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Intuitionistic Level Subgroups in the Dihedral Group D_3

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ABSTRACT. A well known result in fuzzy group theory states that "level subgroups of any fuzzy subgroup of a finite group form a chain". We check the validity of this statement in the intuitionistic fuzzy perspective. We do this using Dihedral Group D_3, which is a non-cyclic group. We prove that D_3 has 100 distinct intuitionistic fuzzy subgroups (IFSGs) upto isomorphism. The intuitionistic level subgroups (ILSGs) of exactly 76 among them make chains, and hence it can be concluded that the result is not true in the intuitionistic fuzzy perspective. We also enlist all the 100 distinct intuitionistic fuzzy subgroups of D_3 upto isomorphism.

1. INTRODUCTION

The introduction of the concept of fuzzy sets as a generalisation of the classical notion of set by Lotfi A. Zadeh [16] in 1965 revolutionized the concept of an element's belongingness to a set by permitting partial belongingness. Following its introduction, because of the wide range of applications for fuzzy sets and relations in areas such as linguistics [5], decision-making [12], clustering [2], etc., scholars all over the globe have been encouraged to fuzzify most of the mathematical ideas and conduct careful study on them. As a part of this progress, many abstract algebraic ideas were fuzzified, because of their critical importance in fields like computer sciences, information sciences, cryptography, coding theory, and so on. The development of a fuzzy approach in group theory was the first step in this direction which was initiated by Rosenfeld [14] by introducing fuzzy subgroups of a group. Following this, several studies on various fuzzy algebraic structures emerged in the literature. Later, in 1983, K. T. Atanassov [1] came up with the concept of intuitionistic fuzzy sets as an abstraction of the theory of the fuzzy set. In 1989, Atanassov's definition of intuitionistic fuzzy sets was applied to group theory by Biswas [3], thereby developing the theory of intuitionistic fuzzy subgroups of a group. Many fresh findings are still being published in this field.

In 1981, P. S. Das [4] conducted research about the level subgroups of fuzzy subgroups of a finite group and proved that they form a chain. In our previous works [7, 8, 9] we have made an attempt to check whether the level subgroups in a group form a chain in the intuitionistic fuzzy context also and arrived at the following conclusions: (1) ILSGs in all IFSGs of $\mathbb{Z}_{p^n}$ form chains (p is a prime and $n \in \mathbb{N}$), (2) among the 36 distinct (non-isomorphic) IFSGs of $\mathbb{Z}_{pq}$ (p and q are distinct primes) ILSGs in exactly 8 IFSGs form chains and (3) among the 64 distinct (non-isomorphic) IFSGs of the Klein-4 Group ILSGs in exactly 24 IFSGs form chains. Here, we try to study the validity of the findings of P. S. Das [4] in intuitionistic fuzzy subgroups of Dihedral group D_3.

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2. Basic Concepts

Throughout this paper we use G to denote a multiplicative group $(G, .)$ unless otherwise stated.

Definition 2.1. [16] A **Fuzzy Subset** A of a non-empty set X is defined to be a function $A : X \to I = [0, 1]$ which assigns a membership degree in $[0, 1]$ to each element of X.

Definition 2.2. [4] For a fuzzy subset A of a non-empty set X and for any $\gamma \in I$, the γ**-cut** of A (or **Level Subset** of A at γ) is the crisp set $A_\gamma = \{x \in X : A(x) \geq \gamma\}$.

Definition 2.3. [14] A fuzzy subset A of a group G is called a **Fuzzy Subgroup** (FSG) of G if the following axioms are satisfied:
(1) $A(xy) \geq min[A(x), A(y)]$, for all $x, y \in G$
(2) $A(x^{-1}) = A(x)$, for all $x \in G$.

Proposition 2.1. [14] *If A is FSG of a group G and e is the identity element in G, then $A(e) \geq A(x), \forall x \in G$.*

Proposition 2.2. [4] *Let G be a group and A be a fuzzy subset of G. Then A is a FSG of G if and only if A_γ is a crisp subgroup of G for $0 \leq \gamma \leq A(e)$.*

Definition 2.4. [4] *If A is a FSG of a group G, then A_γ is a crisp subgroup of G for all γ with $0 \leq \gamma \leq A(e)$ and is called the **Level Subgroup** of A at γ.*

Proposition 2.3. [4] *Let A be a FSG of a finite group G with $Im(A) = \{t_i : i = 1, 2, 3, ..., n\}$. Then the collection $\{A_{t_i} : i = 1, 2, 3, ..., n\}$ contains all level subgroups of A.
Moreover, if $t_1 > t_2 > t_3 > ... > t_n$, then all these level subgroups will form a chain $G_A = A_{t_1} \subset A_{t_2} \subset A_{t_3} \subset ... \subset A_{t_n} = G$, where $G_A = \{x \in G : A(x) = A(e)\}$.*

Definition 2.5. [1] An **Intuitionistic Fuzzy Subset** (IFS) of a set X is denoted by $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in X\}$ where $\mathscr{M}$ and $\mathscr{N}$ are functions from $X \to I$ such that $\mathscr{M}(x)$ and $\mathscr{N}(x)$ represent the *degree of membership* and *degree of non membership* of any element $x \in X$ and satisfy the condition $0 \leq \mathscr{M}(x) + \mathscr{N}(x) \leq 1$.

