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Hesitant Fuzzy Maximal and Minimal Clopen Sets

A. SWAMINATHAN and S. SIVARAJA

ABSTRACT. This article is to study the notions of hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen sets. In this article, we have proved hesitant fuzzy maximal and minimal clopen sets are independent to hesitant fuzzy minimal and maximal open sets(resp. closed sets). Using hesitant fuzzy disconnectedness certain properties of both hesitant fuzzy minimal and maximal clopen sets are discussed.

1. INTRODUCTION

Zadeh [11] established fuzzy set in 1965 and Chang [1] introduced fuzzy topology in 1968. Ittanagi and Wali[4] introduced fuzzy maximal and minimal open sets. The idea of hesitant fuzzy set introduced by Torra [7] in 2010 which is an addendum to fuzzy sets. Deepak et. al. [2] introduced hesitant fuzzy topological space and extended the study to hesitant connectedness and compactness in hesitant fuzzy topology.

In this paper, we have defined both hesitant fuzzy minimal (resp.maximal)clopen sets of hesitant fuzzy topological space and study certain properties of it.

The notions h_1, h_2, h_3,... represent hesitant fuzzy sets.

2. PRELIMINARIES

Definition 2.1. [5] A hesitant fuzzy set h in X is a function $h : X \to P\,[0,1]$, where $P\,[0,1]$ represents the power set of [0,1].

We define the hesitant fuzzy empty set h^0(resp. whole set h^1) is a hesitant fuzzy set in X as follows: $h^0(x) = \phi$ (resp. $h^1(x) = [0,1]), \forall x \in X$. $HS(X)$ stands for collection of hesitant fuzzy set in X.

Definition 2.2. [3] Two hesitant fuzzy set $h_1, h_2 \in HS(X)$ such that $h_1(x) < h_2(x), \forall x \in X$, then h_1 is contained in h_2.

Definition 2.3. [3] Two hesitant fuzzy set h_1 and h_2 of X are said to be equal if $h_1 < h_2$ and $h_2 < h_1$.

Definition 2.4. [5] Let $h \in HS(X)$ for any nonempty set X. Then h^c is the complement of h which is hesitant fuzzy set in X such that $h^c(x) = [h(x)]^c = [0,1]\backslash h(x)$.

Definition 2.5. [6] Let X be a nonempty set. A hesitant fuzzy topology τ of subsets X is said to be hesitant fuzzy topology on X if
(i) $h^0, h^1 \in \tau$.
(ii) $\bigcup\limits_{i\in J} h_i \in \tau$ for each $(h_i)_{i\in J} \in \tau$.
(iii) $h_1 \cap h_2 \in \tau$ for any $h_1,\ h_2 \in \tau$.

"The pair (X,τ) is called hesitant fuzzy topology. The members of τ are called hesitant fuzzy open sets in X. A hesitant fuzzy set h in X is hesitant fuzzy closed set (in short hesitant fuzzy closed) in (X,τ) if $h^c \in \tau$."

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Corresponding author: asnathanway@gmail.com

Definition 2.6. ([10]) A proper hesitant fuzzy open set h_1 of X is an hesitant fuzzy maximal open set if h_2 is an hesitant fuzzy open set such that $h_1 < h_2$, then $h_1 = h_2$ otherwise $h_2 = h^1$.

Definition 2.7. ([10]) A proper hesitant fuzzy open set h_1 of X is an hesitant fuzzy minimal open set if h_2 is an hesitant fuzzy open set such that $h_2 < h_1$, then $h_1 = h_2$ otherwise $h_2 = h^0$.

Definition 2.8. ([10]) A proper hesitant fuzzy closed set h_1 of X is an hesitant fuzzy minimal closed set if h_2 is an hesitant fuzzy closed set such that $h_2 < h_1$, then $h_1 = h_2$ otherwise $h_2 = h^0$.

Definition 2.9. ([10]) A proper hesitant fuzzy closed set h_1 of X is an hesitant fuzzy maximal closed set if h_2 is an hesitant fuzzy closed set such that $h_1 < h_2$, then $h_1 = h_2$ otherwise $h_2 = h^1$.

Theorem 2.1. ([10]) Let h_1 and h_2 be hesitant fuzzy minimal open and hesitant fuzzy maximal open sets respectively in a hesitant fuzzy topology X with $h_2 \nsubseteq h_1$ then $h_1 = h^1 - h_2$.

3. Hesitant Fuzzy Minimal and Maximal Clopen Sets

We now introduce hesitant fuzzy minimal clopen and maximal clopen sets:

Definition 3.10. A proper hesitant fuzzy clopen h_1 of a hesitant fuzzy topology X is said to be hesitant fuzzy minimal clopen set if any other hesitant fuzzy clopen set contained in h_1 is either h^0 or itself.

