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Generalized convergence and generalized sequential spaces

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ABSTRACT. We continue the study of g-convergence given in 2005 [Caldas, M.; Jafari, S. On g-US spaces. *Stud. Cercet. Ştiinţ. Ser. Mat. Univ. Bacău* **14** (2004), 13–19 (2005).] by introducing the sequential g-closure operator and we prove that the product of g-sequential spaces is not g-sequential by giving an example. We further investigate sequential g-continuity in topological spaces and present interesting theorems which are also new for the real case. It is shown that in a topological space the property of being g-sequential implies sequential, g-Fréchet implies Fréchet and g-Fréchet implies g-sequential. However, the inverse conclusions are not true and some counter examples are given. Also, we show that strongly g-continuous image of a g-sequential space is g-sequential, if the map is quotient. Finally, we obtain a necessary and sufficient condition for a topological space to be g-sequential in terms of a sequentially g-quotient map.

1. INTRODUCTION AND PRELIMINARIES

Fast [6] and Schoenberg [9] independently introduced the concept of statistical convergence by extending the convergence of real sequences. Any convergent sequence is statistically convergent but the converse is not true [8]. In 2005, Caldas and Jafari [4] introduced a new type of convergence in terms of g-open sets. Also, they studied sequentially g-closed set and sequential g-continuity by utilizing g-open sets. In this paper, we continue the study of g-convergence sequences. In Section 2, we introduce the sequential g-closure operator using g-convergence and derive some of their properties. In Section 3, we further investigate sequential g-continuity in topological spaces and present interesting theorems which are also new for the real case. In Section 4, we introduce generalized Fréchet spaces and provide relations between sequential, g-sequential, Fréchet and g-Fréchet spaces by giving counter examples. In Section 5, we point out that the space is g-sequential if and only if each quotient map on the space is sequentially g-quotient.

Let (X, τ) be a topological space. A subset A of X is called g-closed [4] if $cl(A) \subset G$ holds whenever $A \subset G$ and G is open in X. A is called a g-open subset of X if its complement $X - A$ is g-closed in X. Every open set is g-open [7]. A subset A of X is called a g-neighborhood of a point $x \in X$ if there exists a g-open set U with $x \in U \subset A$. A topological space (X, τ) is said to be $T_{1/2}$ [7] if every g-closed set in X is closed in X. A sequence of points $\{x_n\}$ in X is said to converge [5] to a point $x \in X$, denoted $\{x_n\} \to x$ if for every open set U of x, there is a $m \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq m$. Let $A \subset X$. A sequence $\{x_n\}$ converging to $x \in X$ is eventually in A if $\{x_n \mid n > p\} \cup \{x\} \subset A$ for some $p \in \mathbb{N}$. A is called sequentially closed [4] if for every sequence $\{x_n\}$ in A with $\{x_n\} \to x$, then $x \in A$. A function $[.]_{seq}$ of the power set $\mathbb{P}(X)$ to itself defined by for each subset A of X, $[A]_{seq} = \{x \in X \mid \{x_n\} \to x$ in (X, τ) for some sequence $\{x_n\}$ of points in $A\}$ is called the sequential closure operator on (X, τ) [1]. Observe that $A \subset [A]_{seq}$ [1].

A sequence $\{x_n\}$ in a space X g-converges to a point $x \in X$ [4] if $\{x_n\}$ is eventually in every g-open set containing x and is denoted by $\{x_n\} \xrightarrow{g} x$ and x is called the g-limit of

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the sequence $\{x_n\}$, denoted by $glim\, x_n$. A subset A of X is called sequentially g-closed [4] if every sequence in A g-converges to a point in A. Let $S[A]$ denote the set of all sequences in A and $c_g(X)$ denote the set of all g-convergent sequences in X. A sequentially g-open subset U (which is the complement of a sequentially g-closed set) is one in which every sequence in X which g-converges to a point in U is eventually in U.

A map $f : X \to Y$ from a topological space (X, τ) into a topological space (Y, σ) is called g-continuous [2] if the inverse image of every closed set in Y is g-closed in X. A map $f : X \to Y$ is said to be strongly g-continuous [2] if the inverse image of every g-closed set in Y is closed in X. A mapping $f : X \to Y$ is said to be sequentially quotient [3] provided that: a set A is sequentially closed in Y if and only if $f^{-1}(A)$ is sequentially closed in X. f is said to be *sequentially g-continuous* [4] if $\{f(x_n)\} \xrightarrow{g} f(x)$ in Y whenever $\{x_n\} \xrightarrow{g} x$ in X.

The following lemmas will be useful in the sequel.

Lemma 1.1. *Let (X, τ) and (Y, σ) be any two topological spaces. Then every strongly g-continuous function from (X, τ) to (Y, σ) is a continuous function.*

Lemma 1.2. *Let (X, τ) be a topological space and $A \subset X$. If A is sequentially closed, then $A = [A]_{seq}$.*

Proof. Suppose that A is sequentially closed and $x \in [A]_{seq}$. Then $\{x_n\} \to x$ for some sequence $\{x_n\}$ of points in A which implies that $x \in A$. Therefore, $[A]_{seq} \subset A$. But $A \subset [A]_{seq}$. Hence $[A]_{s-seq} = A$. □

2. SEQUENTIALLY g-CLOSED SETS

In this section, we define an operator, called sequentially g-closure operator in terms of g-convergent sequences and derive some of their properties.

The following Lemma 2.3(a) shows that every g-convergent sequence is a convergent sequence and Example 2.1 below shows that the converse is not true.

Lemma 2.3. *Let (X, τ) be a topological space. Then the following hold.*

(a) *Every g-convergent sequence is a convergent sequence.*
(b) *If (X, τ) is a $T_{1/2}$ space, then the concepts convergence and g-convergence coincide.*

Proof. (a) Suppose that $\{x_n\}$ is a sequence in X such that $\{x_n\} \xrightarrow{g} x$. Let U be a neighborhood of x. Then U is a g-open neighborhood of x. Therefore, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Thus, $\{x_n\} \to x$.

(b) By (a), every g-convergent sequence is a convergent sequence. Conversely, suppose that $\{x_n\} \to x$. Let U be a g-open neighborhood of x. Since X is a $T_{1/2}$ space, U is an open neighborhood of x. Since $\{x_n\} \to x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. Therefore, $\{x_n\} \xrightarrow{g} x$. Hence the convergence and the g-convergence of sequences coincide in a $T_{1/2}$ space. □

Example 2.1. Consider the topological space (X, τ) where $X = [0, 5)$, $\tau = \{\emptyset, (0, 1), X\}$. Suppose $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges to 0. If $A = (0, 1]$, then A is g-closed and so $X \setminus A$ is g-open. That is, $\{0\} \cup (1, 5)$ is a g-open subset of X such that $\frac{1}{n} \notin \{0\} \cup (1, 5)$ for any n. Hence $\{x_n\}$ does not g-converge to 0.

Lemma 2.4. *Let (X, τ) be a topological space. Then the following hold.*
(a) *Every constant sequence $(x, x, ...,)$ in X g-converges to x.*
(b) *If a sequence $\{x_n\}$ g-converges to x in X, then each subsequence $\{x_{n_k}\}$ of $\{x_n\}$ also g-converges to x.*

Proof. (a) Let $\{x_n\} = (x, x, ...,)$ be a sequence in X and U be a g-open set containing x. Since $x_n = x$ for all $n \in \mathbb{N}, x_n \in U$. Therefore, $\{x_n\} \xrightarrow{g} x$.