Definition 2.6. [15] If $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in G\}$ is an IFS of a set X and α, $\beta \in I$, then the crisp subset of X given by $A_{\alpha,\beta} = \{x \in X : \mathscr{M}(x) \geq \alpha \ and \ \mathscr{N}(x) \leq \beta\}$ is called the **Intuitionistic Level Subset** (ILS) of A at (α, β) (or (α, β)–cut of IFS A).

Definition 2.7. [13] An IFS $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in G\}$ of a group G is called an **Intuitionistic Fuzzy Subgroup** (IFSG) of G if the following axioms are satisfied:
(1) $\mathscr{M}(xy) \geq \wedge[\mathscr{M}(x), \mathscr{M}(y)]$
(2) $\mathscr{M}(x^{-1}) = \mathscr{M}(x)$
(3) $\mathscr{N}(xy) \leq \vee[\mathscr{N}(x), \mathscr{N}(y)]$, and
(4) $\mathscr{N}(x^{-1}) = \mathscr{N}(x)$.

Proposition 2.4. [13] *If $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in G\}$ is an IFSG of a group G and e is the identity element in G, then $\mathscr{M}(e) \geq \mathscr{M}(x)$ and $\mathscr{N}(e) \leq \mathscr{N}(x)$, for all $x \in G$.*

Proposition 2.5. [15] *Let G be a group and $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in G\}$ be an IFS of G. Then A is an IFSG of G if and only if $A_{\alpha,\beta}$ is a crisp subgroup of G for $0 \leq \alpha \leq \mathscr{M}(e)$ and $\mathscr{N}(e) \leq \beta \leq 1$.*

Definition 2.8. [6] If $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in G\}$ is an IFSG of a group G, then $A_{\alpha,\beta}$ is a crisp subgroup of G for all α, β with $0 \leq \alpha \leq \mathscr{M}(e)$ and $\mathscr{N}(e) \leq \beta \leq 1)$ and is called the **Intuitionistic Level Subgroup** (ILSG) of A at (α, β).

Proposition 2.6. [6] *Let $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in G\}$ be an IFSG of a finite group G, $Im(\mathscr{M}) = \{t_i : i = 1, 2, 3, ..., n\}$ and $Im(\mathscr{N}) = \{s_j : j = 1, 2, 3, ..., m\}$. Then the collection*

$$\{ A_{t_i,s_j} : i = 1, 2, 3, ..., n; j = 1, 2, 3, ..., m\}$$

contains all ILSGs of G.

Remark 2.1. Proposition 2.6 states that the intuitionistic fuzzy analogue of the first part of proposition 2.3 holds true. Also, the results obtained in section 4 prove that the intuitionistic fuzzy analogue of the second part need not be true always.

Definition 2.9. [10] Let $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in X\}$ be an IFS of a non-empty set X with $Im(\mathscr{M}) = \{t_i : i = 1, 2, 3, ..., n\}$ and $Im(\mathscr{N}) = \{s_j : j = 1, 2, 3, ..., m\}$ where $1 \geq t_1 > t_2 > ... > t_n \geq 0$ and $0 \leq s_1 < s_2 < ... < s_m \leq 1$. The *finite sequence* consisting of all intuitionistic level subsets of A, given by $\tilde{Ł}(A) = \{A_{t_1,s_1}, A_{t_1,s_2}, ..., A_{t_1,s_m}, A_{t_2,s_1}, A_{t_2,s_2}, ..., A_{t_2,s_m}, ..., A_{t_n,s_1}, A_{t_n,s_2}, ..., A_{t_n,s_m}\}$, is called the **Intuitionistic Level Representation (ILR)** of A.

