On a particular extension of the EV-Theorem

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ABSTRACT. The main aim of the paper is to determine the extreme values of the product $P = a_1a_2 \cdots a_n$ under the constraints $\sum_{i=1}^{n} a_i = S$ and $\sum_{i=1}^{n} \frac{1}{a_i+1} = S_0$ for $n \geq 3$ nonnegative real numbers $a_1, a_2, \ldots, a_n$ and some given constants $S$ and $S_0$. Some interesting applications of our results are provided as well.

1. INTRODUCTION

Let $a_1, a_2, \ldots, a_n$ ($n \geq 3$) be nonnegative real numbers such that
\[
\sum_{i=1}^{n} a_i = \text{fixed}, \quad \sum_{i=1}^{n} \frac{1}{a_i+1} = \text{fixed}.
\]
If we are interested in finding the minimum and the maximum value of the product $P = a_1a_2 \cdots a_n$, then we are tempted to use the EV-Theorem (see [1-3]). To do this, the following substitution is necessary:
\[
\frac{1}{a_i+1} = x_i, \quad a_i = \frac{1}{x_i} - 1, \quad x_i \in (0, 1], \quad i = 1, 2, \ldots, n.
\]
Thus, we need to find the minimum and the maximum value of the product
\[
P = \left( \frac{1}{x_1} - 1 \right) \left( \frac{1}{x_2} - 1 \right) \cdots \left( \frac{1}{x_n} - 1 \right)
\]
for
\[
\sum_{i=1}^{n} x_i = \text{fixed}, \quad \sum_{i=1}^{n} \frac{1}{x_i} = \text{fixed}.
\]
By the EV-Theorem, if $f$ is a real valued function, continue and differentiable on $(0, 1)$, $f(1-) = \pm \infty$ and the joined function $g(x) = f'(\frac{1}{\sqrt{x}})$ is strictly convex for $\frac{1}{\sqrt{x}} \in (0, 1)$, i.e. for $x \in (1, \infty)$, then the sum
\[
S_n = f(x_1) + f(x_2) + \cdots + f(x_n)
\]
attains its maximum (if $S_n$ has a global maximum) for $x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and its minimum (if $S_n$ has a global minimum) for $x_1 \leq x_2 = x_3 = \cdots = x_n$. In our case, the function
\[
f(x) = \ln \left( \frac{1}{x} - 1 \right)
\]

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Under the condition (*), there is a unique set \( (x) \). Proof.

From \( S \), where \( x \geq 0 \), be obtained from Karamata’s inequality [5,6] applied to the convex function \( g(x) = \frac{1}{x+1} \).

Let \( \text{Lemma 2.1} \). Theorem 2.1, we need \( \text{Lemma 2.1} \) and \( \text{Proposition 2.1} \) below.

We get \( S \), there is at least a set \( (x) \). Note that the domain \( D = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n a_i = S, \right\} \) is a non-empty compact set in \( \mathbb{R}^n_+ \) if and only if

\[
\frac{n^2}{S+n} \leq S_0 \leq \frac{(n-1)S+n}{S+1}. 
\]

Note that the domain \( g(x) = \frac{1}{x+1} \), where \( x \geq 0 \):

\[
g(a_1) + g(a_2) + \cdots + g(a_n) \leq g(a_1 + a_2 + \cdots + a_n) + g(0) + \cdots + g(0).
\]

Under the condition (*), there is a unique set \( (a_1, a_2, \ldots, a_n) \) such that \( a_1 \geq a_2 = a_3 = \cdots = a_n \geq 0, \sum_{i=1}^n a_i = S \) and \( \sum_{i=1}^n \frac{1}{a_i+1} = S_0 \). Also, under the condition

\[
\frac{n^2}{S+n} \leq S_0 < \frac{S+n(n-1)}{S+n-1}, \quad S > 0,
\]

there is a unique set \( (a_1, a_2, \ldots, a_n) \) such that \( a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0, \sum_{i=1}^n a_i = S \) and \( \sum_{i=1}^n \frac{1}{a_i+1} = S_0 \). Moreover, for

\[
\frac{S+n(n-1)}{S+n-1} \leq S_0 \leq \frac{(n-1)S+n}{S+1}, \quad S \geq 0,
\]

there is at least a set \( (a_1, a_2, \ldots, a_n) \) such that \( a_n = 0, \sum_{i=1}^n a_i = S \) and \( \sum_{i=1}^n \frac{1}{a_i+1} = S_0 \).

2. MAIN RESULTS

The main results of the paper are given in Theorem 2.1 and Theorem 2.2. To prove Theorem 2.1, we need Lemma 2.1 and Proposition 2.1 below.