Definition 3.11. A proper hesitant fuzzy clopen h_1 of a hesitant fuzzy topology X is said to be hesitant fuzzy maximal clopen set if any other hesitant fuzzy clopen set containing h_1 is either h^1 or itself.

Example 3.1. Let $X = \{a, b, c\}$ and $\tau = \{h^0, h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h^1\}$ where
$h_1(a) = (0, 0.4]$,$h_1(b) = (0, 0.8)$,$h_1(c) = (0.3, 0.9]$
$h_2(a) = (0.4, 1] \vee \{0\}$,$h_2(b) = [0.8, 1] \vee \{0\}$,$h_2(c) = [0, 0.3] \vee (0.9, 1]$
$h_3(a) = (0, 0.3)$,$h_3(b) = (0, 0.7)$,$h_3(c) = (0.3, 0.9]$
$h_4(a) = [0.3, 1) \vee \{0\}$,$h_4(b) = [0.7, 1] \vee \{0\}$,$h_4(c) = [0, 0.3] \vee (0.9, 1]$
$h_5(a) = (0, 1]$,$h_5(b) = (0, 1]$,$h_5(c) = (0.3, 1]$
$h_6(a) = (0, 0.3) \vee (0.4, 1]$,$h_6(b) = (0, 0.7) \vee [0.8, 1]$,$h_6(c) = (0.3, 1]$
$h_7(a) = [0, 0.4]$,$h_7(b) = [0, 0.8)$,$h_7(c) = [0, 0.9]$
$h_8(a) = [0, 0.3)$,$h_8(b) = [0, 0.7)$,$h_8(c) = [0, 0.9]$
and
$h_9(a) = [0, 0.3) \vee (0.4, 1]$,$h_9(b) = [0, 0.7) \vee [0.8, 1]$,$h_9(c) = [0, 1]$.
Here $\{h_1, h_2, h_3, h_4\}$ are hesitant fuzzy clopen sets. Obviously h_3 is hesitant fuzzy minimal clopen set and h_4 is hesitant fuzzy maximal clopen set.Also, h_3 is hesitant fuzzy minimal open and h_5 is hesitant fuzzy maximal open.Hence, hesitant fuzzy minimal closed,hesitant fuzzy maximal closed,hesitant fuzzy minimal open,hesitant fuzzy maximal open and hesitant fuzzy minimal clopen,hesitant fuzzy maximal clopen notions are independent.

It is evident that a hesitant fuzzy maximal open or a hesitant fuzzy maximal closed may not be a hesitant fuzzy maximal clopen set. Therefore, the notion of hesitant fuzzy maximal clopen sets is independent to the notions of hesitant fuzzy maximal open and hesitant fuzzy maximal closed as well as sets. It is also easy to see that if h_4 is both hesitant fuzzy maximal open and hesitant fuzzy maximal closed, then h_4 is hesitant fuzzy maximal clopen. In fact, a hesitant fuzzy clopen set is hesitant fuzzy maximal clopen if it is either hesitant fuzzy maximal open or hesitant fuzzy maximal closed. In [10] , we observed that if a hesitant fuzzy topology with a single proper hesitant fuzzy open set h_1, then h_1 is both hesitant fuzzy maximal open and hesitant fuzzy minimal open. Also, we have observe h_1 is neither hesitant fuzzy maximal closed nor hesitant fuzzy minimal closed. If h_1,h_2 are the only proper hesitant fuzzy open sets such that one is not contained in other, then both are hesitant fuzzy maximal closed,hesitant fuzzy minimal closed sets. In addition, hesitant fuzzy maximal closed or hesitant fuzzy minimal closed sets can exist only in a hesitant fuzzy disconnected space. Clearly theorems 3.2 to corollary 3.2 are obvious, the proofs of them are omitted.

Theorem 3.2. *Let h_1 be a hesitant fuzzy minimal clopen set. Then either $h_1 \wedge h_2 = h^0$ or $h_1 < h_2$ for any hesitant fuzzy clopen set h_2 in X.*

Corollary 3.1. *Let h_1,h_2 be distinct hesitant fuzzy minimal clopen sets in X. Then $h_1 \wedge h_2 = h^0$.*

Theorem 3.3. *Let h_1 be a hesitant fuzzy maximal clopen set. Then either $h_1 \vee h_2 = h^1$ or $h_2 < h_1$ for any hesitant fuzzy clopen set h_2 in X.*

Corollary 3.2. *Let h_1 and h_2 be distinct hesitant fuzzy maximal clopen sets in X. Then $h_1 \vee h_2 = h^1$.*

Lemma 3.1. *If h_1 is hesitant fuzzy minimal clopen in a hesitant fuzzy topology (X, τ), then h_1^c is hesitant fuzzy maximal clopen in X and conversely.*