(b) Suppose $A = \{x_n \mid n \in \mathbb{N}\}$. Let $\{x_{n_k} \mid n_k \in \mathbb{N}\}$ be any subsequence of A and U be any g-open neighbourhood of x. Then there is an $m \in \mathbb{N}$ such that $x_n \in U$ whenever $n \geq m$. But $n \geq n_k$ for each $k \in \mathbb{N}$. So $n_k \geq m$ whenever $k \geq m$. Therefore, $x_{n_k} \in U$ whenever $k \geq m$. Thus, $\{x_{n_k}\} \xrightarrow{g} x$. □

Definition 2.1. Let (X, τ) be a topological space, $A \subset X$ and let $S[A]$ be the set of all sequences in A. Then the sequential g-closure of A, denoted by $[A]_{g_{seq}}$, is defined as

$$[A]_{g_{seq}} = \{x \in X \mid x = glim\ x_n \text{ and } \{x_n\} \in S[A] \cap c_g(X)\}$$

where $c_g(X)$ denotes the set of all g-convergent sequences in X.

The following Theorem 2.1 gives the properties of sequential g-closure operator.

Theorem 2.1. *Let A and B be subsets of a topological space (X, τ). Then the following hold.*

(a) $[\emptyset]_{g_{seq}} = \emptyset$.
(b) $A \subset [A]_{g_{seq}}$.
(c) $[A]_{g_{seq}} \subset cl(A)$.
(d) $A \subset B \Rightarrow [A]_{g_{seq}} \subset [B]_{g_{seq}}$.
(e) $[A]_{g_{seq}} \cup [B]_{g_{seq}} = [A \cup B]_{g_{seq}}$.
(f) $[A]_{g_{seq}} \subset [[A]_{g_{seq}}]_{g_{seq}}$.

Proof. (a) is clear.

(b) Suppose that $l \in A$. Consider the sequence $\{x_n\} = (l, l, ...,)$. Then $\{x_n\} \in S[A] \cap c_g(X)$. Also, $glim\ x_n = l$. Therefore, $l \in [A]_{g_{seq}}$ and hence $A \subset [A]_{g_{seq}}$.

(c) Suppose that $x \in [A]_{g_{seq}}$. Then $x = glim\ x_n$ where $\{x_n\} \in S[A] \cap c_g(X)$. That is, $\{x_n\} \xrightarrow{g} x$ and so by Lemma 2.3 (a), $\{x_n\} \to x$. Thus, $x \in cl(A)$.

(d) Suppose that $x \in [A]_{g_{seq}}$. Then $\{x_n\} \xrightarrow{g} x$ and $\{x_n\} \in S[A] \cap c_g(X)$. Since $A \subset B$, $\{x_n\} \in S[B] \cap c_g(X)$. Therefore, $x \in [B]_{g_{seq}}$. Thus, $[A]_{g_{seq}} \subset [B]_{g_{seq}}$.

(e) Since $A \subset A \cup B$ and by (d), $[A]_{g_{seq}} \subset [A \cup B]_{g_{seq}}$ and also, $[B]_{g_{seq}} \subset [A \cup B]_{g_{seq}}$. Therefore, $[A]_{g_{seq}} \cup [B]_{g_{seq}} \subset [A \cup B]_{g_{seq}}$. Let $x \in [A \cup B]_{g_{seq}}$. Then $\{x_n\} \xrightarrow{g} x$ for some sequence $\{x_n\}$ in $A \cup B$. Clearly, either $\{n \in \mathbb{N} \mid x_n \in A\}$ or $\{n \in \mathbb{N} \mid x_n \in B\}$ is infinite. Without loss of generality, assume that $\{n \in \mathbb{N} \mid x_n \in A\}$ is infinite. Then, it is obvious that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k} \mid n_k \in \mathbb{N}\} \subset A$ where $\{x_{n_k} \mid n_k \in \mathbb{N}\}$ is the range of (x_{n_k}). By Theorem 2.4(b), $\{x_{n_k}\} \xrightarrow{g} x$. Hence $x \in [A]_{g_{seq}} \cup [B]_{g_{seq}}$. Thus, $[A \cup B]_{g_{seq}} = [A]_{g_{seq}} \cup [B]_{g_{seq}}$.

(f) By (b), we have $A \subset [A]_{g_{seq}}$ and so $[A]_{g_{seq}} \subset [[A]_{g_{seq}}]_{g_{seq}}$, by (d). This completes the proof. □

The following Example 2.2 shows that the sequential g-closure operator $[A]_{g_{seq}}$ of a set A does not satisfy the idempotent property.

Example 2.2. Consider $X = \{0\} \cup \bigcup_{i \in \mathbb{N}} X_i$ where $X_i = \{\frac{1}{i}\} \cup \{\frac{1}{i} + \frac{1}{k} \mid k \in \mathbb{N}, k \geq i^2\}$ for each $i \in \mathbb{N}$. Suppose that X is endowed with the following topology.
(i) Each point of the form $\frac{1}{i} + \frac{1}{j}$ where $j \in \mathbb{N}$ is isolated.
(ii) Each neighbourhood of each point of the form $\frac{1}{i}$ contains a set of the form $\{\frac{1}{i}\} \cup \{\frac{1}{i} + \frac{1}{k} \mid k \in \mathbb{N}, k \geq j\}$ where $j \geq i^2$.
(iii) Each neighbourhood of the point 0 contains a set obtained from X by removing a finite number of X_i's and finite number of points of the form $\frac{1}{i} + \frac{1}{j}$, $j \geq i^2$ in all the remaining X_i's.

Take $A = \{\frac{1}{i}\} \cup \{\frac{1}{i} + \frac{1}{k} \mid k \in \mathbb{N}, k \geq i^2\}$. Then $[A]_{g_{seg}} = X \setminus \{0\}$ and so $[[A]_{g_{seq}}]_{g_{seg}} = [X \setminus \{0\}]_{g_{seq}} = X$. Therefore, $[A]_{g_{seq}} \neq [[A]_{g_{seq}}]_{g_{seq}}$.

Theorem 2.2. *Let (X, τ) be a topological space. Then the following hold.*

(a) *Every sequentially closed set is a sequentially g-closed set.*
(b) *$[A]_{g_{seq}} \subset [A]_{seq}$ for $A \subset X$.*

Proof. (a) Suppose $A \subset X$ is sequentially closed. Then $A = [A]_{seq}$, by Lemma 1.2. Let $\{x_n\}$ be a sequence in A such that $\{x_n\} \xrightarrow{g} x$. By Lemma 2.3 (a), $\{x_n\} \to x$ in A and so $x \in [A]_{seq}$. Hence $x \in A$. Thus, A is sequentially g-closed.
(b) Suppose that $x \in [A]_{g_{seq}}$. Then by the definition of sequential g-closure of A, there is a sequence $\{x_n\}$ in A such that $\{x_n\} \xrightarrow{g} x$. By Lemma 2.3 (a), $\{x_n\} \to x$ and so $x \in [A]_{seq}$. Therefore, $[A]_{g_{seq}} \subset [A]_{seq}$. □

The following Example 2.3 shows that the converse of Theorem 2.2 (b) is not true.

Example 2.3. Consider the topological space (X, τ) where $X = [0, 5)$, $\tau = \{\emptyset, (0, 1), X\}$. Suppose $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges to 0. If $A = (0, 1]$, then A is g-closed and so $X \setminus A$ is g-open. That is, $\{0\} \cup (1, 5)$ is a g-open subset of X. But $\frac{1}{n} \notin \{0\} \cup (1, 5)$ for any n. Hence $\{x_n\}$ does not g-converge to 0. Therefore, $[A]_{seq} \nsubseteq [A]_{g_{seq}}$

The following Theorem 2.3 gives the relation between closed set and sequentially g-closed set in any topological space. The converse of Theorem 2.3 (c) is not true in general as shown by Example 2.4 below.