Definition 2.10 (Isomorphic Intuitionistic Fuzzy Subsets). [10]
Two IFSs $A = \{\langle x, \mathscr{M}_A(x), \mathscr{N}_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mathscr{M}_B(x), \mathscr{N}_B(x)\rangle : x \in X\}$ of a non-empty set X are isomorphic, denoted by $A \cong B$, if for all $x, y \in X$
(I_1) $\mathscr{M}_A(x) < \mathscr{M}_A(y) \Leftrightarrow \mathscr{M}_B(x) < \mathscr{M}_B(y)$
(I_2) $\mathscr{M}_A(x) = \mathscr{M}_A(y) \Leftrightarrow \mathscr{M}_B(x) = \mathscr{M}_B(y)$
(I_3) $\mathscr{N}_A(x) < \mathscr{N}_A(y) \Leftrightarrow \mathscr{N}_B(x) < \mathscr{N}_B(y)$
(I_4) $\mathscr{N}_A(x) = \mathscr{N}_A(y) \Leftrightarrow \mathscr{N}_B(x) = \mathscr{N}_B(y)$
where $\mathscr{M}_A(x), \mathscr{M}_B(x)$ denote the membership degrees and $\mathscr{N}_A(x), \mathscr{N}_B(x)$ denote the non-membership degrees of x in A and B respectively.

Definition 2.11. [11] The **Dihedral Group** D_3, also known as the **group of symmetries of an equilateral triangle**, consists of elements k_0, k_1, k_2, m_1, m_2 and m_3, where k_0, k_1, k_2 correspond to rotations of an equilateral triangle through 0°, 120° and 240° and m_1, m_2, m_3 correspond to the reflections of the triangle along its altitutes. The multiplication table for D_3 is shown in Table 1.

*	k_0	k_1	k_2	m_1	m_2	m_3
k_0	k_0	k_1	k_2	m_1	m_2	m_3
k_1	k_1	k_2	k_0	m_2	m_3	m_1
k_2	k_2	k_0	k_1	m_3	m_1	m_2
m_1	m_1	m_3	m_2	k_0	k_2	k_1
m_2	m_2	m_1	m_3	k_1	k_0	k_2
m_3	m_3	m_2	m_1	k_2	k_1	k_0

TABLE 1. Multiplication table for D_3.

3. IFSGs of Dihedral Group

Throughout this section we consider $A = \{\langle x, \mathscr{M}(x), \mathscr{N}(x)\rangle : x \in D_3\}$ to be an arbitrary IFSG of D_3.

Remark 3.2. By proposition 2.4, the identity element in a group should have the greatest membership degree and least non-membership degree. Hence $\mathscr{M}(k_0) \geq \mathscr{M}(x)$ and $\mathscr{N}(k_0) \leq \mathscr{N}(x), \forall\, x \in D_3$.

In the following proposition we prove that *the two non-identity rotations in D_3 have the same membership degrees and same non-membership degrees.*

Proposition 3.7. *$\mathscr{M}(k_1) = \mathscr{M}(k_2)$ and $\mathscr{N}(k_1) = \mathscr{N}(k_2)$.*

Proof. From the multiplication table 1 it can be seen that $k_1 = k_2^2$. Then by first axiom of IFSG we get $\mathscr{M}(k_1) \geq \wedge[\mathscr{M}(k_2), \mathscr{M}(k_2)] = \mathscr{M}(k_2)$. Similarly, since $k_2 = k_1^2$, we get $\mathscr{M}(k_2) \geq \mathscr{M}(k_1)$. Hence $\mathscr{M}(k_1) = \mathscr{M}(k_2)$.
Proceeding similarly with non-membership function $\mathscr{N}$ and using third axiom of IFSG we get $\mathscr{N}(k_1) = \mathscr{N}(k_2)$. □

In the following proposition we prove that *the membership degree of the two non-identity rotations is greater than or equal to the minimum among the membership degrees of any two reflections*. Similarly, *the non-membership degree of the two non-identity rotations is less than or equal to the maximum among the non-membership degrees of any two reflections*.

Proposition 3.8. *For any $i, j = 1, 2, 3$ with $i \neq j$, $\mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \wedge[\mathscr{M}(m_i), \mathscr{M}(m_j)]$ and $\mathscr{N}(k_1) = \mathscr{N}(k_2) \leq \vee[\mathscr{N}(m_i), \mathscr{N}(m_j)]$.*

Proof. From the multiplication table 1 it can be seen that $m_i m_j = k_1$ or k_2 for any $i, j = 1, 2, 3$ with $i \neq j$. Then the required results follow directly from the first and third axioms of IFSG. □

In the next two propositions we prove that *the membership degree of any one reflection will be greater than or equal to the minimum among the membership degrees of any one non-identity rotation and any other reflection*. Similarly, *the non-membership degree of any one reflection will be less than or equal to the maximum among the membership degrees of any one non-identity rotation and any other reflection*.

