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Strongly n-polynomial convexity and related inequalities

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ABSTRACT. In this paper, we introduce and study the concept of strongly n-polynomial convexity functions and their some algebraic properties. We prove two Hermite-Hadamard type inequalities for the newly introduced class of functions. In addition, we obtain some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is strongly n-polynomial convexity. Also, we compare the obtained results with both Hölder, Hölder-İşcan inequalities and power-mean, improved-power-mean integral inequalities and show that the result obtained with Hölder-İşcan and improved power-mean inequalities give better approach than the others.

1. PRELIMINARIES

A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \leq tf\left(x\right) + \left(1-t\right) f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [14, 17, 19, 23, 27] and the references therein.Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [14, 17, 19, 23, 27] and the references therein. We would also like to point out in particular that readers wishing to learn more about the subject of this article and the various types of convexity may refer to the references [3, 1, 2, 4, 6, 7, 9, 10, 8, 11, 24, 20, 21, 25, 26, 29, 28, 30, 31, 33].

Let $f : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f\left(a\right)+f(b)}{2} \tag{1.1}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double inequality is well known as the Hermite-Hadamard inequality [15]. Some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [13, 37]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f.

In [32], Polyak introduced the class of strongly convex functions as follows:

Definition 1.1 ([32]). Let $I \subset \mathbb{R}$ be an interval and c be a positive number. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus c if

$$f\left(ta + (1-t)b\right) \leq tf\left(a\right) + (1-t)f(b) - ct\left(1-t\right)\left(b-a\right)^2$$

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for all $a, b \in I$ and $t \in [0,1]$.

Remark 1.1. It is clear from Definition 1.1 that every strongly convex function is also convex. However, the converse of this statement is generally not true. For example, $f : \mathbb{R} \to \mathbb{R}, f(x) = x$, is a convex function, but no matter how $c > 0$ the function f is not strongly convex with respect to the modulus c. Naturally, the inequalities obtained when we work with strongly convex functions will be better than the inequalities obtained for convex functions.

If a function $f : I \to \mathbb{R}$ is strongly convex with modulus c, then

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}(b-a)^2 \tag{1.2}$$

for all $a, b \in I,\ a < b$.

In this definition, if we take $c = 0$, we get the definition of convexity in the classical sense. Strongly convex functions play an important role in optimization theory and mathematical economics. Since strongly convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just "stronger versions" of known properties of convex functions.

Definition 1.2 ([35]). Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [5, 18] for studies on h-convexity.

Definition 1.3. Let $(X, \|.\|)$ be a real normed space, D stands for a convex subset of X, $h : (0,1) \to (0,\infty)$ is a given function and c is a positive constant. Then we say that a function $f : D \to \mathbb{R}$ is strongly h-convex with module c if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) - ct(1-t)\|x-y\|^2$$

for all $x, y \in D$ and $t \in (0,1)$.

Theorem 1.1. *Let $h : (0,1) \to (0,\infty)$ be a given function. If a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable and strongly h-convex with module $c > 0$, then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{a+b}{2}\right) + \frac{c}{12}(b-a)^2\right] &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq (f(a)+f(b))\int_a^b h(t)dt - \frac{c}{6}(b-a)^2\end{aligned}$$

for all $a, b \in I$, $a < b$.

In [34], Toplu et al., introduced the class of n-polynomial convex functions as follows:

Definition 1.4 ([34]). *Let $n \in \mathbb{N}$. Then a non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial convex function if for every $x, y \in I$ and $t \in [0,1]$,*

$$f(tx + (1-t)y) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]f(x) + \frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right]f(y). \tag{1.3}$$

We will denote by $POLC\,(I)$ the class of all n-polynomial convex functions on interval I. We note that, every n-polynomial convex function is a h-convex function with the function $h(t) = \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]$.