Theorem 2.3. *Let (X, τ) be a topological space and $A \subset X$. Then the following hold.*

(a) *A is sequentially g-closed if and only if $[A]_{g_{seq}} \subset A$.*
(b) *Every closed subset of X is sequentially g-closed.*
(c) *If A is open, then A is sequentially g-open.*

Proof. (a) Suppose $x \in [A]_{g_{seq}}$. Then there exists $\{x_n\} \in S[A] \cap c_g(X)$ such that $x = glim\ x_n$. Since A is a sequentially g-closed subset of X, $x \in A$. Hence $[A]_{g_{seq}} \subset A$. Conversely, let $\{x_n\}$ be a sequence in A such that $\{x_n\} \xrightarrow{g} x$. Then $x \in [A]_{g_{seq}}$. By assumption, $[A]_{g_{seq}} \subset A$ and so $x \in A$. Hence A is sequentially g-closed.
(b) Suppose that A is closed. Then $A = cl(A) \supset [A]_{g_{seq}}$, by Theorem 2.1 (c). By (a), A is sequentially g-closed.
(c) Let A be open. Then $X \setminus A$ is closed in X. By (b), $X \setminus A$ is sequentially g-closed. Hence A is sequentially g-open. □

Example 2.4. Consider $(\mathbb{R}, \tau_c)$ where τ_c is the cocountable topology on $\mathbb{R}$. Here $F \subset \mathbb{R}$ is closed if and only if $F = \mathbb{R}$ or F is countable. Let A be a subset of $\mathbb{R}$. Suppose that a sequence $\{x_n\}$ in A has a g-limit y. Then the g-neighbourhood $(\mathbb{R} \setminus \{x_n \mid n \in \mathbb{N}\}) \cup \{y\}$ of y must contain $\{x_n\}$ for large n. This is only possible when $x_n = y$ for large n. A sequence in any set A, g-converges to an element of A. So every subset of $\mathbb{R}$ is sequentially g-open. As $\mathbb{R}$ is uncountable, not every subset is open. Thus, a sequentially g-open set need not be open.

Theorem 2.4. *Let (X, τ) be a topological space. Then the following hold.*
(a) *Intersection of any collection of sequentially g-closed subsets of X is sequentially g-closed.*
(b) *The union of any collection of sequentially g-open subsets of X is sequentially g-open.*

Proof. (a) Let $\{F_\alpha \mid \alpha \in \Delta\}$ be a collection of sequentially g-closed subsets of X where Δ is an arbitrary index set. If $\cap F_\alpha = \emptyset$, then there is nothing to prove. Suppose $\cap F_\alpha \neq \emptyset$. If $x \in [\cap F_\alpha]_{g_{seq}}$, then there exists a sequence $\{x_n\}$ of points in $\bigcap_{\alpha \in \Delta} F_\alpha$ such that $\{x_n\} \xrightarrow{g} x$.

Therefore, for each $\alpha \in \Delta$, $\{x_n\}$ is in F_α such that $\{x_n\} \xrightarrow{g} x$. Therefore, $x \in [F_\alpha]_{g_{seq}}$ for each $\alpha \in \Delta$. As each F_α is sequentially g-closed, $x \in F_\alpha$ for each $\alpha \in \Delta$. Thus, $x \in \bigcap_{\alpha\in\Delta} F_\alpha$ and so $[\bigcap_{\alpha\in\Delta} F_\alpha]_{g_{seq}} \subset \bigcap_{\alpha\in\Delta} F_\alpha$. By Theorem 2.3 (a), $\bigcap_{\alpha\in\Delta} F_\alpha$ is sequentially g-closed in X.
(b) Since complement of a sequentially g-closed set is a sequentially g-open set, the proof follows from (a). □

Theorem 2.5. *Let (X, τ) be a topological space and A be a subset of X. Then $x \in [A]_{g_{seq}}$ if and only if for every sequentially g-open neighbourhood U of x, $U \cap A \neq \emptyset$.*

Proof. Let $x \in [A]_{g_{seq}}$ and let U be a sequentially g-open neighbourhood of x. Suppose $U \cap A = \emptyset$. Then $A \subseteq X \setminus U$ so that $[A]_{g_{seq}} \subset [X \setminus U]_{g_{seq}}$, by Theorem 2.1 (d). Here $X \setminus U$ is a sequentially g-closed subset and hence $[A]_{g_{seq}} \subset X \setminus U$, by Theorem 2.3 (a). Therefore, $x \in X \setminus U$ and so $x \notin U$ which is a contradiction. Hence $A \cap U \neq \emptyset$ for every sequentially g-open neighbourhood of x.

Conversely, suppose that $x \notin [A]_{g_{seq}}$. Then $U = X \setminus [A]_{g_{seq}}$ is a sequentially g-open neighbourhood of x but does not intersect A, since $A \subset [A]_{g_{seq}}$ (Theorem 2.1 (b)). Hence $U \cap A = \emptyset$. □

Theorem 2.6. *Let (X, τ) be a topological space and A be a subset of X. Then $[A]_{g_{seq}} = \bigcap\{M \mid A \subseteq M$ and M is sequentially g-closed $\}$.*

Proof. Let $x \in [A]_{g_{seq}}$ and M be a sequentially g-closed subset containing A. Then there is a sequence $\{x_n\}$ in A such that $g \lim x_n = x$. Since $\{x_n\}$ is also a sequence in M, $x \in [M]_{g_{seq}}$. Since M is sequentially g-closed, $x \in M$, by Theorem 2.3 (a). Thus, $[A]_{g_{seq}} \subset \bigcap\{M \mid A \subseteq M$ and M is sequentially g-closed $\}$.

Conversely, let $x \notin [A]_{g_{seq}}$. Then there is a sequentially g-open neighbourhood U of x such that $U \cap A = \emptyset$, by Theorem 2.5. Now $U \cap A = \emptyset$ implies that $A \subset (X \setminus U)$. Since $X \setminus U$ is sequentially g-closed, $X \setminus U$ must be some M. As $x \notin X \setminus U$, we have $x \notin \bigcap\{M \mid A \subseteq M$ and M is sequentially g-closed $\}$. Therefore, $\bigcap\{M \mid A \subseteq M$ and M is sequentially$g$-closed$\} \subset [A]_{g_{seq}}$. Hence $[A]_{g_{seq}} = \bigcap\{M \mid A \subseteq M$ and M is sequentially g-closed $\}$. □

3. Sequentially g-continuous maps

Balachandran et al. [2] already studied the relation between continuous, g-continuous and strongly g-continuous maps. In 2004, Caldas et al. [4] introduced the concept of sequentially g-continuous map and studied their properties. In this section, we investigate further properties of sequential g-continuous maps in topological spaces.

Theorem 3.7. *If $f : (X, \tau) \to (Y, \sigma)$ is a sequentially g-continuous map, then the following hold.*
(a) *The inverse image of any sequentially g-closed subset of Y is sequentially g-closed in X.*
(b) *The inverse image of any sequentially g-open subset of Y is sequentially g-open in X.*

Proof. (a) Let F be a sequentially g-closed subset of Y. Suppose that $x \in [f^{-1}(F)]_{g_{seq}}$. Then there is a sequence $\{x_n\} \in f^{-1}(F)$ such that $\{x_n\} \xrightarrow{g} x$. Since f is sequentially g-continuous, $\{f(x_n)\} \xrightarrow{g} f(x)$. Thus, $f(x) \in F$, since F is sequentially g-closed and so $x \in f^{-1}(F)$. Therefore, $[f^{-1}(F)]_{g_{seq}} \subset f^{-1}(F)$. Hence $f^{-1}(F)$ is sequentially g-closed, by Theorem 2.3 (a).
(b) Since complement of a sequentially g-closed set is a sequentially g-open set, the proof follows from (a). □

Definition 3.2. Let (X, τ) and (Y, σ) be two topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a map. Then f is said to be *sequentially g-open* if the image of any sequentially g-open subset of X is sequentially g-open and f is said to be *sequentially g-closed* if the image of any sequentially g-closed subset of X is sequentially g-closed.