Proposition 3.9. *For any $i, j, k = 1, 2, 3$ with $i \neq j \neq k \neq i$,*

$$\mathscr{M}(m_i) \geq \begin{cases} \wedge[\mathscr{M}(k_1), \mathscr{M}(m_j)] \\ \wedge[\mathscr{M}(k_1), \mathscr{M}(m_k)] \\ \wedge[\mathscr{M}(k_2), \mathscr{M}(m_j)] \\ \wedge[\mathscr{M}(k_2), \mathscr{M}(m_k)] \end{cases}$$

Proof. From the multiplication table 1 it can be seen that $m_i = k_1 m_j = k_1 m_k = k_2 m_j = k_2 m_k$ for any $i, j, k = 1, 2, 3$ with $i \neq j \neq k \neq i$. Then the required result follows directly from first axiom of IFSG. □

Proposition 3.10. *For any $i, j, k = 1, 2, 3$ with $i \neq j \neq k \neq i$,*

$$\mathscr{N}(m_i) \leq \begin{cases} \vee[\mathscr{N}(k_1), \mathscr{M}(m_j)] \\ \vee[\mathscr{N}(k_1), \mathscr{M}(m_k)] \\ \vee[\mathscr{N}(k_2), \mathscr{M}(m_j)] \\ \vee[\mathscr{N}(k_2), \mathscr{M}(m_k)] \end{cases}$$

Proof. From the multiplication table 1 it can be seen that $m_i = k_1 m_j = k_1 m_k = k_2 m_j = k_2 m_k$ for any $i, j, k = 1, 2, 3$ with $i \neq j \neq k \neq i$. Then the required result follows directly from third axiom of IFSG. □

Proposition 3.11. *For any $i, j, k = 1, 2, 3$ with $i \neq j \neq k \neq i$, either $\mathscr{M}(k_0) \geq \mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \mathscr{M}(m_i) = \mathscr{M}(m_j) = \mathscr{M}(m_k)$ or $\mathscr{M}(k_0) \geq \mathscr{M}(m_k) \geq \mathscr{M}(m_i) = \mathscr{M}(m_j) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$.*

Proof. From remark 3.2 and propositions 3.7 and 3.8 we have

$$\mathscr{M}(k_0) \geq \mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \wedge[\mathscr{M}(m_p), \mathscr{M}(m_q)] \tag{3.1}$$

for any $p, q = 1, 2, 3$ with $p \neq q$. Without loss of generality, now suppose $\mathscr{M}(m_i) \leq \mathscr{M}(m_j) \leq \mathscr{M}(m_k)$ where $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k \neq i$. Then equation 3.1

implies $\mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \wedge[\mathscr{M}(m_i), \mathscr{M}(m_k)] = \mathscr{M}(m_i)$ and $\mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \wedge[\mathscr{M}(m_j), \mathscr{M}(m_k)] = \mathscr{M}(m_j)$. Thus,

$$\mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \mathscr{M}(m_i), \mathscr{M}(m_j) \tag{3.2}$$

Equation 3.2 and proposition 3.9 together implies $\mathscr{M}(m_i) \geq \mathscr{M}(m_j)$. Hence our assumption becomes

$$\mathscr{M}(m_i) = \mathscr{M}(m_j) \leq \mathscr{M}(m_k) \tag{3.3}$$

Now it is enough to get the relation between $\mathscr{M}(m_k)$ and $\mathscr{M}(k_1) = \mathscr{M}(k_2)$.
Case (i): Suppose $\mathscr{M}(m_k) \leq \mathscr{M}(k_1) = \mathscr{M}(k_2)$. Then proposition 3.9 implies $\mathscr{M}(m_i) \geq \mathscr{M}(m_k)$ and hence by equation 3.3 we get $\mathscr{M}(m_i) = \mathscr{M}(m_k)$. As a result, the hierarchy of membership degrees in this case becomes:

$$\mathscr{M}(k_0) \geq \mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \mathscr{M}(m_i) = \mathscr{M}(m_j) = \mathscr{M}(m_k)$$

Case (ii): Next suppose $\mathscr{M}(m_k) \geq \mathscr{M}(k_1) = \mathscr{M}(k_2)$. Then proposition 3.9 implies $\mathscr{M}(m_i) \geq \mathscr{M}(k_1), \mathscr{M}(k_2)$ and hence by equation 3.2 we get $\mathscr{M}(m_i) = \mathscr{M}(k_1), \mathscr{M}(k_2)$. Then equations 3.2 and 3.3 imply $\mathscr{M}(m_k) \geq \mathscr{M}(k_1) = \mathscr{M}(k_2) = \mathscr{M}(m_i) = \mathscr{M}(m_j)$. As a result, the hierarchy of membership degrees in this case becomes:

$$\mathscr{M}(k_0) \geq \mathscr{M}(m_k) \geq \mathscr{M}(m_i) = \mathscr{M}(m_j) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$$

This completes the proof. □

Proposition 3.12. *For any $i, j, k = 1, 2, 3$ with $i \neq j \neq k \neq i$, either $\mathscr{N}(k_0) \leq \mathscr{N}(k_1) = \mathscr{N}(k_2) \leq \mathscr{N}(m_i) = \mathscr{N}(m_j) = \mathscr{N}(m_k)$ and $\mathscr{N}(k_0) \leq \mathscr{N}(m_k) \leq \mathscr{N}(m_i) = \mathscr{N}(m_j) = \mathscr{N}(k_1) = \mathscr{N}(k_2)$.*