Lemma 2.1. Let \( a, b, c \) be nonnegative real numbers such that \( a \geq b \geq c \) and

\[
a + b + c = S > 0, \quad \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = S_0,
\]

where \( S_0 \in (1, 3) \) and \( \frac{9}{S+3} < S_0 < \frac{2S+3}{S+1} \). For fixed \( S \) and \( S_0 \), the range of \( b \) is an interval \( [m, M] \) with \( m < M \). In addition, \( b = m \) for \( b = c \), and \( b = M \) for either \( a = b \) or \( c = 0 \).

Proof. From

\[
\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} < 1 + 1 + 1 = 3
\]

we get \( S_0 < 3 \), by the AM-HM inequality

\[
[(a+1) + (b+1) + (c+1)][\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}] \geq 9
\]
we get $S_0 \geq \frac{9}{S+3}$, and from Karamata’s inequality
\[
\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq \frac{1}{a+b+c+1} + \frac{1}{0+1} + \frac{1}{0+1}
\]
we get
\[
S_0 \leq \frac{2S+3}{S+1}.
\]
The equalities $S_0 = \frac{9}{S+3}$ and $S_0 = \frac{2S+3}{S+1}$ involve $a = b = c = \frac{S}{3}$ and $S = a > b = c = 0$, respectively. Therefore, in these cases, $m = M$. Next, according to the statement conditions, we may consider $a$ and $c$ ($a > c$) as functions of $b$. From
\[
a' + 1 + c' = 0, \quad \frac{a'}{(a+1)^2} + \frac{1}{(b+1)^2} + \frac{c'}{(c+1)^2} = 0,
\]
we get
\[
a'(b) = \frac{(b-c)(b+c+2)(a+1)^2}{(a-c)(a+c+2)(b+1)^2} \leq 0, \quad c'(b) = \frac{(a-b)(a+b+2)(c+1)^2}{(a-c)(a+c+2)(b+1)^2} \leq 0.
\]
Let us define the nonnegative functions
\[
f_1(b) = b - c(b), \quad f_2(b) = a(b) - b, \quad f_3(b) = c(b).
\]
Since
\[
f_1'(b) = 1 - c'(b) > 0, \quad f_2'(b) = a'(b) - 1 < 0, \quad f_3'(b) = c'(b) \leq 0,
\]
these functions are strictly increasing, decreasing and decreasing, respectively. The inequality $f_1(b) \geq 0$ (with $f_1$ increasing) involves $b \geq m$, where $m$ is a root of the equation $c(b) = b$, the inequality $f_2(b) \geq 0$ (with $f_2$ decreasing) involves $b \leq b_2$, where $b_2$ is a root of the equation $a(b) = b$, and the inequality $f_3(b) \geq 0$ (with $f_3$ decreasing) involves $b \leq b_3$, where $b_3$ is a root of the equation $c(b) = 0$. Therefore, $M = \min\{b_2, b_3\}$ and $b \in [m, M]$, with $b = m$ for $b = c$, and $b = M$ for either $a = b$ or $c = 0$.

**Proposition 2.1.** Let $a_1, b_1, c_1$ be fixed nonnegative real numbers,
\[
S = a_1 + b_1 + c_1, \quad S_0 = \frac{1}{a_1+1} + \frac{1}{b_1+1} + \frac{1}{c_1+1},
\]
and let $a, b, c$ be nonnegative real numbers such that $a \geq b \geq c$ and
\[
a + b + c = S, \quad \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = S_0.
\]
For $S_0 > 1$, the product $P = abc$ achieves its maximum for $a \geq b = c$, and its minimum for either $a = b \geq c > 0$ or $c = 0$.

**Proof.** If $S = 0$, then $a = b = c = 0$ and the conclusion follows. Consider further $S > 0$. As shown at Lemma 2.1, in the special cases $S_0 = 3$, $S_0 = \frac{9}{S+3}$ and $S_0 = \frac{2S+3}{S+1}$, a single set $(a, b, c)$ verifies the given equations. This set has respectively $a = b = c = 0$, $a = b = c = \frac{S}{3}$ and $S = a > b = c = 0$, satisfying the extremum conditions in the statement ($b = c$ and either $a = b$ or $c = 0$). Consider further that
\[
S_0 < 3, \quad S_0 > \frac{9}{S+3}, \quad S_0 < \frac{2S+3}{S+1},
\]
when $b \in [m, M]$, $m < M$. Thus, we may consider $a$ and $c$ as functions of $b$. We will show that $P'(b) \leq 0$. From
\[
P'(b) = a'bc + ac + abc'
\]
and the expressions of $a'$ and $c'$ determined in the proof of Lemma 2.1, we write the inequality $P'(b) \leq 0$ as