Theorem 3.8. *Let $f, h : (X, \tau) \to (X, \tau)$ be any two maps. Then the following hold.*
(a) *If f and h are sequentially g-continuous, then so is $hf = h \circ f$.*
(b) *If f and h are sequentially g-closed, then so also is hf.*
(c) *If f is sequentially g-continuous, hf is sequentially g-closed and f is onto, then h is sequentially g-closed.*
(d) *If hf is sequentially g-closed, h is sequentially g-continuous and h is one-to-one, then f is sequentially g-closed.*

Proof. (a) Let $\{x_n\}$ be a sequence in X such that $\{x_n\} \xrightarrow{g} x \in X$. Since f is sequentially g-continuous at x, $\{f(x_n)\} \xrightarrow{g} f(x)$ and so $\{h(f(x_n))\} \xrightarrow{g} h(f(x))$, since h is sequentially g-continuous at $f(x)$. Therefore, the map hf is sequentially g-continuous.

(b) Let A be a sequentially g-closed subset of X. Then $f(A)$ is a sequentially g-closed subset of X. Since h is sequentially g-closed, $h(f(A))$ is a sequentially g-closed subset of X. Therefore, hf is sequentially g-closed.

(c) Let A be a sequentially g-closed subset of X. Since f is sequentially g-continuous, $f^{-1}(A)$ is sequentially g-closed, by Theorem 3.7(a). Since hf is sequentially g-closed and f is onto, $hf(f^{-1}(A)) = h(A)$ is sequentially g-closed.

(d) Let A be a sequentially g-closed subset of X. Since hf is sequentially g-closed, $hf(A)$ is sequentially g-closed. Since h is sequentially g-continuous and one-to-one, $h^{-1}h(f(A)) = f(A)$ is sequentially g-closed. Therefore, f is a sequentially g-closed. □

Theorem 3.9. *Let (X, τ) be a topological space. Then the following hold.*
(a) *If $f : X \to X$ is a sequentially g-continuous map, then so is the restriction $f : A \to X$ to a subset A.*
(b) *The identity map $f : X \to X$ is sequentially g-continuous.*
(c) *For a subset $A \subseteq X$, the inclusion map $f : A \to X$ is sequentially g-continuous.*
(d) *The constant map $f : X \to X$ is sequentially g-continuous.*

Proof. (a) Let $\{x_n\}$ be a sequence in A with $\{x_n\} \xrightarrow{g} x$. Then $\{x_n\}$ is a sequence in X. Since f is sequentially g-continuous. $\{f(x_n)\} \xrightarrow{g} f(x)$. Hence $f : A \to X$ is a sequentially g-continuous map.

(b) Suppose $\{x_n\} \xrightarrow{g} x$ where $\{x_n\} \in S[X]$. Then $\{f(x_n)\} \xrightarrow{g} f(x)$, since $f(x) = x$ and $f(x_n) = x_n$ for all $n \in \mathbb{N}$. Therefore, the identity map is sequentially g-continuous.

(c) Let $\{x_n\}$ be a sequence in A with $\{x_n\} \xrightarrow{g} x \in A$. Since $\{f(x_n)\} = \{x_n\}$, we have $\{f(x_n)\} \xrightarrow{g} x$. Therefore, the inclusion map is sequentially g-continuous.

(d) Let $f : X \to X$ be a constant map with $f(x) = x_0$ for all $x \in X$ and let $\{x_n\}$ be a sequence in X g-converging to x. Then $\{f(x_n)\} = (x_0, x_0, ...,) \xrightarrow{g} x_0$, by Lemma 2.4(a). Therefore, $\{f(x_n)\} \xrightarrow{g} f(x)$. □

Theorem 3.10. *Let (X, τ) be a topological group under addition $+$ and let $f, h : (X, \tau) \to (X, \tau)$ be any two maps. If f and h are sequentially g-continuous, then also is $f + h$.*

Proof. Let $\{x_n\}$ be a sequence in X with $\{x_n\} \xrightarrow{g} x$ in X. Since the maps f and h are sequentially g-continuous, $\{f(x_n)\} \xrightarrow{g} f(x)$ and $\{h(x_n)\}) \xrightarrow{g} h(x)$. Therefore, $\{(f + h)(x_n)\} = \{f(x_n)\} + \{h(x_n)\} \xrightarrow{g} f(x) + h(x) = (f + h)(x)$. That is, $f + h$ is sequentially g-continuous. □

Theorem 3.11. *Let (X, τ) be a topological group and let $f : X \to X$ be an additive map on X onto itself. Then f is sequentially g-continuous at the origin if and only if f is sequentially g-continuous at every point in X.*

Proof. Let the additive map $f : X \to X$ be sequentially g-continuous everywhere. Then $\{f(x_n)\} \xrightarrow{g} f(0)$ whenever $\{x_n\} \xrightarrow{g} 0$. Hence $\{f(x_n)\} \xrightarrow{g} 0$, since $f(0) = 0$. Therefore, f is sequentially g-continuous at 0. Conversely, let $a \in X$ and $\{x_n\}$ be a sequence in X with $glim\ x_n = a$. Since the constant sequence g-converges, by Lemma 2.4(a), $(a) \xrightarrow{g} a$. Therefore, the sequence $\{x_n - a\} \xrightarrow{g} 0$ and so by assumption, $\{f(x_n - a)\} \xrightarrow{g} f(0) = 0$. Since f is additive, $\{f(x_n) - f(a)\} \xrightarrow{g} 0$. Hence $\{f(x_n)\} \xrightarrow{g} f(a)$. □

Theorem 3.12. *Let (X, τ) be a topological group and $a \in X$ be fixed. The map $f_a : X \to X$, defined by $f_a(x) = a + x$ is sequentially g-continuous, sequentially g-closed and sequentially g-open. We denote the inverse of the map f_a by f_{-a}.*

Proof. Let x be a point in X with $\{x_n\} \xrightarrow{g} x$. Then the sequence $\{a + x_n\}$ is g-convergent to $a + x$, by Lemma 2.4(a). Since $f_a(x_n) = a + x_n$, $\{f_a(x_n)\} \xrightarrow{g} a + x$ which implies that $\{f_a(x_n)\} \xrightarrow{g} f_a(x)$. Therefore, f_a is sequentially g-continuous. Since the inverse of f_a is f_{-a}, by Theorem 3.7 (a), the map f_a is sequentially g-closed and by Theorem 3.7 (b), f_a is sequentially g-open. □

Theorem 3.13. *Let (X, τ) be a topological group and $A, B \subset X$. If one of the sets A and B is sequentially g-open, then the sum $A + B$ is also a sequentially g-open set.*

Proof. Suppose that B is a sequentially g-open subset of X and A is any subset. By Theorem 3.12, $a + B$ is sequentially g-open for any $a \in A$. Since $A + B = \bigcup_{a \in A} a + B$, by Theorem 2.4 (b), $A + B$ is sequentially g-open. □

Theorem 3.14. *Let (X, τ) be a topological group, Y be Hausdorff space and $f : X \to Y$ be a sequentially g-continuous map. Then $A = \{x \in X \mid f(x) = 0\}$, the kernel of f is a sequentially g-closed subset of X.*

Proof. If $x \in [A]_{g_{seq}}$, then there exists a sequence $\{x_n\}$ of points in A such that $g \lim x_n = x$. Since $x_n \in A$ for all $n \in \mathbb{N}$, the sequence $\{f(x_n)\}$ is the constant sequence with $f(x_n) = 0$ for all n. Therefore, $g \lim f(x_n) = 0$, by Lemma 2.4(a). Since f is sequentially g-continuous, $g \lim f(x_n) = f(x)$. Since Y is Hausdorff, $f(x) = 0$ and so $x \in A$. By Theorem 2.3(a), A is sequentially g-closed. □