Theorem 1.2 ([34]). *Let $f : [a, b] \to \mathbb{R}$ be a n-polynomial convex function. If $a < b$ and $f \in L\,[a, b]$, then the following Hermite-Hadamrd type inequalities hold:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right) f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{f\,(a)+f(b)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}. \tag{1.4}$$

In [16], İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 1.3 (Hölder-İşcan integral inequality [16]). *Let $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a, b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \quad &\leq \quad \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ &\qquad \left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}. \end{aligned}\tag{1.5}$$

An refinement of power-mean integral inequality as a different version of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.4 (Improved power-mean integral inequality [22]). *Let $q \geq 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a, b]$ then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \quad &\leq \quad \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ &\qquad \left.+\left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

The main purpose of this paper is to introduce the concept of strongly n-polynomial convex functions and establish some results connected with the right-hand side of new inequalities similar to the Hermite-hadamard inequality for these classes of functions. Some applications to special means of positive real numbers are also given.

2. The definition of strongly n-polynomial convex functions

In this section, we introduce a concept which is called strongly n-polynomial convexity (which is a special case of the definition given in [11]) and we give by setting some algebraic properties for the strongly n-polynomial convex functions, as follows:

Definition 2.5. Let $n \in \mathbb{N}$, $I \subset \mathbb{R}$ be an interval and c be a positive number. A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called strongly n-polynomial convex function with modulus c if for every $x, y \in I$ and $t \in [0, 1]$,

$$f\,(tx+(1-t)y) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right] f(x) + \frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right] f(y) - ct\,(1-t)\,(x-y)^2\,. \tag{2.6}$$

We will denote by $SPOLC\,(I)$ the class of all strongly n-polynomial convex functions on interval I.

We note that, every strongly n-polynomial convex function is a strongly h-convex function with the function $h(t)=\frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]$. Therefore, if $f,g\in SPOLC\,(I)$, then

i.) $f+g\in SPOLC\,(I)$ *and for* $c\in\mathbb{R}$ $c\geq 0$ *and* $cf\in SPOLC\,(I)$ *(see* [35], *Proposition 9).*

ii.) if f and g be a similarly ordered functions on I , then $fg\in SPOLC\,(I)$ *(see* [35], *Proposition 10).*

Also, if $f:I\to J$ is a convex and $g\in SPOLC\,(J)$ and nondecreasing, then $g\circ f\in SPOLC\,(I)$ *(see* [35], *Theorem 15).*

We especially note that;

Remark 2.2. If we take $n=1$ in the inequality (2.6), then the strongly 1-polynomial convexity reduces to the clasical strongly convexity.

Remark 2.3. If we take $c=0$ in the inequality (2.6), then the strongly n-polynomial convexity reduces to the n-polynomial convexity.

Remark 2.4. If we take $n=1$ and $c=0$ in the inequality (2.6), then the strongly n-polynomial convexity reduces to the clasical convexity.

Remark 2.5. Let the function $f:I\subset\mathbb{R}\to[0,\infty)$ be a strongly 2-polynomial convex function if for every $x,y\in I$ and $t\in[0,1]$,

$$f\left(tx+(1-t)y\right)\leq\frac{3t-t^2}{2}f(x)+\frac{2-t-t^2}{2}f(y)-ct\left(1-t\right)\left(x-y\right)^2.$$

It is easily seen that

$$t\leq\frac{3t-t^2}{2}\text{ and }1-t\leq\frac{2-t-t^2}{2}$$

for all $t\in[0,1]$. This shows that every nonnegative strongly convex function is also a strongly 2-polynomial convex function.

More generally, we can give the following remark together with proof:

Remark 2.6. Every nonnegative strongly convex function is also a strongly n-polynomial convex function. Indeed, this case is clear from the following inequalities

$$t\leq\frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\text{ and }1-t\leq\frac{1}{n}\sum_{s=1}^{n}[1-t^s]$$

for all $t\in[0,1]$ and $n\in\mathbb{N}$ [34].

Example 2.1. Let $f(x)=x^2$ and $[a,b]=[-1,1]$. Then the function f is strongly n-polinomial convex with modulus $c=1$.

