(\frac{\pi(-m+3n+1)}{l}\right)$$
$$\left.\left.+\sin\left(\frac{\pi(m+3n+2)}{l}\right)\right)\right) \tag{5.21}$$

Proof. Use equation (5.12) and set $k = 2, a = 1$ simplify using entry (4) in Table below (64:12:7) in [3]. □

5.11. **Definite integrals with logarithm in the denominator.**

Theorem 5.8. *For all* $m, n, l \in \mathbb{C}$

$$\int_0^\infty \left(\frac{\log(x)\left(x^{2n}-x^{2m}\right)}{\left(x^{2l}-1\right)\left(a^2+\log^2(x)\right)}+\frac{ia\left(x^{2m}-x^{2n}\right)}{\left(x^{2l}-1\right)\left(a^2+\log^2(x)\right)}dx\right.$$
$$= e^{\frac{i\pi}{l}}\left(e^{\frac{2i\pi n}{l}}\Phi\left(e^{\frac{i(2n+1)\pi}{l}},1,\frac{al}{\pi}+1\right)-e^{\frac{2i\pi m}{l}}\Phi\left(e^{\frac{i(2m+1)\pi}{l}},1,\frac{al}{\pi}+1\right)\right)\Bigg) \tag{5.22}$$

Proof. Use equation (5.12) and set $k = -1, a = e^{2ai}$ and simplify. □

Proposition 5.2.

$$\int_0^\infty \frac{\left(x-x^{2/3}\right)\log(x)}{\left(x^3-1\right)\left(\log^2(x)+\pi^2\right)}dx = \frac{1}{4}\left(4+\sqrt{3}\pi-8\cos\left(\frac{\pi}{9}\right)+\log\left(\frac{2\left(1+\sin\left(\frac{\pi}{18}\right)\right)}{9\left(2-2\sin\left(\frac{\pi}{18}\right)\right)}\right)\right) \tag{5.23}$$

and

$$\int_0^\infty \frac{x^{2/3}-x}{\left(x^3-1\right)\left(\log^2(x)+\pi^2\right)}dx$$
$$= \frac{\pi+8\sin\left(\frac{\pi}{9}\right)+2\sqrt{3}\left(\tanh^{-1}\left(\sin\left(\frac{\pi}{18}\right)\right)-2\right)}{4\pi} \tag{5.24}$$

Proof. Use equation (5.22) and set $a = \pi, l = 3/2, n = 1/2, m = 1/3$ and rationalize the real and imaginary parts and simplify using equation (9.559) in [1] and entry (1) in Table below (64:12:7) in [3]. □

Proposition 5.3.

$$\int_0^\infty \frac{\left(x-x^{2/3}\right)\log(x)}{\left(x^4-1\right)\left(4\log^2(x)+\pi^2\right)}dx = \frac{1}{96}\left(-\pi+24\log(2)-6\sqrt{3}\log\left(2+\sqrt{3}\right)\right) \tag{5.25}$$

and

$$\int_0^\infty \frac{x^{2/3}-x}{\left(x^4-1\right)\left(4\log^2(x)+\pi^2\right)}dx = \frac{\sqrt{3}\pi-6\cosh^{-1}(2)}{48\pi} \tag{5.26}$$

Proof. Use equation (5.22) and set $a = \pi/2, l = 2, n = 1/2, m = 1/3$ and rationalize the real and imaginary parts and simplify using entry (1) in Table below (64:12:7) in [3]. □

Proposition 5.4.

$$\int_0^\infty \frac{\left(\sqrt{x}-1\right)x}{\left(x^4-1\right)\left(4\log^2(x)+\pi^2\right)}dx = -\frac{\pi-4\log\left(2+\sqrt{2}\right)}{16\sqrt{2}\pi} \tag{5.27}$$

and

$$\int_0^\infty \frac{\left(x-x^{3/2}\right)\log(x)}{\left(x^4-1\right)\left(\log^2(x)+\frac{\pi^2}{4}\right)}dx = \log(2)-\frac{\log\left(2+\sqrt{2}\right)}{2\sqrt{2}}-\frac{\tan^{-1}\left(\frac{1}{1+\sqrt{2}}\right)}{\sqrt{2}} \tag{5.28}$$

Proof. Use equation (5.22) and set $a=\pi/2, l=2, n=1/2, m=3/4$ and rationalize the real and imaginary parts and simplify. Use equation (9.559) in [1] and entry (1) in Table below (64:12:7) in [3]. □

5.12. **Definite integrals of product logarithmic functions in terms of fundamental constants.**