Theorem 3.15. *Let (X, τ) be a topological group, Y be Hausdorff space and $f, h : X \to Y$ be sequentially g-continuous maps. Then $A = \{x \in X \mid f(x) = h(x)\}$ is a sequentially g-closed subset of X.*

Proof. If $x \in [A]_{g_{seq}}$, then there exists a sequence $\{x_n\}$ of the points in A such that $g \lim x_n = x$. Since $(h-f)(x_n) = h(x_n) - f(x_n)$, $\{(h-f)(x_n)\}$ is a constant sequence with $(h-f)(x_n) = 0$ for all n. Therefore, $g \lim(h - f)(x_n) = 0$. By Theorem 3.10, $h - f$ is sequentially g-continuous, $g \lim(h - f)(x_n) = (h - f)(x)$. Thus, $(h - f)(x) = 0$ so that $h(x) = f(x)$. Thus, $x \in A$. Hence A is sequentially g-closed. □

4. g-SEQUENTIAL AND g-FRÉCHET SPACES

In this section, we define g-sequential and g-Fréchet spaces and derive some of their properties. We prove that each g-Fréchet space is a g-sequential space and each g-Fréchet space is a Fréchet space. However, we show that the converse implications of the above statements are not true by giving counter examples.

Definition 4.3. A topological space (X, τ) is said to be *g-sequential* if every sequentially g-closed set in X is a closed set. A topological space (X, τ) is said to be *g-Fréchet* if $cl(A) \subset [A]_{g_{seq}}$ for each $A \subset X$.

The following Theorem 4.5 shows that every quotient of a g-sequential space is a g-sequential space. Theorem 4.16 below gives relation between sequential, Fréchet, g-sequential and g-Fréchet spaces.

Lemma 4.5. *Every quotient of a g-sequential space is g-sequential.*

Proof. Let f be a quotient map of a g-sequential space X onto a space Y. Let U be sequentially g-open subset of Y. We prove that U is open in Y. Enough to show that $f^{-1}(U)$ is sequentially g-open in X. Let $\{x_n\}$ be a sequence in X g-converging to $x \in f^{-1}(U)$. Then $\{x_n\} \to x \in X$ in $f^{-1}(U)$, by Theorem 2.3 (a). Thus, $\{f(x_n)\} \to f(x) \in Y$ in U, as f is continuous. Thus, $\{f(x_n)\}$ is eventually in U. Therefore, there exists $n_0 \in \mathbb{N}$ such that $f(x_n) \in U$ for $n \geq n_0$ which implies that $f^{-1}f(x_n) \subset f^{-1}(U)$ for $n \geq n_0$ and so $x_n \in f^{-1}(U)$ for $n \geq n_0$. Thus, the set $\{x_n \mid n \geq n_0\}$ is eventually in $f^{-1}(U)$. Therefore, $f^{-1}(U)$ is sequentially g-open in X. Since X is g-sequential, $f^{-1}(U)$ is open in X which implies that U is open in Y, since f is quotient. Hence Y is a g-sequential space. □

Theorem 4.16. *Let (X, τ) be a topological space. Then the following hold.*
(a) *If X is a g-sequential space, then X is a sequential space.*
(b) *If X is a g-Fréchet space, then X is a Fréchet space.*
(c) *If X is a g-Fréchet space, then X is a g-sequential space and hence X is a sequential space.*

Proof. (a) Let A be a sequentially open subset of X. Then $X \setminus A$ is a sequentially closed subset of X. By Lemma 2.2 (a), $X \setminus A$ is sequentially g-closed. Hence $X \setminus A$ is closed, by our hypothesis and so A is open in X. Therefore, X is a sequential space.
(b) Let $x \in cl(A)$. Since X is g-Fréchet, $cl(A) \subset [A]_{g_{seq}}$ and so $x \in [A]_{g_{seq}}$. There is a sequence $\{x_n\}$ in A such that $\{x_n\} \xrightarrow{g} x$. By Theorem 2.3 (a), $\{x_n\} \to x$ and so $x \in [A]_{seq}$. Hence $cl(A) \subset [A]_{seq}$. Thus, X is a Fréchet space.
(c) Suppose A is sequentially g-closed. By Theorem 2.3 (a), $[A]_{g_{seq}} \subset A$. Since X is g-Fréchet, $cl(A) \subset [A]_{g_{seq}} \subset A \subset cl(A)$. Hence $A = cl(A)$. Therefore, A is closed. Thus, X is a g-sequential space.
By (a), every g-sequential space is a sequential space. This completes the proof. □

We show that the reverse implications of Theorem 4.16 are not true by giving examples. The following Example 4.5 gives a sequential space which is not a g-sequential space.

Example 4.5. Let $S = \{x_n \mid n \in \mathbb{N}\}$ be a sequence such that $x_n \neq x_m$ if $n \neq m$. Take $x \notin S$ and let $X = S \cup \{x\}$. The topology on X is defined in the following sense.
(1) Each point x_n is isolated.
(2) Each open neighbourhood of the point x is a set U of the form $U = \{x\} \cup M$ where M is a subset of S.
Clearly, x is a unique non isolated point in X.
We prove that X is sequential but no subsequence of S g-converges to x.

Let Y be a sequentially closed subset of X. To prove that Y is closed in X. If $x \in Y$ or Y is a finite set, then it is obvious that Y is closed. So we assume that $x \notin Y$ and Y is infinite. Let $Y = \{x_{n_k} \mid k \in \mathbb{N}\}$. It is enough to verify that $y \in Y$ if $y \in cl(Y)$. Suppose that $y \in cl(Y)$. Since Y is a sequentially closed subset of X, Y converges to y. Therefore, $y \in Y$. It follows that $cl(Y) = Y$. Hence X is sequential.

Next, we prove that no subsequence of S g-converge to x. Assume that $\{x_{n_k}\}$ is a subsequence of S. Put $U = X \setminus \{x_{n_k} \mid k \in \mathbb{N}\}$. Then U is a g-open neighbourhood of x and for any $k \in \mathbb{N}$, $x_{n_k} \notin U$ which implies that $\{x_{n_k}\}$ does not g-converge to x. Since S is a non-closed subset of X and $x \in cl(S) = X$, X is not a g-sequential space. Thus, there exists a sequential space which is not a g-sequential space.

The following Example 4.6 gives a Fréchet space which is not a g-Fréchet space.

Example 4.6. For each $n \in \mathbb{N}$, let $S_n = \{x_{n,m} \mid m \in \mathbb{N}\}$ be a sequence such that $x_{n,m} \neq x_{n,l}$ if $m \neq l$. Take $\infty \notin \bigcup\{S_n \mid n \in \mathbb{N}\}$ and let $X = \bigcup\{S_n \mid n \in \mathbb{N}\} \cup \{\infty\}$. The topology on X is defined in the following sense.

(1) Each point $x_{n,m}$ is isolated.

(2) Each open neighbourhood of the point ∞ is a set U of the form $U = \{\infty\} \cup \{M_n \mid n \in \mathbb{N}\}$ where M_n is a dense subsequence of S_n.

We prove that the space X is a Fréchet space.

Assume that Y is an arbitrary subset of X. Take $y \in cl(Y)$. If $y \in \cup\{S_n \mid n \in \mathbb{N}\}$, then it is obvious that $y \in Y$. For each $k \in \mathbb{N}$, put $x_k = y$. Then the sequence $\{x_k\}$ is a subset of Y converges to y. Therefore, we assume that $y = \infty$ and $\infty \notin Y$. Put $V = X \setminus Y$. Then V is an open neighbourhood of ∞ and $V \cap Y = \emptyset$ which is a contradiction to $\infty \in cl(Y)$. It follows that S_n converges to ∞. Therefore, X is a Fréchet space.