3. Hermite-Hadamard inequality for strongly n-polynomial convex functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for strongly n-polynomial convex functions. In this section, we will denote by $L\,[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 3.5. *Let $f:[a,b]\to\mathbb{R}$ be a strongly n-polynomial convex function with modulus c. If $a<b$ and $f\in L\,[a,b]$, then the following Hermite-Hadamard type inequalities hold:*

$$\begin{aligned}&\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[f\left(\frac{a+b}{2}\right)+\frac{c}{12}\left(b-a\right)^2\right]\\ \leq\;&\frac{1}{b-a}\int_a^b f(x)dx\leq\left(\frac{f\left(a\right)+f(b)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}-\frac{c}{6}\left(b-a\right)^2.\end{aligned}\tag{3.7}$$

Proof. By using the strongly n-polynomial convex function of f, we get

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) \\
= \;& f\left(\frac{[ta+(1-t)b]+[(1-t)a+tb]}{2}\right) \\
= \;& f\left(\frac{1}{2}\left[ta+(1-t)b\right]+\frac{1}{2}\left[(1-t)a+tb\right]\right) \\
\leq \;& \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(1-\frac{1}{2}\right)^{s}\right] f\left(ta+(1-t)b\right)+\frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^{s}\right] f\left((1-t)a+tb\right) \\
& -\frac{c}{4}\left[(2t-1)\,b-(1-2t)\,a\right]^{2} \\
= \;& \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^{s}\right]\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right]-\frac{c}{4}\left(2t-1\right)\left(b-a\right)^{2}.
\end{aligned}$$

By taking integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) \;\leq\; & \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^{s}\right]\left[\int_0^1 f\left(ta+(1-t)b\right)dt+\int_0^1 f\left((1-t)a+tb\right)dt\right] \\
& -\frac{c}{4}\left(b-a\right)^{2}\int_0^1\left(2t-1\right)^{2}dt \\
= \;& \frac{2}{b-a}\left(\frac{n+2^{-n}-1}{n}\right)\int_a^b f(x)dx-\frac{c}{12}\left(b-a\right)^{2}.
\end{aligned}$$

By using the property of the strongly n-polynomial convex function f, if the variable is changed as $x = ta+(1-t)b$, then

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)du=\int_0^1 f\left(ta+(1-t)b\right)dt \\
\leq \;& \int_0^1\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^{s}\right]f(a)+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^{s}\right]f(b)-ct\left(1-t\right)\left(b-a\right)^{2}\right]dt \\
= \;& \frac{f\left(a\right)}{n}\int_0^1\sum_{s=1}^{n}\left[1-(1-t)^{s}\right]dt+\frac{f\left(b\right)}{n}\int_0^1\sum_{s=1}^{n}\left[1-t^{s}\right]dt-c\left(b-a\right)^{2}\int_0^1 t\left(1-t\right)dt \\
= \;& \left[\frac{f\left(a\right)+f(b)}{n}\right]\sum_{s=1}^{n}\frac{s}{s+1}-\frac{c}{6}\left(b-a\right)^{2},
\end{aligned}$$

where

$$\int_0^1\left[1-(1-t)^{s}\right]dt=\int_0^1\left[1-t^{s}\right]=\frac{s}{s+1},\;\int_0^1\left(2t-1\right)^{2}dt=\frac{1}{3},\;\int_0^1 t\left(1-t\right)dt=\frac{1}{6}.$$

This completes the proof of theorem. □

Remark 3.7. In case of $n = 1$ and $c = 0$, (3.7) coincides with the the inequality (1.1).

Remark 3.8. In case of $n = 1$, (3.7) coincides with the the inequality (1.2).

Remark 3.9. In case of $c = 0$, (3.7) coincides with the the inequality (1.4).

4. New inequalities for strongly n-polynomial convex functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard type integral inequalities for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is strongly n-polynomial convex function. Dragomir and Agarwal [12] used the following lemma:

Lemma 4.1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt.$$

Theorem 4.6. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a, b]$. If $|f'|$ is a strongly n-polynomial convex function with modulus c on interval $[a, b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{4.8}$$
$$\leq \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right]A\left(\left|f'(a)\right|, \left|f'(b)\right|\right) - \frac{c}{12}(b-a)^3.$$