Theorem 5.9. *For all* $k,m,n,l\in\mathbb{C}$

$$\begin{aligned}
&\int_0^\infty \frac{\log(x)\left(x^{2m}+x^{2n}\right)\log^k\left(ax^2\right)}{x^{2l}-1}dx \\
&= -\frac{2^{k-1}\pi^{k+1}e^{\frac{i\pi}{l}}\left(\frac{i}{l}\right)^k}{l^2}\left(e^{\frac{2i\pi m}{l}}\left(2\pi\Phi\left(e^{\frac{i(2m+1)\pi}{l}},-k-1,1-\frac{il\log(a)}{2\pi}\right)\right.\right. \\
&\qquad\left.+il\log(a)\Phi\left(e^{\frac{i(2m+1)\pi}{l}},-k,1-\frac{il\log(a)}{2\pi}\right)\right) \\
&\qquad+e^{\frac{2i\pi n}{l}}\left(2\pi\Phi\left(e^{\frac{i(2n+1)\pi}{l}},-k-1,1-\frac{il\log(a)}{2\pi}\right)\right. \\
&\qquad\left.\left.+il\log(a)\Phi\left(e^{\frac{i(2n+1)\pi}{l}},-k,1-\frac{il\log(a)}{2\pi}\right)\right)\right)
\end{aligned} \tag{5.29}$$

Proof. Form two equations by first taking first partial derivative equation (5.12) with respect to n, then again take the first partial derivative of equation (5.12) with respect to m. Then add these two equations and simplify. □

Proposition 5.5. *For all* $k,n,l\in\mathbb{C}$

$$\begin{aligned}
\int_0^\infty \frac{x^{2n}\log(x)\log^k\left(ax^2\right)}{x^{2l}-1}dx = &-\frac{2^{k-1}\pi^{k+1}\left(\frac{i}{l}\right)^k e^{\frac{i(2\pi n+\pi)}{l}}}{l^2}\left(2\pi\Phi\left(e^{\frac{i(2\pi n+\pi)}{l}},-k-1,1\right.\right. \\
&\left.\left.-\frac{il\log(a)}{2\pi}\right)+il\log(a)\Phi\left(e^{\frac{i(2\pi n+\pi)}{l}},-k,1-\frac{il\log(a)}{2\pi}\right)\right)
\end{aligned} \tag{5.30}$$

Proof. Use equation (5.29) set $m=n$ and simplify. □

Proposition 5.6. *For all* $k\in\mathbb{C}$

$$\begin{aligned}
\int_0^\infty \frac{x\log(x)\log^k\left(ax^2\right)}{x^4-1}dx = i^k 2^{k-2}\pi^{k+1}&\left(2\pi\zeta\left(-k-1,\frac{\pi-i\log(a)}{2\pi}\right)\right. \\
&-2\pi\zeta\left(-k-1,1-\frac{i\log(a)}{2\pi}\right) \\
&\left.+i\log(a)\left(\zeta\left(-k,\frac{\pi-i\log(a)}{2\pi}\right)-\zeta\left(-k,1-\frac{i\log(a)}{2\pi}\right)\right)\right)
\end{aligned} \tag{5.31}$$

Proof. Use equation (5.30) set $n = 1/2, l = 2$ and simplify using entry (4) in Table below (64:12:7) in [3]. □

Theorem 5.10. *For all* $k \in \mathbb{C}$

$$\int_0^\infty \frac{x \log(x) \log\left(\log\left(x^2\right)\right) \log^k\left(x^2\right)}{x^4-1} dx = 2^{-k-4} e^{\frac{i\pi k}{2}} \left(\left((4\pi)^{k+2} - (2\pi)^{k+2}\right) \zeta'(-k-1) + \zeta(k+2)\left(i\pi\left(2^{k+2}-1\right) + 2^{k+3}\log(2\pi) - 2\log(\pi)\right) \cos\left(\frac{\pi k}{2}\right) \Gamma(k+2)\right) \tag{5.32}$$

Proof. Use equation (5.31) and take the first partial derivative with respect to k and set $a = 1$ and simplify using equation (64:12:1) and entry (2) in Table below (64:7) in [3]. □

Proposition 5.7.

$$\int_0^\infty \frac{x \log(x) \log\left(x^2\right) \log\left(\log\left(x^2\right)\right)}{x^4-1} dx = -\frac{7}{16} i\pi\zeta(3) \tag{5.33}$$

Proof. Use equation (5.32) and set $k = 1$ and simplify in terms of Aprey's constant section (1.6) in [6]. □

Proposition 5.8.

$$\int_0^\infty \frac{x \log(x) \log\left(\log\left(x^2\right)\right)}{\left(x^4-1\right)\log\left(x^2\right)} dx = \frac{1}{16}\pi(\pi - 2i\log(2)) \tag{5.34}$$

Proof. Use equation (5.32) and set $k = -1$ and simplify using equation (6.8) in [7]. □

Proposition 5.9.

$$\int_0^\infty \frac{x \log(x) \log\left(\log\left(x^2\right)\right)}{x^4-1} dx = \frac{1}{32}\pi^2 \left(8 \log\left(\frac{\sqrt[3]{2}\sqrt[4]{\pi}}{A^3}\right) + 2 + i\pi\right) \tag{5.35}$$

Proof. Use (5.32) and set $k = 0$ and simplify using equation (2.15), pp. 135-145 in [6]. □