$$ab(a-b)(a+b+2)(c+1)^2 + bc(b-c)(b+c+2)(a+1)^2 \geq ac(a-c)(a+c+2)(b+1)^2.$$  

Replacing $a-c$ with $(a-b) + (b-c)$, the inequality becomes as follows:

$$a(a-b)A \geq c(b-c)B,$$

where

$$A = b(S + 2 - c)(c + 1)^2 - c(S + 2 - b)(b + 1)^2,$$

$$B = a(S + 2 - b)(b + 1)^2 - b(S + 2 - a)(a + 1)^2.$$  

Since

$$A = (S + 2)b(c + 1)^2 - c(b + 1)^2 + bc[(b + 1)^2 - (c + 1)^2]$$

and

$$B = (S + 2)a(b + 1)^2 - b(a + 1)^2 + ab[(a + 1)^2 - (b + 1)^2]$$

we have

$$a(a-b)A - c(b-c)B = (a-b)(b-c)(a-c)(S + 2 - abc).$$

Thus, we only need to show that $S + 2 - abc \geq 0$. Indeed, from $S_0 > 1$, we get

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} > 1,$$

which is equivalent to $S + 2 - abc > 0$. Since $P'(b) \leq 0$, the function $P(b)$ is strictly decreasing, therefore the product $P$ achieves its maximum for $b = m$, when $a \geq b = c$, and its minimum for $b = M$, when either $a = b \geq c$ or $c = 0$ (see Lemma 2.1).  

**Theorem 2.1.** Let $c_1, c_2, \ldots, c_n$ ($n \geq 3$) be fixed nonnegative real numbers,

$$S = \sum_{i=1}^{n} c_i, \quad S_0 = \sum_{i=1}^{n} \frac{1}{c_i + 1},$$

and let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and

$$\sum_{i=1}^{n} a_i = S, \quad \sum_{i=1}^{n} \frac{1}{a_i + 1} = S_0.$$  

If $S_0 > 1$ for $n = 3$ and $S_0 \geq n - 2$ for $n \geq 4$, then

(a) the product $P = a_1a_2 \cdots a_n$ achieves its maximum for $a_1 \geq a_2 = a_3 = \cdots = a_n$;

(b) the product $P = a_1a_2 \cdots a_n$ achieves its minimum for either $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$ or $a_n = 0$.

**Proof.** Since the domain

$$D = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}_+^n : \sum_{i=1}^{n} a_i = S, \sum_{i=1}^{n} \frac{1}{a_i + 1} = S_0 \right\}$$

is a non-empty compact set in $\mathbb{R}_+^n$, the product $P$ achieves its maximum and minimum. For $n = 3$, the conclusion follows from Proposition 2.1. For $n \geq 4$, we use the contradiction method.

(a) Assume, for the sake of contradiction, that $P$ achieves its maximum at $(b_1, b_2, \ldots, b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n$ and $b_2 > b_1$. Let $x_1, x_2, x_n$ be nonnegative real numbers such that $x_1 \geq x_2 \geq x_n$ and

$$x_1 + x_2 + x_n = b_1 + b_2 + b_n,$$
where $S_m < M$. We have

$$S_0 = \sum_{i=1}^{n} \frac{1}{b_i + 1} \leq \frac{1}{1 + b_1} + \frac{n - 2}{1 + b_n} \leq S_3 + n - 3,$$

hence

$$S_3 \geq S_0 - n + 3 \geq n - 2 - n + 3 = 1.$$

The equality $S_3 = 1$ holds only if $S_0 = n - 2$ and $b_3 = b_4 = \cdots = b_n = 0$. This is not possible since it leads to the contradiction

$$n - 2 = \sum_{i=1}^{n} \frac{1}{b_i + 1} = \frac{1}{1 + b_1} + \frac{1}{1 + b_2} + n - 2.$$

Therefore, we have $S_3 > 1$. According to Proposition 2.1, the product $x_1 x_2 x_n$ achieves its maximum for $x_2 = x_n$. So, we have $x_1 x_2 x_n > b_1 b_2 b_n$, which contradicts the assumption that the product achieves its maximum at $(b_1, b_2, \ldots, b_n)$.