Next we prove that the space X is not a g-Fréchet space. Suppose $\infty \in cl(S_1)$. By Example 4.5, we know that no subsequence of S_1 g-converges to ∞. Hence the space X is not a g-Fréchet space. Therefore, there is a Fréchet space which is not a g-Fréchet space.

The following Example 4.7 gives a g-sequential space which is not a g-Fréchet space.

Example 4.7. Let $L = \mathbb{R} \setminus \{0\}$ be the set of real numbers with 0 removed and let $M = \{0\} \cup \{\frac{1}{i} \mid i \in \mathbb{N}\}$ and $Y = (L \times \{0\}) \cup (M \times \{1\})$ have its usual topology as a subspace of $\mathbb{R}^2$. In particular, Y is a g-sequential space. Now let $f : Y \to X$ be the projection of Y onto its first coordinate space, the set $X = \{0\} \cup L$, that is, X is the set of all real numbers. The topology on X is generated by the usual topology with all sets of the form $\{0\} \cup U$ where U is open in $\mathbb{R}$ and contains $\{\frac{1}{i} \mid i \in \mathbb{N}\}$. As a quotient of a g-sequential space, X is g-sequential, by Lemma 4.5. Now we prove that X is not g-Fréchet. Let $A = X \setminus M$. Then $[A]_{g_{seq}} = \mathbb{R} \setminus \{0\}$. We show that $0 \in cl(A)$ but no sequence in A g-converges to 0. Let N be a neighbourhood of 0. Then N contains a point $\frac{1}{i}$ for some $i \in \mathbb{N}$. Away from 0, the topology of X is locally 'as usual'. At any rate, there exists some $\delta > 0$ such that $(\frac{1}{i} - \delta, \frac{1}{i} + \delta) \subseteq N$. Now there exists some irrational $x \in N$ and x being irrational, $x \in A$. Hence $A \cap N \neq \emptyset$. This shows that $0 \in cl(A)$. Suppose that $\{x_n\}$ is any sequence in A. For each $n \in \mathbb{N}$, $x_n \neq 0$. Therefore, $\inf_{j \in \mathbb{N}} \mid x_n - \frac{1}{j} \mid > 0$. Consider the sequence (α_i) where $\alpha_i = \inf_{j \in \mathbb{N}} \mid x_n - \frac{1}{j} \mid$, and let $W = \{0\} \cup \bigcup_{i \in \mathbb{N}} (\frac{1}{i} - \delta_i, \frac{1}{i} + \delta_i)$. Then W is a g-neighbourhood of 0 that contains none of the member of the sequence $\{x_n\}$. Therefore, $\{x_n\}$ cannot be g-convergent to 0. It follows that no sequence in A g-converging to 0. So X is not a g-Fréchet space. Therefore, there exists a g-sequential space which is not a g-Fréchet space.

Theorem 4.17. *If the product space is g-sequential, so is each of its factors.*

Proof. Let $X = \Pi_{\alpha \in \Lambda} X_\alpha$ be a g-sequential space. It is enough to prove that for $\beta \in \Lambda$, if U_β is not open in X_β, then U_β is not sequentially g-open in X_β. Suppose that U_β is not open in X_β. Let $U \subset X$ such that $\pi_\beta(U) = U_\beta$ and $\pi_\alpha(U) = U_\alpha$ for $\alpha \neq \beta$. Then U is not open in X. Since X is g-sequential, U is not a sequentially g-open subset of X. Therefore, there exists a sequence $\{x_{\alpha_n}\}$ in X not in U g-converging to $(x_\alpha) \in U$. In particular, for $\alpha = \beta$, $\{x_{\beta_n}\}$ g-converges to $(x_\beta) \in U$ and $\{x_{\beta_n}\} \notin U_\beta$, that is, U_β is not sequentially g-open. Therefore, X_β is a g-sequential space. □

The following Example 4.8 shows that the product of g-sequential spaces need not be a g-sequential space.

Example 4.8. Let $\mathbb{Q}'$ be rationals, $\mathbb{Q}$, with integers identified and let $X = \mathbb{Q} \times \mathbb{Q}'$. Since $\mathbb{Q}$ is a $T_{\frac{1}{2}}$ space, every g-open set is open and hence $\mathbb{Q}$ and $\mathbb{Q}'$ are g-sequential. Therefore, X is the product of two g-sequential spaces. But X contains a sequentially g-open set W which is not open. Now we describe W as follows: Let $\{x_n\} \subseteq \mathbb{R}$ be a sequence of irrational numbers less than one, g-converging monotonically downwards to 0. For $n = 0, 1, \dots,$ let T_n be the integer of the plane triangle determined by the points $(x_n, n), (1, n+1/2)(1, n-1/2)$. Let T_n' be the reflection of T_n on the y-axis and R_n be the interior of the rhombus determined by the points $(-x_n, n), (0, n+1/2), (x_n, n)$ and $(0, n-1/2)$. Then $W_n = T_n \cup R_n \cup T_n'$ is an open subset of the plane. Considering X as a subset of the plane with the horizontal integer lines identified, let $W = X \cap \bigcup\limits_0^{\infty} W_n$. If $\pi_1 : X \to \mathbb{Q}$ and $\pi_2 : X \to \mathbb{Q}'$ are the canonical projections, for any neighborhoods U and U' of 0 in $\mathbb{Q}$ and $\mathbb{Q}'$, respectively, $\pi_1^{-1}(U) \cap \pi_2^{-1}(U') \nsubseteq W$. Hence $(0,0)$ is not an interior point of W and hence W cannot be open.

Now suppose $(y_n) \subseteq X \setminus W$ and $\{y_n\} \xrightarrow{g} y \in W$. If $\pi_2(y) \neq 0$, g-convergence in X is g-convergence in $\mathbb{Q} \times \mathbb{Q}$ which is contradiction to $\{y_n\} \xrightarrow{g} y \in W$. Hence $\pi_2(y) = 0$. If $\pi_1(y) \neq 0$, then W can be replaced by a scaled down version of itself with y at the symmetric position. Hence we may assume that $y = (0,0)$. But $\{y_n\} \xrightarrow{g} (0,0)$ implies $\{\pi_2(y_n)\} \xrightarrow{g} 0$ in $\mathbb{Q}'$ which can occur if and only if some subsequence g-converges in $\mathbb{Q}$ to some integer k. But this restrict $\{y_n\}$ eventually to arbitrary small $\pi_2(k-\epsilon, k+\epsilon)$. Since $\{y_n\} \xrightarrow{g} (0,0)$ in X, eventually put $\{y_n\}$ in W. Hence W is sequentially g-open. Therefore, $\mathbb{Q} \times \mathbb{Q}'$ is not g-sequential.

Proposition 4.1. *In a topological space, the following statements hold.*
(a) *The disjoint topological sum of any family of g-sequential spaces is g-sequential.*
(b) *The disjoint topological sum of any family of g-Fréchet spaces is g-Fréchet.*

Proof. (a) Let X be the disjoint sum of the family $\{X_i\}_{i\in\Lambda}$ of g-sequential spaces. Suppose U is not open in X. Then for some $i \in \Lambda$, $U \cap X_i$ is not open in X_i, so $U \cap X_i$ is not sequentially g-open in X_i for some $i \in \Lambda$. Thus, there is a point $x \in U \cap X_i$ and a sequence $\{x_n\} \subset X_i \setminus U$ g-converges to x in X_i and also in X. Therefore, U is not sequentially g-open. Hence X is a g-sequential space.
(b) Let X be the disjoint sum of the family of g-Fréchet spaces $\{X_i \mid i \in \Lambda\}$. Suppose U is closed in X. Then for some $i \in \Lambda$, $U \cap X_i$ is closed in X_i and so $U \cap X_i$ is sequentially g-closed in X_i, since each X_i is g-Fréchet. So $U \cap X_i$ is closed for some $i \in \Lambda$, since $[A]_{g_{seq}} \subset cl(A)$. Thus, there is a point $x \in U \cap X_i$ and a sequence $\{x_n\}$ in X_i g-converges to x in X_i and also in X. Hence U is sequentially g-closed. □