Proof. Similar to proof of proposition 3.11. □

Remark 3.3. Proposition 3.11 implies that the hierarchy of membership degrees in any IFSG of D_3 should be as follows:

$$\begin{aligned} &\mathscr{M}(k_0) \geq \mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \mathscr{M}(m_i) = \mathscr{M}(m_j) = \mathscr{M}(m_k) \\ &\mathscr{M}(k_0) \geq \mathscr{M}(m_k) \geq \mathscr{M}(m_i) = \mathscr{M}(m_j) = \mathscr{M}(k_1) = \mathscr{M}(k_2) \end{aligned} \tag{3.4}$$

for any $i, j, k = 1, 2, 3$ where $i \neq j \neq k \neq i$.
The possibilities that arise from inequalities in (3.4) are listed in Table 2.

1	$\mathscr{M}(k_0) > \mathscr{M}(k_1) = \mathscr{M}(k_2) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3)$
2	$\mathscr{M}(k_0) = \mathscr{M}(k_1) = \mathscr{M}(k_2) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3)$
3	$\mathscr{M}(k_0) > \mathscr{M}(k_1) = \mathscr{M}(k_2) = \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3)$
4	$\mathscr{M}(k_0) = \mathscr{M}(k_1) = \mathscr{M}(k_2) = \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3)$
5	$\mathscr{M}(k_0) > \mathscr{M}(m_1) > \mathscr{M}(m_2) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
6	$\mathscr{M}(k_0) = \mathscr{M}(m_1) > \mathscr{M}(m_2) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
7	$\mathscr{M}(k_0) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
8	$\mathscr{M}(k_0) = \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
9	$\mathscr{M}(k_0) > \mathscr{M}(m_2) > \mathscr{M}(m_1) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
10	$\mathscr{M}(k_0) = \mathscr{M}(m_2) > \mathscr{M}(m_1) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
11	$\mathscr{M}(k_0) > \mathscr{M}(m_2) = \mathscr{M}(m_1) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
12	$\mathscr{M}(k_0) = \mathscr{M}(m_2) = \mathscr{M}(m_1) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
13	$\mathscr{M}(k_0) > \mathscr{M}(m_3) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
14	$\mathscr{M}(k_0) = \mathscr{M}(m_3) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
15	$\mathscr{M}(k_0) > \mathscr{M}(m_3) = \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
16	$\mathscr{M}(k_0) = \mathscr{M}(m_3) = \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$

TABLE 2. The possible inequalities obtained from inequalities in (3.4).

It can be noted that, in the Table 2, the inequalities numbered as 7, 8, 11, 12, 15, 16 are same as those numbered as 3, 4 and hence can be omitted. As a result, there are exactly 10 possible hierarchies of membership degrees in any IFSG A in D_3, which are listed in the Table 3.

Similarly, by proposition 3.12 the hierarchy of non-membership degrees in any IFSG of D_3 should be:

$$\begin{aligned} &\mathscr{N}(k_0) \leq \mathscr{N}(k_1) = \mathscr{N}(k_2) \leq \mathscr{N}(m_i) = \mathscr{N}(m_j) = \mathscr{N}(m_k) \\ &\mathscr{N}(k_0) \leq \mathscr{N}(m_k) \leq \mathscr{N}(m_j) = \mathscr{N}(m_j) = \mathscr{N}(k_1) = \mathscr{N}(k_2) \end{aligned} \tag{3.5}$$

for any $i, j, k = 1, 2, 3$ where $i \neq j \neq k \neq i$. As in the case of membership degrees, it can be obtained from inequality (3.5) that, there are exactly 10 possible hierarchies of non-membership degrees in any IFSG A in D_3, which are listed in the Table 4.

	Membership degrees
1	$\mathscr{M}(k_0) > \mathscr{M}(k_1) = \mathscr{M}(k_2) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3)$
2	$\mathscr{M}(k_0) = \mathscr{M}(k_1) = \mathscr{M}(k_2) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3)$
3	$\mathscr{M}(k_0) > \mathscr{M}(m_1) > \mathscr{M}(m_2) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
4	$\mathscr{M}(k_0) = \mathscr{M}(m_1) > \mathscr{M}(m_2) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
5	$\mathscr{M}(k_0) > \mathscr{M}(m_2) > \mathscr{M}(m_1) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
6	$\mathscr{M}(k_0) = \mathscr{M}(m_2) > \mathscr{M}(m_1) = \mathscr{M}(m_3) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
7	$\mathscr{M}(k_0) > \mathscr{M}(m_3) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
8	$\mathscr{M}(k_0) = \mathscr{M}(m_3) > \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(k_1) = \mathscr{M}(k_2)$
9	$\mathscr{M}(k_0) > \mathscr{M}(k_1) = \mathscr{M}(k_2) = \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3)$
10	$\mathscr{M}(k_0) = \mathscr{M}(k_1) = \mathscr{M}(k_2) = \mathscr{M}(m_1) = \mathscr{M}(m_2) = \mathscr{M}(m_3)$