(b) Assume, for the sake of contradiction, that $P$ achieves its minimum at $(b_1, b_2, \ldots, b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n > 0$ and $b_1 > b_{n-1}$. Let $x_1, x_{n-1}, x_n$ be nonnegative real numbers such that $x_1 \geq x_{n-1} \geq x_n$ and

$$x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n,$$

$$\frac{1}{1 + x_1} + \frac{1}{1 + x_{n-1}} + \frac{1}{1 + x_n} = \frac{1}{1 + b_1} + \frac{1}{1 + b_{n-1}} + \frac{1}{1 + b_n} := S_3.$$

We have

$$S_0 = \sum_{i=1}^{n} \frac{1}{b_i + 1} \leq \frac{1}{1 + b_1} + \frac{n - 2}{1 + b_{n-1}} + \frac{1}{1 + b_n} \leq S_3 + n - 3,$$

hence

$$S_3 \geq S_0 - n + 3 \geq n - 2 - n + 3 = 1.$$

The equality $S_3 = 1$ holds only if $S_0 = n - 2$ and $b_2 = b_3 = \cdots = b_n = 0$. This is not possible since it leads to the contradiction

$$n - 2 = \sum_{i=1}^{n} \frac{1}{b_i + 1} = \frac{1}{1 + b_1} + n - 1.$$

Therefore, we have $S_3 > 1$. According to Proposition 2.1, the product $x_1 x_{n-1} x_n$ achieves its minimum for $x_1 = x_{n-1} > x_n > 0$ or $x_n = 0$. Thus, we have $x_1 x_{n-1} x_n > b_1 b_{n-1} b_n$, which contradicts the assumption that the product achieves its minimum at $(b_1, b_2, \ldots, b_n)$.

\[
\text{Lemma 2.2.} \text{ Let } a, b, c \text{ be nonnegative real numbers such that } a \geq b \geq c \text{ and }
\]

$$a + b + c = S, \quad \frac{1}{a + 1} + \frac{1}{b + 1} + \frac{1}{c + 1} = S_0,$$

where $S_0 < 1$ and $S_0 > \frac{9}{S + 3}$. For fixed $S$ and $S_0$, the range of $b$ is an interval $[m, M]$ with $m < M$. In addition, $b = m$ for $b = c$, and $b = M$ for $a = b$.

\[\text{Proof.} \text{ It is not possible to have } c = 0 \text{ since this involves the contradiction }
\]

$$1 > S_0 = \frac{1}{a + 1} + \frac{1}{b + 1} + 1.$$
By the AM-HM inequality

\[(a + 1) + (b + 1) + (c + 1) \geq 9,\]

we get

\[S_0 = \frac{a + b + c}{a + 1 + b + 1 + c + 1} \geq \frac{9}{S + 3}.\]

The equality \(S_0 = \frac{9}{S + 3}\) involves \(a = b = c = \frac{S}{3}\), hence \(m = M\). For \(S_0 > \frac{9}{S + 3}\), we may consider \(a\) and \(c\) as functions of \(b\). Furthermore, the proof is identical to that of Lemma 2.1, but without using the function \(f_3(b)\) (because it cannot decrease to zero).

\[\square\]

**Proposition 2.2.** Let \(a_1, b_1, c_1\) be fixed nonnegative real numbers,

\[S = a_1 + b_1 + c_1, \quad S_0 = \frac{1}{a_1 + 1} + \frac{1}{b_1 + 1} + \frac{1}{c_1 + 1},\]

and let \(a, b, c\) be nonnegative real numbers such that \(a \geq b \geq c\) and

\[a + b + c = S, \quad \frac{1}{a + 1} + \frac{1}{b + 1} + \frac{1}{c + 1} = S_0.\]

For \(S_0 < 1\), the product \(P = abc\) achieves its maximum for \(a = b \geq c\), and its minimum for \(a \geq b = c\).

**Proof.** If \(S = 0\), then \(a = b = c = 0\) and the conclusion follows. Consider further \(S > 0\).

As shown at Lemma 2.2, in the special case \(S_0 = \frac{9}{S + 3}\), the given equations are satisfied for \(a = b = c = \frac{S}{3}\). Consider further that \(S_0 > \frac{9}{S + 3}\), when \(a > c\) and \(b \in [m, M]\), \(m < M\).

Thus, we may consider \(a\) and \(c\) as functions of \(b\). We will show that \(P'(b) \geq 0\). As shown in the proof of Proposition 2.1, this inequality is equivalent to

\[(a - b)(b - c)(a - c)(S + 2 - abc) \leq 0.\]

Thus, we only need to show that \(S + 2 - abc \leq 0\). Indeed, from \(S_0 < 1\) we get

\[\frac{1}{a + 1} + \frac{1}{b + 1} + \frac{1}{c + 1} < 1,\]

which is equivalent to \(S + 2 - abc < 0\). Since \(P'(b) \geq 0\), the function \(P(b)\) is strictly increasing, therefore the product \(P\) achieves its maximum for \(b = M\), when \(a = b \geq c\), and its minimum for \(b = m\), when \(a \geq b = c\) (see Lemma 2.2).

\[\square\]