Theorem 4.18. *Let (X, τ) be a g-sequential space, (Y, σ) be a topological space and let $f : X \to Y$ be a map. Then f is strongly g-continuous if and only if f is sequentially g-continuous.*

Proof. Suppose f is strongly g-continuous and $\{x_n\} \xrightarrow{g} x$ in X. Let V be a g-open set containing $f(x)$. Then $f^{-1}(V)$ is an open set containing x, by hypothesis. Therefore, $f^{-1}(V)$ is a g-open set containing x. Since $\{x_n\} \xrightarrow{g} x$, $\{x_n\}$ is eventually in $f^{-1}(V)$. That is, there exists $n_0 \in \mathbb{N}$ such that $x_n \in f^{-1}(V)$ for all $n \geq n_0$ and so $f(x_n) \in ff^{-1}(V) \subset V$. Thus, $\{f(x_n)\}$ is eventually in V, by Lemma 1.1. Therefore, $\{f(x_n)\} \xrightarrow{g} f(x)$.

Suppose that f is not strongly g-continuous. Then there is a g-open set $U \subset Y$ such that $f^{-1}(U)$ is not open in X. Thus, $f^{-1}(U)$ is also not sequentially g-open, since X is a

g-sequential space. Therefore, there is a sequence $\{x_n\}$ in $X \setminus f^{-1}(U)$ that g-converges to a point $y \in f^{-1}(U)$. But $\{f(x_n)\}$ is a sequence in $Y \setminus U$, a closed set and so $f(x)$ can not be a g-limit of $\{f(x_n)\}$. Hence f does not preserve g-convergence. Therefore, f is not sequentially g-continuous. □

Proposition 4.2. *The strongly g-continuous open (or closed) image of a g-sequential space is a g-sequential space.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an open and a strongly g-continuous map. Let X be a g-sequential space. Suppose $f(X)$ is not a g-sequential space. Then there exists a sequentially g-open subset U of $f(X)$ which is not open. Since f is an open map, $f^{-1}(U)$ is not open. Now X is a g-sequential space and $f^{-1}(U)$ is not open implies that $f^{-1}(U)$ is not sequentially g-open. Therefore, there exists a point $x \in f^{-1}(U)$ and a sequence $\{x_n\} \notin f^{-1}(U)$ such that $\{x_n\}$ g-converges to x. By Theorem 4.18, $\{f(x_n)\} \xrightarrow{g} f(x)$ and $f(x) \in U$. But $f(x_n) \notin U$ which is a contradiction to the fact that U is sequentially g-open. □

Theorem 4.19. *Each sequentially g-open (sequentially g-closed) subspace of a g-sequential space is g-sequential.*

Proof. Let (X, τ) be a g-sequential space. Suppose that Y is a sequentially g-open subspace of X. Then Y is open in X, since X is g-sequential. Let U be an arbitrary sequentially g-open subset of Y. We prove that U is sequentially g-open in X. Let $\{x_n\}$ be a sequence in X which g-converges to $x \in U$. Then $x \in Y$ and since Y is a sequentially g-open subset of X, $\{x_n\}$ is eventually in Y. That is, there exists $k_1 \in \mathbb{N}$ such that $\{x_n \mid n > k_1\} \subset Y$. Since U is a sequentially g-open subset of Y, there exists $k_2 \in \mathbb{N}$ such that $\{x_n \mid n > k_2\} \subset U$. Then $\{x_n \mid n > k\} \subset U$ where $k = max\{k_1, k_2\}$. Therefore, U is sequentially g-open in X and hence open in X implies that U is open in Y, since Y is open in X. Therefore, Y is a g-sequential space.

If Y is a sequentially g-closed subset of X, then Y is closed in X, since X is a g-sequential space. Let A be a sequentially g-closed subset of Y and $\{x_n\}$ be a sequence in A g-converging to $x \in X$. Since Y is closed, $x \in Y$. Hence $x \in A$. Therefore, A is a sequentially g-closed set in X and so A is closed in X, as X is a g-sequential space. Since Y is closed in X, A is closed in Y. □

The following Corollary 4.1 shows that strongly g-continuous image of a g-sequential space is g-sequential, if the map is quotient.

Corollary 4.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a quotient map from X onto a space Y and f be strongly g-continuous. If X is a g-sequential space, then Y is a g-sequential space.*

Proof. Suppose that X is a g-sequential space. Let G be any sequentially g-open set in Y. We prove that $f^{-1}(G)$ is sequentially g-open in X. Let $\{x_n\}$ be a sequence in X which g-converges to a point x in $f^{-1}(G)$. Then $\{f(x_n)\} \xrightarrow{g} f(x)$, by Theorem 4.18. Since $f(x) \in G$, there exists $k \in \mathbb{N}$ such that $\{f(x_n) \mid n > k\}$ is eventually in G. Since f is onto, $x = f^{-1}(f(x)) \in f^{-1}(G)$. Hence $\{f^{-1}(f(x_n)) \mid n > k\}$ is eventually in $f^{-1}(G)$ which implies that $\{x_n \mid n > k\}$ is eventually in $f^{-1}(G)$ and so $f^{-1}(G)$ is sequentially g-open in X. Since X is g-sequential, $f^{-1}(G)$ is open X. Therefore, G is an open subset of Y, by the definition of quotient map. □

5. Sequentially g-quotient map

In this section, we introduce the concept of sequentially g-quotient map and study their properties. Also,we give a characterization for a sequentially g-quotient map. Finally, we obtain a necessary and sufficient condition for a topological space to be g-sequential in terms of a sequentially g-quotient map.

Definition 5.4. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be *sequentially g-quotient* if it satisfies the following: A is sequentially g-closed in Y if and only if $f^{-1}(A)$ is sequentially g-closed in X.

The following Example 5.9 shows the existence of sequentially g-quotient map and Theorem 5.20 provides an equivalent condition for a sequentially g-quotient map.

Example 5.9. Let X be the topological sum of the collection $\{I, S_\alpha \mid \alpha \in I\}$ where $I = [0, 1]$ and each S_α is a g-convergent sequence with its g-limit x_α for each $\alpha \in I$ and let Y be the space obtained from X by identifying the g-limit point of S_α with α. Let $f : X \to Y$ be the natural map. Let $\{y_n\}$ be a g-converging sequence in Y. Then there is a subsequence of $\{y_n\}$ which is either contained in S_α or in I. Therefore, the sequence $\{f^{-1}(y_n)\} \cap S_\alpha$ or $\{f^{-1}(y_n)\} \cap I$ must g-converges whose image is a subsequence of $\{y_n\}$. Therefore, f is sequentially g-quotient.