Proof. Using Lemma 4.1 and the inequality

$$\left|f'(ta+(1-t)b)\right| \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]\left|f'(a)\right| + \frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right]\left|f'(b)\right| - ct(1-t)(b-a)^2,$$

we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|dt \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\left(\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]\left|f'(a)\right| + \frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right]\left|f'(b)\right| - ct(1-t)(b-a)^2\right)dt \\
\leq\ & \frac{b-a}{2}\left(\begin{array}{c} \frac{\left|f'(a)\right|}{n}\int_0^1 |1-2t|\sum_{s=1}^{n}\left[1-(1-t)^s\right]dt \\ +\frac{\left|f'(b)\right|}{n}\int_0^1 |1-2t|\sum_{s=1}^{n}\left[1-t^s\right]dt - c(b-a)^2\int_0^1 t(1-t)\,dt \end{array}\right) \\
=\ & \frac{b-a}{2}\left(\begin{array}{c} \frac{\left|f'(a)\right|}{n}\sum_{s=1}^{n}\int_0^1 |1-2t|\left[1-(1-t)^s\right]dt \\ +\frac{\left|f'(b)\right|}{n}\sum_{s=1}^{n}\int_0^1 |1-2t|\left[1-t^s\right]dt - c(b-a)^2\int_0^1 t(1-t)\,dt \end{array}\right) \\
=\ & \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right]A\left(\left|f'(a)\right|, \left|f'(b)\right|\right) - \frac{c}{12}(b-a)^3,
\end{aligned}$$

where

$$\int_0^1 t(1-t)\,dt = \frac{1}{6},$$

$$\int_0^1 |1-2t|\left[1-(1-t)^s\right]dt = \int_0^1 |1-2t|\left[1-t^s\right]dt = \frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}$$

and A is the arithmetic mean. This completes the proof of theorem. □

Corollary 4.1. *If we take $n = 1$ and $c = 0$ in the inequality (4.8), then we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right).$$

This inequality coincides with the inequality in [12].

Corollary 4.2. *If we take $c = 0$ in the inequality (4.8), then we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right]A\left(\left|f'(a)\right|,\left|f'(b)\right|\right).$$

This inequality coincides with the inequality in [34].

Corollary 4.3. *If we take $n = 1$ in the inequality (4.8), then we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)-\frac{c}{12}(b-a)^3.$$

Theorem 4.7. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and assume that $f' \in L[a,b]$. If $|f'|^q$ is a strongly n-polynomial convex function with modulus c on interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (4.9)\\ \leq\ &\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)-\frac{c}{6}(b-a)^2\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$\left|f'(ta+(1-t)b)\right|^q \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]\left|f'(a)\right|^q+\frac{1}{n}\sum_{s=1}^{n}\left[1-t^s\right]\left|f'(b)\right|^q-ct(1-t)(b-a)^2$$

which is the strongly n-polynomial convex function of $|f'|^q$, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &\frac{b-a}{2}\int_0^1|1-2t|\left|f'(ta+(1-t)b)\right|dt\\ \leq\ &\frac{b-a}{2}\left(\int_0^1|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\ \leq\ &\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\begin{array}{c}\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1\left[1-(1-t)^s\right]dt+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1\left[1-t^s\right]dt\\ -c(b-a)^2\int_0^1 t(1-t)\,dt\end{array}\right)^{\frac{1}{q}}\\ =\ &\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)-\frac{c}{6}(b-a)^2\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 |1-2t|^p \, dt &= \frac{1}{p+1}, \\
\int_0^1 [1-(1-t)^s] \, dt &= \int_0^1 [1-t^s] \, dt = \frac{s}{s+1}
\end{aligned}$$

and A is the arithmetic mean. This completes the proof of theorem. □

Corollary 4.4. *If we take $n = 1$ and $c = 0$ in the inequality (4.9), then we get the following inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} A^{\frac{1}{q}} \left(|f'(a)|^q, |f'(b)|^q \right).$$

This inequality coincides with the inequality in [12].

Corollary 4.5. *If we take $c = 0$ in the inequality (4.9), then we get the following inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{n} \sum_{s=1}^{n} \frac{s}{s+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(|f'(a)|^q, |f'(b)|^q \right).$$

This inequality coincides with the inequality in [34].