6. Table of integrals

$f(x)$	$\int_0^\infty f(x)dx$
$\frac{\log^k(x)\left(x^{2n}-x^{2m}\right)}{x^{2l}-1}$	$\frac{\pi^{k+1}\left(\frac{i}{l}\right)^{k-1}}{l^2}\left(\text{Li}_{-k}\left(e^{\frac{i(2m+1)\pi}{l}}\right)-\text{Li}_{-k}\left(e^{\frac{i(2n+1)\pi}{l}}\right)\right)$
$\frac{x^{2n}-x^{2m}}{x^{2l}-1}$	$\frac{\pi\left(\cot\left(\frac{2\pi m+\pi}{2l}\right)-\cot\left(\frac{2\pi n+\pi}{2l}\right)\right)}{2l}$
$\frac{\log\left(x^2\right)\left(x^{2n}-x^{2m}\right)}{x^{2l}-1}$	$\frac{\pi^2\left(\csc^2\left(\frac{2\pi n+\pi}{2l}\right)-\csc^2\left(\frac{2\pi m+\pi}{2l}\right)\right)}{2l^2}$
$-\frac{(x-1)x^{n-2}\log(x)}{x^{2n}-1}$	$\frac{\pi^2\tan^2\left(\frac{\pi}{2n}\right)}{4n^2}$
$-\frac{\left(x^2-1\right)x^{m-1}\log(x)}{x^{2n}-1}$	$\frac{\pi^2\left(\csc^2\left(\frac{\pi m}{2n}\right)-\csc^2\left(\frac{\pi(m+2)}{2n}\right)\right)}{4n^2}$
$\frac{x^{2m}-x^{2n}-x^{2p}+x^{2q}}{(x^{2l}-1)\log(x)}$	$\log\left(\frac{\left(\cos\left(\frac{\pi(n-p)}{l}\right)-\cos\left(\frac{\pi(n+p+1)}{l}\right)\right)e^{-\frac{i\pi(m-n-p+q)}{l}}}{\cos\left(\frac{\pi(m-q)}{l}\right)-\cos\left(\frac{\pi(m+q+1)}{l}\right)}\right)$
$\frac{\left(x-x^{2/3}\right)\log(\log(x))}{x^4-1}$	$\frac{\pi\left(4\left((1+2i)+i\sqrt{3}\right)\text{Li}_0'\left((-1)^{5/6}\right)+\left(-\sqrt{3}+(2+i)\right)\pi+\left(4+4i\sqrt{3}\right)\log\left(\frac{\pi}{2}\right)\right)}{8\left(\sqrt{3}+(2-i)\right)}$
$\frac{\left(x-x^{2/3}\right)\log(x)}{(x^3-1)(\log^2(x)+\pi^2)}$	$\frac{1}{4}\left(4+\sqrt{3}\pi-8\cos\left(\frac{\pi}{9}\right)+\log\left(\frac{2\left(1+\sin\left(\frac{\pi}{18}\right)\right)}{9\left(2-2\sin\left(\frac{\pi}{18}\right)\right)}\right)\right)$
$\frac{x^{2/3}-x}{(x^3-1)(\log^2(x)+\pi^2)}$	$\frac{\pi+8\sin\left(\frac{\pi}{9}\right)+2\sqrt{3}\left(\tanh^{-1}\left(\sin\left(\frac{\pi}{18}\right)\right)-2\right)}{4\pi}$
$\frac{\left(x-x^{2/3}\right)\log(x)}{(x^4-1)(4\log^2(x)+\pi^2)}$	$\frac{1}{96}\left(-\pi+24\log(2)-6\sqrt{3}\log\left(2+\sqrt{3}\right)\right)$
$\frac{x^{2/3}-x}{(x^4-1)(4\log^2(x)+\pi^2)}$	$\frac{\sqrt{3}\pi-6\cosh^{-1}(2)}{48\pi}$
$\frac{\left(\sqrt{x}-1\right)x}{(x^4-1)(4\log^2(x)+\pi^2)}$	$-\frac{\pi-4\log\left(2+\sqrt{2}\right)}{16\sqrt{2}\pi}$
$\frac{x\log(x)\log\left(x^2\right)\log\left(\log\left(x^2\right)\right)}{x^4-1}$	$-\frac{7}{16}i\pi\zeta(3)$
$\frac{x\log(x)\log\left(\log\left(x^2\right)\right)}{(x^4-1)\log(x^2)}$	$\frac{1}{16}\pi(\pi-2i\log(2))$
$\frac{x\log(x)\log\left(\log\left(x^2\right)\right)}{x^4-1}$	$\frac{1}{32}\pi^2\left(8\log\left(\frac{\sqrt[3]{2}\sqrt[4]{\pi}}{A^3}\right)+2+i\pi\right)$

7. Discussion

In this work the authors looked at deriving definite integrals involving the logarithmic function, function of powers and polynomials in terms of the Lerch function. One of the goals was to supply a table for easy reading by researchers and to have these results added to existing textbooks.

The results presented were numerically verified for both real and imaginary values of the parameters in the integrals using Mathematica by Wolfram. The authors considered various ranges of these parameters for real, integer, negative and positive values. The authors compared the evaluation of the definite integral to the evaluated Special function and ensured agreement.

8. Conclusion

In this paper the authors used their contour integral method [2] to evaluate definite integrals using the Lerch function. The contour we used was specific to solving integral representations in terms of the Lerch function. We expect that other contours and integrals can be derived using this method.

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