Theorem 5.20. *Let (X, τ) and (Y, σ) be two topological spaces and let $f : X \to Y$ be a strongly g-continuous map. Then the following are equivalent.*
(a) *f is a sequentially g-quotient map.*
(b) *If $\{x_n\}$ is a sequence in Y g-converging to x, then $\{f^{-1}(x_n)\} \xrightarrow{g} f^{-1}(x)$.*

Proof. (a)⇒(b) Suppose f is a sequentially g-quotient map. Let $\{x_n\} \xrightarrow{g} x$ in Y. If $A = \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$, then A is a sequentially g-closed set in Y. Since f is a sequentially g-quotient map, $f^{-1}(A)$ is a sequentially g-closed set in X. That is, $\{f^{-1}(x_n)\} \xrightarrow{g} f^{-1}(x)$ and $f^{-1}(x) \in f^{-1}(A)$.
(b)⇒(a) Let B be sequentially g-closed in Y. We prove that $f^{-1}(B)$ is sequentially g-closed in X. Let $\{x_n\}$ be a sequence in $f^{-1}(B)$ g-converging to x. Then there is some $y_n \in B$ such that $f(x_n) = y_n$. Let V be a g-open neighbourhood of $f(x)$. Since f is strongly g-continuous, $f^{-1}(V)$ is a g-open neighbourhood of x. Therefore, there exists $n_0 \in \mathbb{N}$ such that $x_n \in f^{-1}(V)$ for all $n \geq n_0$ and so $f(x_n) \in V$. Thus, $y_n \in V$ for all $n \geq n_0$. Hence $\{y_n\} \xrightarrow{g} f(x)$ and so $x \in f^{-1}(B)$. Therefore, $f^{-1}(B)$ is sequentially g-closed in X. □

Theorem 5.21. *Let (X, τ) be a $T_{1/2}$ space and $f : (X, \tau) \to (Y, \sigma)$ be a sequentially g-quotient map. If f is a sequentially g-closed map, then f is a sequentially closed map.*

Proof. Suppose A is a sequentially closed set in X. By Theorem 2.2 (a), A is sequentially g-closed set in X. By hypothesis, $f(A)$ is sequentially g-closed in Y. To prove that $f(A)$ is sequentially closed in Y. Let $\{y_n\}$ be a sequence in $f(A)$ such that $\{y_n\} \to y$. Since f is sequentially g-quotient, $\{f^{-1}(y_n)\} \xrightarrow{g} f^{-1}(y)$, by Theorem 5.20. That is, $\{f^{-1}(y_n)\}$ is a sequence in A g-converging to $f^{-1}(y)$. Since X is $T_{1/2}$, $\{f^{-1}(y_n)\} \to f^{-1}(y)$ in A, by Lemma 2.3 (a). Hence $f^{-1}(y) \in A$ so that $y \in f(A)$. Hence $f(A)$ is sequentially closed in Y. Therefore, f is a sequentially closed map. □

Theorem 5.22. *Let (X, τ) be a topological space. Then X is g-sequential if and only if each quotient map on X is sequentially g-quotient.*

Proof. Let X be a g-sequential space and let $f : X \to Y$ be any quotient map. Then Y is a g-sequential space, by Lemma 4.5. Let U be any non sequentially g-closed subset of Y. Then U is not closed and $f^{-1}(U)$ is not closed. Since X is g-sequential, $f^{-1}(U)$ is not sequentially g-closed in X.

Conversely, suppose that X is not g-sequential. Let A be a sequentially g-closed subset of X which is not closed. Consider the map $f : X \to Y$ where $Y = \{0, 1\}$ defined by

$$f(x) = \begin{cases} 0, \text{ if } x \in A \\ 1, \text{ if } x \in X - A \end{cases}$$

Let Y has the quotient topology induced by f. Since $f^{-1}(\{1\}) = X - A$ is not open in X, $\{1\}$ is not open in Y and also, not g-open in Y. Thus, the constant sequence $(0, 0, ...,) \xrightarrow{g} 1$. The set $\{0\}$ is not sequentially g-closed, but $f^{-1}(\{0\})$ is sequentially g-closed. Hence f is a quotient map on X and f is not sequentially g-quotient. □

Recall that a class of map is said to be *hereditary* [1] if whenever $f : X \to Y$ is in the class, then for each subspace L of Y, the restriction of f to $f^{-1}(L)$ is in the class. The following Theorem 5.23 shows that every sequentially g-quotient map is a hereditarily sequentially g-quotient map.

Theorem 5.23. *Sequentially g-quotient maps are hereditarily sequentially g-quotient.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a sequentially g-quotient map and L be a subspace of Y. Take $h = f|_{f^{-1}(L)}$ such that $h : f^{-1}(L) \to L$ be a map. Given a sequence $\{x_n\}$ g-convergence to y in L, there exists a sequence $\{x_n\} \in f^{-1}(y_n) \subset f^{-1}(L)$ such that $\{x_n\} \xrightarrow{g} x \in f^{-1}(y) \subset f^{-1}(L)$, since f is a sequentially g-quotient map. Hence $\{y_n\} \xrightarrow{g} y$ in Y. Therefore, h is a sequentially g-quotient map. □

Proposition 5.3. *Finite product of sequentially g-quotient maps is sequentially g-quotient.*

Proof. Let $\Pi_{i=1}^N f_i : \Pi_{i=1}^N X_i \to \Pi_{i=1}^N Y_i$ be a map where each $f_i : X_i \to Y_i$ is a sequentially g-quotient map for $i = 1, 2, ...N$. Let $\{(y_{i,n})\}$ be a sequence g-converges to (y_i) in $\Pi_{i=1}^N Y_i$. Since each f_i is a sequentially g-quotient map, there exists a sequence $\{x_{i,n}\}$ in X_i such that $\{x_{i,n}\} \xrightarrow{g} x_i$ and $f_i(x_{i,n}) = y_{i,n}$. Let $(x_i) \in \Pi_{i=1}^N X_i$ and U be a g-open set containing (x_i). Then $(x_{i,n}) \in U$ implies that $\{(x_{i,n})\}_{n\in\mathbb{N}} \xrightarrow{g} (x_i)$. Therefore, $\Pi_{i=1}^N f_i$ is a sequentially g-quotient map. □

Proposition 5.4. *Let $f : X \to Y$ and $g : Y \to Z$ be two strongly g-continuous maps. Then the following hold.*

(a) *If f and g are sequentially g-quotient, then $g \circ f$ is sequentially g-quotient.*
(b) *If X is a g-sequential space and $g \circ f$ is sequentially g-quotient, then g is sequentially g-quotient.*

Proof. (a) Let A be a g-converging sequence in Z with its g-limit point z. Since g is sequentially g-quotient, there exists a g-converging sequence B in Y with its g-limit point $y \in g^{-1}(z)$ such that $g(B) = A$, by Theorem 5.20. Also, f is a sequentially g-quotient map implies that there exists a g-converging sequence C in X with its g-limit point $x \in f^{-1}(y)$ and $f(C) = B$. That is, there exists a g-converging sequence C in X with its g-limit point $x \in (g \circ f)^{-1}(z)$ such that $(g \circ f)(C) = A$. Hence $g \circ f$ is sequentially g-quotient.
(b) Let A be a g-convergent sequence in Z with its g-limit z. Since $g \circ f$ is sequentially g-quotient, there exists a g-convergent sequence C in X with its g-limit point $x \in (g\circ f)^{-1}(z)$ such that $(g \circ f)(C) = A$. By Theorem 4.18, $f(C)$ is a g-convergent sequence in Y with its g-limit $f(x) = y \in g^{-1}(z)$ such that $g(f(C)) = A$. Therefore, g is a sequentially g-quotient map. □

Theorem 5.24. *If (Y, σ) is a g-sequential space, then every sequentially g-quotient map onto Y is quotient.*

Proof. Let Y be a g-sequential space and $f : (X, \tau) \to (Y, \sigma)$ be a sequentially g-quotient map onto Y. Suppose that $f^{-1}(U)$ is open in X and U is not open in Y. Then $Y \setminus U$ is not closed in Y. Therefore, by hypothesis, there exists $y \in U$ such that $\{y_n\} \xrightarrow{g} y$ and

$\{y_n\} \in Y \setminus U$. Since f is sequentially g-quotient, there exists a sequence $\{x_n\} \xrightarrow{g} x$ such that $x \in f^{-1}(y) \subset f^{-1}(U)$ and $\{x_n\} \in f^{-1}(y_n) \subset f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$. Therefore, $X \setminus f^{-1}(U)$ is not a sequentially g-closed set, since $x \notin X \setminus f^{-1}(U)$. Therefore, $f^{-1}(U)$ is not a sequentially g-open set which is a contradiction to $f^{-1}(U)$ is open. Hence f is a quotient map. □

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