**Theorem 2.2.** Let \(c_1, c_2, \ldots, c_n (n \geq 3)\) be fixed nonnegative real numbers,

\[S = \sum_{i=1}^{n} c_i, \quad S_0 = \sum_{i=1}^{n} \frac{1}{c_i + 1},\]

and let \(a_1, a_2, \ldots, a_n\) be nonnegative real numbers such that \(a_1 \geq a_2 \geq \cdots \geq a_n\) and

\[\sum_{i=1}^{n} a_i = S, \quad \sum_{i=1}^{n} \frac{1}{a_i + 1} = S_0.\]

If \(S_0 < 1\) for \(n = 3\) and \(S_0 \leq 1\) for \(n \geq 4\), then

(a) the product \(P = a_1 a_2 \cdots a_n\) achieves its maximum for \(a_1 = a_2 = \cdots = a_{n-1} \geq a_n\);

(b) the product \(P = a_1 a_2 \cdots a_n\) achieves its minimum for \(a_1 \geq a_2 = a_3 = \cdots = a_n\).
Remark 2.2. We may reformulate Theorem 2.1 and Theorem 2.2 as follows:

Theorem 2.1'. Let \( 1 < m < n \) and the product \( b \) achieves its minimum at \( x = x_n \).

In this case we have \( x = x_n \).

Remark 2.1. We have \( a = 0 \).

Proof. Since the domain

\[
D = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}_{++}^n : \sum_{i=1}^{n} a_i = S, \sum_{i=1}^{n} \frac{1}{a_i + 1} = S_0 \right\}
\]

is a non-empty compact set in \( \mathbb{R}_{++}^n \), the product \( P \) achieves its maximum and minimum. For \( n = 3 \), the conclusion turns out from Proposition 2.2. For \( n \geq 4 \), we use the contradiction method.

(a) Assume, for the sake of contradiction, that \( P \) has the maximum value for a set \((b_1, b_2, \ldots, b_n)\) with \( b_1 \geq b_2 \geq \cdots \geq b_n \) and \( b_1 > b_{n-1} \), which satisfies the given two equations. Let \( x_1, x_{n-1}, x_n \) be positive real numbers such that \( x_1 \geq x_{n-1} \geq x_n \) and

\[
\begin{align*}
x_1 + x_{n-1} + x_n &= b_1 + b_{n-1} + b_n, \\
\frac{1}{1 + x_1} + \frac{1}{1 + x_{n-1}} + \frac{1}{1 + x_n} &= \frac{1}{1 + b_1} + \frac{1}{1 + b_{n-1}} + \frac{1}{1 + b_n} := S_3.
\end{align*}
\]

We have

\[
S_3 < S_0 \leq 1.
\]

According to Proposition 2.2, the product \( x_1 x_{n-1} x_n \) achieves its maximum for \( x_1 = x_{n-1} \). In this case we have \( x_1 x_{n-1} x_n > b_1 b_{n-1} b_n \), which contradicts the assumption that the product achieves its minimum at \((b_1, b_2, \ldots, b_n)\).

(b) Similarly, we can prove that \( P \) achieves its minimum for \( a_1 \geq a_2 = a_3 = \cdots = a_n \). \( \square \)

Remark 2.1. The problem of determining the maximum and minimum value of the product \( P = a_1 a_2 \cdots a_n \) remains an open one for \( 1 < S_0 < n - 2 \) (see Theorem 2.1) or \( 1 < m < n - 2 \) (see Theorem 2.1').

Remark 2.2. We may reformulate Theorem 2.1 and Theorem 2.2 as follows:

Theorem 2.1'. Let \( c_1, c_2, \ldots, c_n (n \geq 3) \) be fixed nonnegative real numbers such that

\[
\sum_{i=1}^{n} \frac{1}{(n - m)c_i + m} = 1,
\]

where \( 1 < m \leq 3 \) for \( n = 3 \) and \( n - 2 \leq m \leq n \) for \( n \geq 4 \). If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers such that \( a_1 \geq a_2 \geq \cdots \geq a_n \) and

\[
\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} c_i, \quad \sum_{i=1}^{n} \frac{1}{(n - m)a_i + m} = 1,
\]

then

(a) the product \( P = a_1 a_2 \cdots a_n \) achieves its maximum for \( a_1 \geq a_2 = a_3 = \cdots = a_n \); 
(b) the product \( P = a_1 a_2 \cdots a_n \) achieves its minimum for either \( a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0 \) or \( a_n = 0 \).