TABLE 3. The possible hierarchies of membership degrees in any IFSG A in D_3

	Non-membership degrees
1	$\mathscr{N}(k_0) < \mathscr{N}(k_1) = \mathscr{N}(k_2) < \mathscr{N}(m_1) = \mathscr{N}(m_2) = \mathscr{N}(m_3)$
2	$\mathscr{N}(k_0) = \mathscr{N}(k_1) = \mathscr{N}(k_2) < \mathscr{N}(m_1) = \mathscr{N}(m_2) = \mathscr{N}(m_3)$
3	$\mathscr{N}(k_0) < \mathscr{N}(m_1) < \mathscr{N}(m_2) = \mathscr{N}(m_3) = \mathscr{N}(k_1) = \mathscr{N}(k_2)$
4	$\mathscr{N}(k_0) = \mathscr{N}(m_1) < \mathscr{N}(m_2) = \mathscr{N}(m_3) = \mathscr{N}(k_1) = \mathscr{N}(k_2)$
5	$\mathscr{N}(k_0) < \mathscr{N}(m_2) < \mathscr{N}(m_1) = \mathscr{N}(m_3) = \mathscr{N}(k_1) = \mathscr{N}(k_2)$
6	$\mathscr{N}(k_0) = \mathscr{N}(m_2) < \mathscr{N}(m_1) = \mathscr{N}(m_3) = \mathscr{N}(k_1) = \mathscr{N}(k_2)$
7	$\mathscr{N}(k_0) < \mathscr{N}(m_3) < \mathscr{N}(m_1) = \mathscr{N}(m_2) = \mathscr{N}(k_1) = \mathscr{N}(k_2)$
8	$\mathscr{N}(k_0) = \mathscr{N}(m_3) < \mathscr{N}(m_1) = \mathscr{N}(m_3) = \mathscr{N}(k_1) = \mathscr{N}(k_2)$
9	$\mathscr{N}(k_0) < \mathscr{N}(k_1) = \mathscr{N}(k_2) = \mathscr{N}(m_1) = \mathscr{N}(m_2) = \mathscr{N}(m_3)$
10	$\mathscr{N}(k_0) = \mathscr{N}(k_1) = \mathscr{N}(k_2) = \mathscr{N}(m_1) = \mathscr{N}(m_2) = \mathscr{N}(m_3)$

TABLE 4. The possible hierarchies of non-membership degrees in any IFSG A in D_3

Proposition 3.13. *Given $t_1, t_2, t_3, s_1, s_2, s_3 \in I$ with $1 \geq t_1 > t_2 > t_3 \geq 0$ and $0 \leq s_1 < s_2 < s_3 \leq 1$, there exist exactly 100 non-isomorphic IFSGs of D_3 with membership degrees t_1, t_2, t_3 and non-membership degrees s_1, s_2, s_3.*

Proof. It is clear from tables 4 and 3 that, there are 10 possible hierarchies of membership degrees and 10 possible hierarchies of non-membership degrees in any IFSG of D_3. This means, membership degrees can be assigned in 10 different ways, following which non-membership degrees can also be assigned in 10 different ways. Hence, by the fundamental principle of counting, exactly $10 \times 10 = 100$ distinct (non-isomorphic) IFSGs can be defined in D_3 using the given set of membership and non-membership degrees. $\square$

4. ILSGs in Dihedral Group

Proposition 4.14. *If $\mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \mathscr{M}(m_k)$ and $\mathscr{N}(k_1) = \mathscr{N}(k_2) \leq \mathscr{N}(m_k)$ for all $k = 1, 2, 3$, then the ILSGs of A form a chain.*