Corollary 4.6. *If we take $n = 1$ in the inequality (4.9), then we get the following inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(A \left(|f'(a)|^q, |f'(b)|^q \right) - \frac{c}{6} (b-a)^2 \right)^{\frac{1}{q}}.$$

Theorem 4.8. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $q \geq 1$ and assume that $f' \in L[a,b]$. If $|f'|^q$ is a strongly n-polynomial convex function with modulus c on the interval $[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \qquad (4.10) \\
\leq\ & \frac{b-a}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\frac{2}{n} \sum_{s=1}^{n} \frac{(s^2+s+2)\, 2^s - 2}{(s+1)(s+2)2^{s+1}} A \left(|f'(a)|^q, |f'(b)|^q \right) - \frac{c}{16} (b-a)^2 \right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Assume first that $q > 1$. From Lemma 4.1, Hölder integral inequality and the property of the strongly n-polynomial convex function of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left[\frac{1}{n}\sum_{s=1}^{n}[1-(1-t)^s]\,|f'(a)|^q\right.\right. \\
&\left.\left.+\frac{1}{n}\sum_{s=1}^{n}[1-t^s]\,|f'(b)|^q\,dt - ct\,(1-t)\,(b-a)^2\right]\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 |1-2t|\,[1-(1-t)^s]\,dt\right. \\
&\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 |1-2t|\,[1-t^s]\,dt - c\,(b-a)^2\int_0^1 t\,(1-t)\,|1-2t|\,dt\right]^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}A\left(|f'(a)|^q,|f'(b)|^q\right)-\frac{c}{16}\,(b-a)^2\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 t(1-t)\,|1-2t|dt &= \frac{1}{16}, \\
\int_0^1 |1-2t|\,[1-(1-t)^s]\,dt &= \int_0^1 |1-2t|\,[1-(1-t)^s]\,dt = \frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}.
\end{aligned}$$

For $q = 1$ we use the estimates from the proof of Theorem 4.6, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 4.7. *Under the assumption of Theorem 4.8 with $q = 1$, we get the conclusion of Theorem 4.6.*

Corollary 4.8. *If we take $n = 1$, $c = 0$ and $q = 1$ in the inequality (4.10), we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}A\left(|f'(a)|,|f'(b)|\right).$$

This inequality coincides with the inequality in [12].

Corollary 4.9. *If we take $c = 0$ in the inequality (4.10), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right).
\end{aligned}$$

This inequality coincides with the inequality in [34]. *Also, if we take $q = 1$ in the above inequality, we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right).$$

Also, this inequality coincides with the inequality in [34].

Corollary 4.10. *If we take $n = 1$ in the inequality (4.10), we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)-\frac{c}{8}(b-a)^2\right)^{\frac{1}{q}}.$$

Also, if we take $q = 1$ in the above inequality, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)-\frac{c}{8}(b-a)^2\right)$$

Now, we will prove the Theorem 4.7 by using Hölder-İşcan integral inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 4.7.

Theorem 4.9. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and assume that $f' \in L[a,b]$. If $|f'|^q$ is a strongly n-polynomial convex function on interval $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{4.11}$$

$$\leq \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}-\frac{c}{12}(b-a)^2\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}-\frac{c}{12}(b-a)^2\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 4.1, Hölder-İşcan integral inequality and the strongly n-polynomial convexity of $|f'|^q$, we get

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\,|f'\,(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\int_0^1 t\,|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\,|f'\,(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 (1-t)\,[1-(1-t)^s]\,dt\right. \\
&\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 (1-t)\,[1-t^s]\,dt-c\,(b-a)^2\int_0^1 t\,(1-t)^2\,dt\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 t\,[1-(1-t)^s]\,dt\right. \\
&\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 t\,[1-t^s]\,dt-c\,(b-a)^2\int_0^1 t^2\,(1-t)\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}-\frac{c}{12}\,(b-a)^2\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}-\frac{c}{12}\,(b-a)^2\right)^{\frac{1}{q}}
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 (1-t)\,|1-2t|^p\,dt &= \int_0^1 t\,|1-2t|^p\,dt=\frac{1}{2\,(p+1)}, \\
\int_0^1 t\,(1-t)^2\,dt &= \int_0^1 t^2\,(1-t)\,dt=\frac{1}{12}, \\
\int_0^1 (1-t)\,[1-(1-t)^s]\,dt &= \int_0^1 t\,[1-t^s]\,dt=\frac{s}{2(s+2)}, \\
\int_0^1 (1-t)\,[1-t^s]\,dt &= \int_0^1 t\,[1-(1-t)^s]\,dt=\frac{s(s+3)}{2(s+1)(s+2)}
\end{aligned}
$$

This completes the proof of theorem. □

Corollary 4.11. *If we take $c = 0$ in the inequality (4.11), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}\right)^{\frac{1}{q}}.
\end{aligned}$$

This inequality coincides with the inequality in [34].