Theorem 2.2'. Let \( c_1, c_2, \ldots, c_n (n \geq 3) \) be fixed nonnegative real numbers such that

\[
\sum_{i=1}^{n} \frac{1}{(n - m)c_i + m} = 1,
\]

where \( 0 < m < 1 \) for \( n = 3 \) and \( 0 < m \leq 1 \) for \( n \geq 4 \). If \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers such that \( a_1 \geq a_2 \geq \cdots \geq a_n \) and

\[
\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} c_i, \quad \sum_{i=1}^{n} \frac{1}{(n - m)a_i + m} = 1,
\]

then

(a) the product \( P = a_1 a_2 \cdots a_n \) achieves its maximum for \( a_1 \geq a_2 = a_3 = \cdots = a_n \); 
(b) the product \( P = a_1 a_2 \cdots a_n \) achieves its minimum for either \( a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0 \) or \( a_n = 0 \).
then

(a) the product $P = a_1a_2 \cdots a_n$ achieves its maximum for $a_1 = a_2 = \cdots = a_{n-1} \geq a_n$;
(b) the product $P = a_1a_2 \cdots a_n$ achieves its minimum for $a_1 \geq a_2 = a_3 = \cdots = a_n$.

3. APPLICATIONS

**Application 3.1.** If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are nonnegative real numbers such that

$$\sum_{i=1}^{n} \frac{1}{a_i + n - 1} = 1,$$

then

$$(n-2)(a_1 + a_2 + \cdots + a_n) + a_1a_2 \cdots a_n \geq (n-1)^2.$$

**Proof.** Consider $a_1 \geq a_2 \geq \cdots \geq a_n$. According to Theorem 2.1' (case $m = n - 1$), for fixed $a_1 + a_2 + \cdots + a_n$, the product $a_1a_2 \cdots a_n$ has the minimum value for either $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$ or $a_n = 0$. Thus, it suffices to consider these cases.

**Case 1:** $a_1 = a_2 = \cdots = a_{n-1} \geq a_n > 0$. We need to show that if

$$\frac{n-1}{x+1} + \frac{1}{y+1} = 1,$$

which leads to

$$y = \frac{n-1 - (n-2)x}{x}, \quad 0 < y \leq x < \frac{n-1}{n-2},$$

then

$$(n-2)[(n-1)x + y] + x^{n-1}y \geq (n-1)^2,$$

which is equivalent to

$$(n-2)y + x^{n-1}y \geq (n-1)[n-1 - (n-2)x].$$

Since $n-1 - (n-2)x = xy$, we only need to show that

$$n-2 + x^{n-1} \geq (n-1)x,$$

which is just the AM-GM inequality.

**Case 2:** $a_n = 0$. We need to show that

$$\sum_{i=1}^{n-1} \frac{1}{a_i + n - 1} = \frac{n-2}{n-1}$$

involves

$$(n-2)(a_1 + a_2 + \cdots + a_{n-1}) \geq (n-1)^2.$$

This follows immediately from the AM-HM inequality

$$\left[ \sum_{i=1}^{n-1} (a_i + n - 1) \right] \left( \sum_{i=1}^{n-1} \frac{1}{a_i + n - 1} \right) \geq (n-1)^2.$$

The proof is completed. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = a_2 = \cdots = a_{n-1} = \frac{n-1}{n-2}$ and $a_n = 0$ (or any cyclic permutation).

**Application 3.2.** If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are nonnegative real numbers such that

$$\sum_{i=1}^{n} \frac{1}{2a_i + n - 2} = 1,$$
then
\[ a_1 + a_2 + \cdots + a_n - n \geq 2^{n-1}(a_1a_2\cdots a_n - 1). \]

**Proof.** Consider \( a_1 \geq a_2 \geq \cdots \geq a_n \). For \( n = 3 \), the inequality is an identity. For \( n \geq 4 \), according to Theorem 2.1’ (case \( m = n - 2 \)), for fixed \( a_1 + a_2 + \cdots + a_n \), the product \( a_1a_2\cdots a_n \) attains its maximum value when \( a_1 \geq a_2 = a_3 = \cdots = a_n \). Thus, we only need to show that

\[ y + (n-1)x - n \geq 2^{n-1}(yx^{n-1} - 1) \]

for

\[ \frac{1}{2y+n-2} + \frac{n-1}{2x+n-2} = 1, \]

which implies

\[ y = \frac{n-2-(n-3)x}{2x-1}, \quad \frac{1}{2} < x \leq y. \]

The required inequality is equivalent to

\[ \frac{n-2-(n-3)x + (2x-1)[(n-1)x-n]}{2^{n-1}} \geq \]

\[ (n-2)(x^{n-1} - 1) - (n-3)(x^n - 1) - 2(x-1), \]