Proof. From equations 3.4 and 3.5 we have

$$\mathscr{M}(k_0) \geq \mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \mathscr{M}(m_i) = \mathscr{M}(m_j) = \mathscr{M}(m_k)$$
$$\mathscr{N}(k_0) \leq \mathscr{N}(k_1) = \mathscr{N}(k_2) \leq \mathscr{N}(m_i) = \mathscr{N}(m_j) = \mathscr{N}(m_k)$$

for $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k \neq i$. Hence there exist non-negative real numbers $t_1 \geq t_2 \geq t_3$ and $s_1 \leq s_2 \leq s_3$ in I, such that $\mathscr{M}(k_0) = t_1, \mathscr{M}(k_1) = \mathscr{M}(k_2) = t_2, \mathscr{M}(m_i) = \mathscr{M}(m_j) = \mathscr{M}(m_k) = t_3$ and $\mathscr{N}(k_0) = s_1, \mathscr{N}(k_1) = \mathscr{N}(k_2) = s_2, \mathscr{N}(m_i) = \mathscr{N}(m_j) = \mathscr{N}(m_k) = s_3$. Then ILR of A is:

$$\begin{aligned}\tilde{Ł}(A) &= \{A_{t_1,s_1}, A_{t_1,s_2}, A_{t_1,s_3}, A_{t_2,s_1}, A_{t_2,s_2}, A_{t_2,s_3}, A_{t_3,s_1}, A_{t_3,s_2}, A_{t_3,s_3}\} \\ &= \{\{k_0\}, \{k_0\}, \{k_0\}, \{k_0\}, \{k_0, k_1, k_2\}, \{k_0, k_1, k_2\}, \{k_0\}, \{k_0, k_1, k_2\}, D_3\}\end{aligned}$$

Hence, the distinct ILSGs of A are: $\{k_0\}, \{k_0, k_1, k_2\}, D_3$ which form the chain $A_{t_1,s_1} \subset A_{t_2,s_2} \subset A_{t_3,s_3}$ as shown in figure 1. □

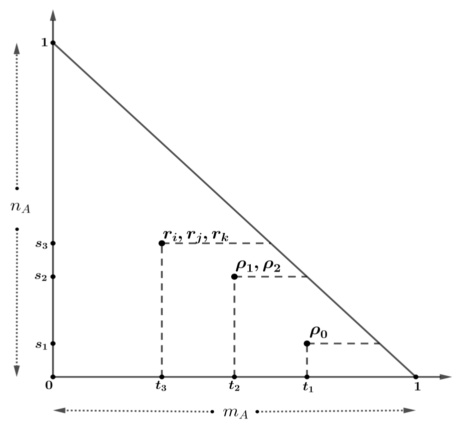

FIGURE 1. ILSGs in proof of proposition 4.14

Proposition 4.15. *If $\mathscr{M}(k_1) = \mathscr{M}(k_2) \leq \mathscr{M}(m_k)$ and $\mathscr{N}(k_1) = \mathscr{N}(k_2) \geq \mathscr{N}(m_k)$ for all $k = 1, 2, 3$, then the ILSGs of A form a chain.*

Proof. Similar to proof of proposition 4.14. □

Proposition 4.16. *If $\mathscr{M}(k_1) = \mathscr{M}(k_2) > \mathscr{M}(m_k)$ and $\mathscr{N}(k_1) = \mathscr{N}(k_2) > \mathscr{N}(m_k)$ for some $k = 1, 2, 3$, then the ILSGs of A will not form a chain.*

Proof. From equations (3.4) and (3.5) we have

$$\mathscr{M}(k_0) \geq \mathscr{M}(k_1) = \mathscr{M}(k_2) > \mathscr{M}(m_k) = \mathscr{M}(m_i) = \mathscr{M}(m_j)$$
$$\mathscr{N}(k_0) \leq \mathscr{N}(m_k) < \mathscr{N}(k_1) = \mathscr{N}(k_2) = \mathscr{N}(m_i) = \mathscr{N}(m_j)$$

for $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k \neq i$. Hence there exist non-negative real numbers $t_1 \geq t_2 \geq t_3$ and $s_1 \leq s_2 \leq s_3$ in I, such that $\mathscr{M}(k_0) = t_1, \mathscr{M}(k_1) = \mathscr{M}(k_2) = t_2, \mathscr{M}(m_i) =$

$\mathscr{M}(m_j) = \mathscr{M}(m_k) = t_3$ and $\mathscr{N}(k_0) = s_1, \mathscr{N}(m_k) = s_2, \mathscr{N}(k_1) = \mathscr{N}(k_2) = \mathscr{N}(m_i) = \mathscr{N}(m_j) = s_3$. Then ILR of A is:

$$\begin{aligned}\tilde{Ł}(A) &= \{A_{t_1,s_1}, A_{t_1,s_2}, A_{t_1,s_3}, A_{t_2,s_1}, A_{t_2,s_2}, A_{t_2,s_3}, A_{t_3,s_1}, A_{t_3,s_2}, A_{t_3,s_3}\}\\ &= \{\{k_0\}, \{k_0\}, \{k_0\}, \{k_0\}, \{k_0\}, \{k_0, k_1, k_2\}, \{k_0\}, \{k_0, m_k\}, D_3\}\end{aligned}$$

Hence, the distinct ILSGs of A are: $\{k_0\}, \{k_0, k_1, k_2\}, \{k_0, m_k\}, D_3$ which does not form a chain as shown in figure 2. □