Corollary 4.12. *If we take $n = 1$ and $c = 0$ in (4.11), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{4\left(p+1\right)^{\frac{1}{p}}}\left[\left(\frac{|f'(a)|^q+2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'(a)\right|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This inequality coincides with the inequality of Theorem 3.2 in [16].

Corollary 4.13. *If we take $n = 1$ in the inequality (4.11), we get the following inequality:*

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq\ & \frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+2\left|f'(b)\right|^q}{6}-\frac{c}{12}\left(b-a\right)^2\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left(\frac{2\left|f'(a)\right|^q+|f'(b)|^q}{6}-\frac{c}{12}\left(b-a\right)^2\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 4.10. The inequality (4.11) gives better results than (4.9). Let us show that

$$\begin{aligned}
&\frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}-\frac{c}{12}\left(b-a\right)^2\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}-\frac{c}{12}\left(b-a\right)^2\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}A\left(|f'(a)|^q,|f'(b)|^q\right)-\frac{c}{6}\left(b-a\right)^2\right)^{\frac{1}{q}}.
\end{aligned}$$

Using concavity of $h : [0, \infty) \to \mathbb{R}$, $h(x) = x^{\lambda}, 0 < \lambda \leq 1$ by sample calculation we get

$$
\begin{aligned}
& \frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}-\frac{c}{12}\,(b-a)^2\right)^{\frac{1}{q}} \\
& +\frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}-\frac{c}{12}\,(b-a)^2\right)^{\frac{1}{q}} \\
\leq\; & \frac{b-a}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}} 2\left[\frac{1}{2}\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}+\frac{1}{2}\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}-\frac{1}{2}\frac{c}{6}\,(b-a)^2\right]^{\frac{1}{q}} \\
=\; & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}A\left(|f'(a)|^q, |f'(b)|^q\right)-\frac{c}{6}\,(b-a)^2\right)^{\frac{1}{q}},
\end{aligned}
$$

which is the required.

Theorem 4.10. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, $q \geq 1$ and assume that $f' \in L\,[a, b]$. If $|f'|^q$ is a strongly n-polynomial convex function on the interval $[a, b]$, then the following inequality holds*

$$
\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (4.12) \\
\leq\; & \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)-\frac{c}{32}\,(b-a)^2\right)^{\frac{1}{q}} \\
& +\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)-\frac{c}{32}\,(b-a)^2\right)^{\frac{1}{q}}.
\end{aligned}
$$

Proof. Assume first that $q > 1$. From Lemma 4.1, improved power-mean integral inequality and the property of the strongly n-polynomial convex function of $|f'|^q$, we obtain

$$
\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\; & \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|1-2t|\,|f'\,(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
& +\frac{b-a}{2}\left(\int_0^1 t\,|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|1-2t|\,|f'\,(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 (1-t)\,|1-2t|\,[1-(1-t)^s]\,dt\right.\\
&\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 (1-t)\,|1-2t|\,[1-t^s]\,dt-c\,(b-a)^2\int_0^1 t\,(1-t)^2\,|1-2t|\,dt\right)^{\frac{1}{q}}\\
&+\frac{b-a}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}\int_0^1 t\,|1-2t|\,[1-(1-t)^s]\,dt\right.\\
&\left.+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}\int_0^1 t\,|1-2t|\,[1-t^s]\,dt-c\,(b-a)^2\int_0^1 t^2\,(1-t)\,|1-2t|\,dt\right)^{\frac{1}{q}}\\
&= \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)-\frac{c}{32}\,(b-a)^2\right)^{\frac{1}{q}}\\
&+\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)-\frac{c}{32}\,(b-a)^2\right)^{\frac{1}{q}}
\end{aligned}
$$

where

$$
\int_0^1 t\,(1-t)^2\,|1-2t|\,dt=\int_0^1 t^2\,(1-t)\,|1-2t|\,dt=\frac{1}{32},
$$

$$
\begin{aligned}
K_1(s) \quad &: \quad = \int_0^1 (1-t)\,|1-2t|\,[1-(1-t)^s]\,dt=\int_0^1 t\,|1-2t|\,[1-t^s]\,dt\\
&= \frac{\left(s^2+s+2\right)2^s-2}{2^{s+2}(s+2)(s+3)},\\
K_2(s) \quad &: \quad = \int_0^1 t\,|1-2t|\,[1-(1-t)^s]\,dt=\int_0^1 (1-t)\,|1-2t|\,[1-t^s]\,dt\\
&= \frac{(s+5)\left[\left(s^2+s+2\right)2^s-2\right]}{2^{s+2}(s+1)(s+2)(s+3)}.
\end{aligned}
$$