(*)

or

\[ \frac{(n-1)(x-1)^2}{2^{n-2}} \geq (x-1)f(x), \]

where

\[ f(x) = (n-2)(x^{n-2} + x^{n-3} + \cdots + x + 1) - (n-3)(x^{n-1} + x^{n-2} + \cdots + x + 1) - 2 \]

\[ = (n-2)[(x^{n-2} - 1) + (x^{n-3} - 1) + \cdots + (x - 1)] - (n-3)[(x^{n-1} - 1) + (x^{n-2} - 1) + \cdots + (x - 1)] \]

\[ = (x-1)g(x), \]

\[ g(x) = (n-2)[x^{n-3} + 2x^{n-4} + \cdots + (n-2)] - (n-3)[x^{n-2} + 2x^{n-3} + \cdots + (n-1)] \]

\[ = -(n-3)x^{n-2} - (n-4)x^{n-3} - \cdots - x^2 + 1. \]

So, we only need to show that

\[ \frac{n-1}{2^{n-2}} \geq g(x). \]

Since \( g \) is a decreasing function, it suffices to show that

\[ \frac{n-1}{2^{n-2}} \geq g\left(\frac{1}{2}\right). \]

This is true if the inequality (*) holds for \( x = \frac{1}{2} \). It is easy to show that this last inequality is an identity.

For \( n \geq 4 \), the equality occurs when \( a_1 = a_2 = \cdots = a_n = 1 \).

**Application 3.3.** If \( a_1, a_2, \ldots, a_n \ (n \geq 3) \) are nonnegative real numbers such that

\[ \sum_{i=1}^{n} \frac{1}{(n-1)a_i + 1} = 1, \]

then

\[ a_1 + a_2 + \cdots + a_n - n \leq k(a_1a_2\cdots a_n - 1), \quad k = \left(\frac{n-1}{n-2}\right)^{n-1}. \]

**Proof.** Consider \( a_1 \geq a_2 \geq \cdots \geq a_n \). For \( n = 3 \), the inequality is an identity. Consider further \( n \geq 4 \). According to Theorem 2.2’ (case \( m = 1 \)), for fixed \( a_1 + a_2 + \cdots + a_n \), the
product $a_1a_2\cdots a_n$ attains its minimum when $a_1 \geq a_2 = a_3 = \cdots = a_n$. We need to show that if
\[
\frac{1}{(n-1)y+1} + \frac{n-1}{(n-1)x+1} = 1,
\]
which leads to
\[
y = \frac{1}{(n-1)x-n+2}, \quad \frac{n-2}{n-1} < x \leq y,
\]
then
\[
y + (n-1)x - n \leq k(yx^{n-1} - 1),
\]
which is equivalent to
\[
1 + [(n-1)x - n + 2][(n-1)x - n] \leq k[x^{n-1} - (n-1)x + n - 2], \quad (**)
\]
or
\[
(n-1)^2(x-1)^2 \leq kf(x), \quad f(x) = x^{n-1} - (n-1)(x-1).
\]
Since
\[
f(x) = (x-1)(x^{n-2} + x^{n-3} + \cdots + x - n + 2) = (x-1)^2g(x),
\]
where
\[
g(x) = x^{n-3} + 2x^{n-4} + \cdots + (n-2),
\]
we only need to show that
\[
(n-1)^2 \leq kg(x).
\]
Since $g$ is an increasing function, it suffices to show that
\[
(n-1)^2 \leq kg \left( \frac{n-2}{n-1} \right).
\]
This inequality is true if the inequality (** holds for $x = \frac{n-2}{n-1}$. Indeed, in this case, (** is an identity.

For $n \geq 4$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

**Remark 3.3.** By the AM-HM inequality
\[
\left[ \sum_{i=1}^{n-1} ((n-1)a_i + 1) \right] \left( \sum_{i=1}^{n-1} \frac{1}{(n-1)a_i + 1} \right) \geq n^2,
\]
we get $a_1 + a_2 + \cdots + a_n \geq n$. As a consequence, the inequality in Application 3.3 involves
\[
a_1a_2\cdots a_n \geq 1.
\]
Actually, the following stronger inequality holds for $n \geq 4$:
\[
a_1a_2\cdots a_n \geq \frac{a_1 + a_2 + \cdots + a_n}{n}.
\]
Indeed, denoting $p = a_1a_2\cdots a_n$ ($p \geq 1$), the inequality in Application 3.3 leads to
\[
n a_1a_2\cdots a_n - (a_1 + a_2 + \cdots + a_n) \geq np - k(p-1) - n = (n-k)(p-1) \geq 0.
\]