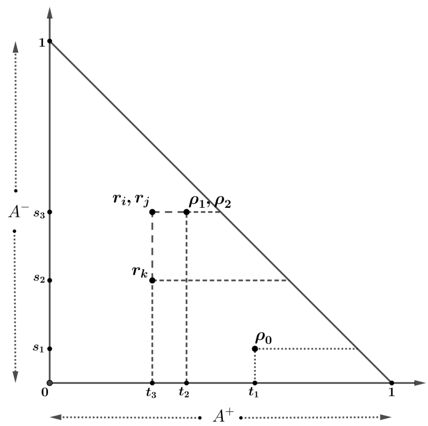

FIGURE 2. ILSGs in proof of proposition 4.16

Proposition 4.17. *If $\mathscr{M}(k_1) = \mathscr{M}(k_2) < \mathscr{M}(m_k)$ and $\mathscr{N}(k_1) = \mathscr{N}(k_2) < \mathscr{N}(m_k)$ for some $k = 1, 2, 3$, then the ILSGs of A will not form a chain.*

Proof. Similar to proof of proposition 4.16. □

Remark 4.4. Propositions 4.14 and 4.16 say that, the only possibilities where the ILSGs **do not** form a chain are when the membership and non-membership degrees of any two elements follow the same hierarchical ordering (other than equality).

Combining the above two propositions we get the following result.

Theorem 4.1. *The ILSGs of A form a chain if, and only if, $\mathscr{M}(k_1) = \mathscr{M}(k_2) \geq \mathscr{M}(m_k)$ and $\mathscr{N}(k_1) = \mathscr{N}(k_2) \leq \mathscr{N}(m_k)$ for all $k = 1, 2, 3$.*

Remark 4.5. We may denote the IFSGs of D_3 by $A(i, j),\ i, j = 1, 2, 3, ..., 10$, where the membership degrees in $A(i, j)$ are chosen as per the i^{th} row of Table 3 and the non-membership degrees are chosen as per the j^{th} row of Table 4.

Remark 4.6. According to theorem 4.1, corresponding to each of the hierarchies of membership levels in rows 1 and 2 of Table 3, exactly six hierarchies of non-membership levels in rows 3,4,5,6,7,8 of Table 4 will form IFSGs of D_3 whose ILSGs do not form a chain and corresponding to each of the hierarchies of non-membership levels in rows 1 and 2 of Table 4, exactly six hierarchies of membership levels in rows 3,4,5,6,7,8 of Table 3 will form IFSGs of D_3 whose ILSGs do not form a chain. Hence the total number of IFSGs of D_3 (upto isomorphism) whose ILSGs do not form a chain is $2 \times 6 + 2 \times 6 = 24$, which are given in Table 5.

$A(1,3)$	$A(1,4)$	$A(1,5)$	$A(1,6)$	$A(1,7)$	$A(1,8)$
$A(2,3)$	$A(2,4)$	$A(2,5)$	$A(2,6)$	$A(2,7)$	$A(2,8)$
$A(3,1)$	$A(4,1)$	$A(5,1)$	$A(6,1)$	$A(7,1)$	$A(8,1)$
$A(3,2)$	$A(4,2)$	$A(5,2)$	$A(6,2)$	$A(7,2)$	$A(8,2)$

TABLE 5. IFSGs of D_3 whose ILSGs do not form a chain.

Proposition 4.18. *The probability that the ILSGs of a randomly defined IFSG of D_3 forms a chain is 19/25.*

Proof. As stated in proposition 3.13, 100 distinct IFSGs can be defined on D_3 (upto isomorphism). By theorem 4.1 and remark 4.6, ILSGs corresponding to exactly 24 among them will not form a chain. Hence the proportion of IFSGs of which the ILSGs form a chain is $\frac{76}{100} = \frac{19}{25}$. □

Remark 4.7. We know that *every non-cyclic group of order 6 is isomorphic to Dihedral group D_3*. Hence it can be concluded that, 100 non-isomorphic IFSGs can be defined on any non-cyclic group of order 6, among which 76 IFSGs have the property that *the ILSGs form a chain*.

5. CONCLUSION

The aim of our research is to check whether the intuitionistic level subgroups of an intuitionistic fuzzy subgroup of any group form a chain a chain or not. In this paper, this investigation is carried out in the Dihedral group D_3. We have proved that, any IFSG of the Dihedral group D_3 can have at most three membership and non-membership levels. We have also proved that, D_3 has 100 distinct IFSGs upto isomorphism and that for only 76 among them the ILSGs form chains. We have also provided an enlisting of all the 100 distinct IFSGs of D_3 upto isomorphism. We are trying to generalize these and get an estimate of IFSGs of D_n whose ILSGs form chains.

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