For $q=1$ we use the estimates from the proof of Theorem 4.6, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 4.14. *If we take $c=0$ in the inequality (4.12), we get the following inequality:*

$$
\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)\right)^{\frac{1}{q}}\\
&+\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)\right)^{\frac{1}{q}}.
\end{aligned}
$$

This inequality coincides with the inequality in [34]. *Also, if we take $q=1$ in the above inequality, we get the following inequality:*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{2^{s+1}(s+1)(s+2)}A\left(|f'(a)|\,,|f'(b)|\right).
$$

Corollary 4.15. *If we take $n=1$ in the inequality (4.12), we get the following inequality:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{16}+\frac{3|f'(b)|^q}{16}-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\\ &+\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{3|f'(a)|^q}{16}+\frac{|f'(b)|^q}{16}-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}.\end{aligned}$$

Also, if we take $q=1$ in the above inequality, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(A\left(|f'(a)|,|f'(b)|\right)-\frac{c}{8}(b-a)^2\right).$$

Corollary 4.16. *If we take $n=1$ and $c=0$ in the inequality (4.12), we get the following inequality:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left[\left(\frac{|f'(a)|^q}{16}+\frac{3|f'(b)|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q}{16}+\frac{|f'(b)|^q}{16}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Also, if we take $q=1$ in the above inequality, we get the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}A\left(|f'(a)|,|f'(b)|\right).$$

Remark 4.11. *The inequality (4.12) gives better result than the inequality (4.10). Let us show that*

$$\begin{aligned}&\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\\ &+\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\\ \leq\ &\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}A\left(|f'(a)|^q,|f'(b)|^q\right)-\frac{c}{16}(b-a)^2\right)^{\frac{1}{q}}.\end{aligned}$$

If we use the concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\leq 1$, we get

$$\begin{aligned}&\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_1(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_2(s)-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\\ &+\frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|f'(a)|^q}{n}\sum_{s=1}^{n}K_2(s)+\frac{|f'(b)|^q}{n}\sum_{s=1}^{n}K_1(s)-\frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}\\ \leq\ &\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{2}{n}\sum_{s=1}^{n}\left[K_1(s)+K_2(s)\right]A\left(|f'(a)|^q,|f'(b)|^q\right)-\frac{c}{16}(b-a)^2\right)^{\frac{1}{q}},\end{aligned}$$

where

$$K_1(s)+K_2(s)=\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}$$

which completes the proof of remark.

5. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers a, b with $b > a$:

1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a,b \geq 0,$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a,b \geq 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0,$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b \\ a, & a = b \end{cases} ; \quad a,b > 0$$

5. The p-logaritmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R}\backslash\{-1,0\} \\ a, & a = b \end{cases} ; \quad a,b > 0.$$

6.The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b > 0,$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 5.1. *Let $a, b \in [-1,1]$ with $a < b$. Then, the following inequalities are obtained:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[A^2(a,b) + \frac{c}{12}(b-a)^2\right] \leq L_2^2(a,b) \leq A(a^2,b^2)\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1} - \frac{c}{6}(b-a)^2.$$

Proof. The assertion follows from the inequalities (3.7) for the function

$$f(x) = x^2, \;\; x \in [-1,1].$$

Because, $f(x) = x^2, \;\; x \in [-1,1]$ is a strongly n -polinomial convex function. □

Proposition 5.2. *Let $a, b \in [-1,1]$ with $a < b$. Then, the following inequalities are obtained:*

$$\frac{2}{3}\left[A\left(a^{\frac{3}{2}}, b^{\frac{3}{2}}\right) - L_{\frac{3}{2}}^{\frac{3}{2}}(a,b)\right] \leq \frac{b-a}{n}\sum_{s=1}^{n}\left[\frac{(s^2+s+2)\,2^s-2}{(s+1)(s+2)2^{s+1}}\right]A(a^2,b^2) - \frac{c}{12}(b-a)^3.$$