**Application 3.4.** If $a_1, a_2, \ldots, a_n$ ($n \geq 3$) are nonnegative real numbers such that
\[
\sum_{i=1}^{n} \frac{1}{(n-1)a_i + 1} = 1.
\]
then
\[
a_1 + a_2 + \cdots + a_n \geq n^{-\sqrt[n]{a_1a_2\cdots a_n}}.
\]
Proof. Consider \(a_1 \geq a_2 \geq \cdots \geq a_n\). If \(n \geq 4\), we may apply Theorem 2.2' for \(m = 1\). So, for fixed \(a_1 + a_2 + \cdots + a_n\), the product \(a_1a_2 \cdots a_n\) has the maximum value when \(a_1 \geq a_2 = a_3 = \cdots = a_n\), and we only need to show that if

\[
\frac{n - 1}{(n - 1)x + 1} + \frac{1}{(n - 1)y + 1} = 1, \quad x \leq y,
\]

that is

\[
y = \frac{1}{(n - 1)x - n + 2}, \quad x > \frac{n - 2}{n - 1},
\]

then

\[
(n - 1)x + y \geq n \sqrt[n-1]{x^n y},
\]

that is

\[
\left[\frac{(n - 1)x + y}{nx}\right]^{n-1} \geq y, \quad \left(1 - \frac{x - y}{nx}\right)^{n-1} \geq y.
\]

By Bernoulli’s inequality, it suffices to show that

\[
1 - \frac{(n - 1)(x - y)}{nx} \geq y,
\]

which is equivalent to

\[
x \geq (nx - n + 1)y,
\]

\[
(n - 1)(x - 1)^2 \geq 0.
\]

For \(n = 3\), we need to show that

\[
a_1 + a_2 + a_3 \geq 3\sqrt[3]{a_1a_2a_3}
\]

for

\[
\frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \frac{1}{2a_3 + 1} = 1,
\]

that is

\[
4a_1a_2a_3 = a_1 + a_2 + a_3 + 1.
\]

Denote \(t = \sqrt[3]{a_1a_2a_3}\). From AM-GM inequality, we have

\[
4t^3 = a_1 + a_2 + a_3 + 1 \geq 3t + 1,
\]

hence \(t \geq 1\). Finally, we get

\[
a_1 + a_2 + a_3 - 3\sqrt[3]{a_1a_2a_3} = 4t^3 - 1 - 3t\sqrt{t} = (t\sqrt{t} - 1)(4t\sqrt{t} + 1) \geq 0.
\]

The equality occurs for \(a_1 = a_2 = \cdots = a_n = 1\).

Application 3.5. If \(a, b, c, d\) are nonnegative real numbers such that

\[
\frac{1}{a + 1} + \frac{1}{b + 1} + \frac{1}{c + 1} + \frac{1}{d + 1} = 2
\]

then

\[
(a + b + c + d)^2 + 4 \geq 5(abc + bcd + cda + dab).
\]

Proof. Consider \(a \geq b \geq c \geq d\) and write the hypothesis in the form

\[
a + b + c + d + 2 = abcd + 2(abc + bcd + cda + dab).
\]

If the sum \(a + b + c + d\) is fixed, then the expression \(abc + bcd + cda + dab\) has the maximum value when the product \(abcd\) has the minimum value, that is when either \(a = b = c \geq d > 0\) or \(d = 0\) (Theorem 2.1). Thus, it suffices to consider these cases.

Case 1: \(a = b = c \geq d > 0\). We need to prove that

\[
(3a + d)^2 + 4 \geq 5a^2(a + 3d)
\]
for
\[
\frac{3}{a+1} + \frac{1}{d+1} = 2,
\]
that is
\[
d = \frac{2-a}{2a-1}, \quad \frac{1}{2} < a \leq 2.
\]

Write the required inequality as follows:
\[
(3a + d)^2 - 16 \geq 5(a^3 + 3a^2 d - 4),
\]
\[
\frac{12(a-1)^2(a+1)(3a-1)}{(2a-1)^2} \geq \frac{10(a-1)^2(a^2 + 2)}{2a-1}.
\]

It is true if
\[
6(a+1)(3a-1) \geq 5(2a-1)(a^2 + 2).
\]

Indeed, we have
\[
6(a+1)(3a-1) - 5(2a-1)(a^2 + 2) \geq 6(a+1)(3a-1) - 5(2a-1)(2a+2) = 2(a+1)(2-a) \geq 0.
\]

**Case 2:** \(d = 0\). Let \(s = a + b + c\). We need to show that
\[
s^2 + 4 \geq 5abc
\]
for
\[
\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1,
\]
that is \(abc = s + 2\). From
\[
s^3 \geq 27abc = 27(s + 2),
\]
we get \((s - 6)(s + 3)^2 \geq 0\), hence \(s \geq 6\). Finally,
\[
s^2 + 4 - 5abc = s^2 + 4 - 5(s + 2) = (s - 6)(s + 1) \geq 0.
\]

The equality occurs for \(a = b = c = d = 1\), and also for \(a = b = c = 2\) and \(d = 0\) (or any cyclic permutation).

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