Proof. The assertion follows from the inequalities (4.8) for the function $f(x) = \frac{2}{3}x^{\frac{3}{2}}$. Because, the function

$$f'(x) = x^2, \;\; x \in [-1,1]$$

is a strongly n-polinomial convex function. □

Proposition 5.3. *Let $a,b\in[-1,1]$ with $a<b$ and $q>1$. Then, the following inequalities are obtained:*

$$\begin{aligned}
&\frac{q}{2+q}\left[A\left(a^{\frac{2+q}{q}},b^{\frac{2+q}{q}}\right)-L_{\frac{2+q}{q}}^{\frac{2+q}{q}}(a,b)\right]\\
\leq\ &\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}A\left(a^2,b^2\right)-\frac{c}{6}\left(b-a\right)^2\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. The assertion follows from the inequalities (4.9) for the function $f(x)=\frac{q}{2+q}x^{\frac{2+q}{q}}$. Because, the function

$$\left|f'(x)\right|^q=x^2\in[-1,1].$$

is a strongly n-polinomial convex function. □

Proposition 5.4. *Let $a,b\in[-1,1]$ with $a<b$ and $q\geq 1$. Then, the following inequalities are obtained:*

$$\begin{aligned}
&\frac{q}{2+q}\left[A\left(a^{\frac{2+q}{q}},b^{\frac{2+q}{q}}\right)-L_{\frac{2+q}{q}}^{\frac{2+q}{q}}(a,b)\right]\\
\leq\ &\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{\left(s^2+s+2\right)2^s-2}{(s+1)(s+2)2^{s+1}}A\left(a^2,b^2\right)-\frac{c}{16}\left(b-a\right)^2\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. The assertion follows from the inequalities (4.10) for the function $f(x)=\frac{q}{2+q}x^{\frac{2+q}{q}}$. Because, the function

$$\left|f'(x)\right|^q=x^2\in[-1,1].$$

is a strongly n-polinomial convex function. □

Proposition 5.5. *Let $a,b\in[-1,1]$ with $a<b$ and $q>1$. Then, the following inequalities are obtained:*

$$\begin{aligned}
&\frac{q}{2+q}\left[A\left(a^{\frac{2+q}{q}},b^{\frac{2+q}{q}}\right)-L_{\frac{2+q}{q}}^{\frac{2+q}{q}}(a,b)\right]\\
\leq\ &\frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left(\frac{a^2}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{b^2}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}-\frac{c}{12}\left(b-a\right)^2\right)^{\frac{1}{q}}\\
&+\frac{b-a}{2}\left(\frac{1}{2\left(p+1\right)}\right)^{\frac{1}{p}}\left(\frac{a^2}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{b^2}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}-\frac{c}{12}\left(b-a\right)^2\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. The assertion follows from the inequalities (4.11) for the function $f(x)=\frac{q}{2+q}x^{\frac{2+q}{q}}$. Because, the function

$$\left|f'(x)\right|^q=x^2\in[-1,1].$$

is a strongly n-polinomial convex function. □

Proposition 5.6. *Let $a, b \in [-1,1]$ with $a < b$ and $q \geq 1$. Then, the following inequalities are obtained:*

$$\begin{aligned}
&\frac{q}{2+q}\left[A\left(a^{\frac{2+q}{q}}, b^{\frac{2+q}{q}}\right) - L_{\frac{2+q}{q}}^{\frac{2+q}{q}}(a,b)\right] \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{a^2}{n}\sum_{s=1}^{n}K_1(s) + \frac{b^2}{n}\sum_{s=1}^{n}K_2(s) - \frac{c}{32}(b-a)^2\right)^{\frac{1}{q}} \\
& + \frac{b-a}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{a^2}{n}\sum_{s=1}^{n}K_2(s) + \frac{b^2}{n}\sum_{s=1}^{n}K_1(s) - \frac{c}{32}(b-a)^2\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. The assertion follows from the inequalities (4.12) for the function $f(x) = \frac{q}{2+q}x^{\frac{2+q}{q}}$. Because, the function

$$|f'(x)|^q = x^2 \in [-1,1].$$

is a strongly n-polinomial convex function. □

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