Proof. For any two proper hesitant fuzzy clopen sets h_1 and h_2, then $h_1^c < h_2$ implies that $h_2^c < h_1$. Since h_1 is hesitant fuzzy minimal clopen set, we have $h_2^c = h_1$ otherwise $h_2^c = h^0$ which gives $h_1^c = h_2$ or $h_2 = h^1$. Therefore h_1^c is a hesitant fuzzy maximal clopen set. Similarly follows the converse. □

Theorem 3.4. *If h_1 is a hesitant fuzzy minimal clopen and h_2 is a hesitant fuzzy maximal clopen set in (X, τ), then $h_1 < h_2$ or $h_1 < h_2^c$.*

Proof. Proof is similar to that of theorem 3.1 in [8]. □

Theorem 3.5. *If h_1 is both hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen of a hesitant fuzzy topology (X, τ), then*
(i) h_1 and h_1^c are the only hesitant fuzzy sets in X which are both hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen.
(ii) The only proper hesitant fuzzy clopen sets in X are h_1 and h_1^c.

Proof. (i) Since lemma 3.1, for any hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen set h_1 in X, h_1^c is also both hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen in X. Let h_2 be any other hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen distinct from h_1 in X. By deploying lemma 3.1, h_2^c is also both hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen in X. Since h_1 and h_2 are hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen set, by corollary 3.1 and corollary 3.2, we have $h_1 \wedge h_2 = h^0$ and $h_1 \vee h_2 = (h)^1$. As for any $h_1 \neq h_2$, h_2 and h_2^c are identical to h_1 and h_1^c respectively. This completes the proof for all possible combinations of h_1,h_2,h_1^c,h_2^c.

(ii) Let h_3 be a hesitant fuzzy clopen in X. Clearly, either $h_1 \vee h_3 = h^1$ or $h_3 < h_1$ for a hesitant fuzzy maximal clopen set h_1. Again, $h_1 \wedge h_3 = h^0$ or $h_1 < h_3$ for a hesitant fuzzy minimal clopen set h_1. Therefore,
$h_1 \vee h_3 = h^1$ and $h_1 \wedge h_3 = h^0 \Rightarrow h_3 = h_1^c$.
$h_1 \vee h_3 = h^1$ and $h_1 < h_3 \Rightarrow h_3 = h^1$ and
$h_3 < h_1$ and $h_1 \wedge h_3 = h^0$ implies that $h_3 = h^0$. This completes the proof. □

Theorem 3.6. *In a hesitant fuzzy topology (X, τ), hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen sets appears in pairs.*

Proof. By theorem 3.5, if h_1 is both hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen in X, then h_1^c is also hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen and these pairs of sets in X are unique. By lemma 3.1, if h_1^c is hesitant fuzzy minimal(resp.max)clopen in X, then h_1^c is a hesitant fuzzy maximal(resp.min)clopen in X. □

Theorem 3.7. *If h_1 is a hesitant fuzzy maximal open and h_2 is a hesitant fuzzy minimal open set of a hesitant fuzzy topology (X, τ) with $h_1 \not< h_2$, then h_1 is a hesitant fuzzy maximal clopen and h_2 is a hesitant fuzzy minimal clopen set.*

Proof. In accordance with fuzzy maximality of h_1 by theorem 2.1, $h_1 = (h_2)^c$. Hence h_1 and h_2 are hesitant fuzzy clopen set. Clearly h_1 is hesitant fuzzy maximal clopen because it is both hesitant fuzzy clopen and hesitant fuzzy maximal open set. Similarly we can prove that h_1 is hesitant fuzzy minimal clopen set. □

Theorem 3.8. *If h_3 is a hesitant fuzzy maximal clopen in X, then $h_3 \vee h_4$ is not a proper hesitant fuzzy clopen set distinct from h_3 for any proper hesitant fuzzy open(or hesitant fuzzy closed) set h_4 in X.*

Proof. Suppose that $h_3 \vee h_4$ is a proper hesitant fuzzy clopen set in X. Clearly either $h_3 \vee h_4 = (h^1)$ or $h_3 \vee h_4 = h_3$ as h_3 is a hesitant fuzzy maximal clopen in X. Hence $h_4 < h_3$ as $h_3 \vee h_4 = h_3$. □

Theorem 3.9. *If h_3 is hesitant fuzzy minimal clopen in X, then $h_3 \wedge h_4$ is not a proper hesitant fuzzy clopen set disticnt from h_3 for any proper hesitant fuzzy open(or hesitant fuzzy closed) set h_4 in X.*

Proof. Obvious. □

According to theorem 3.8 and theorem 3.9, the form of hesitant fuzzy clopen sets in hesitant fuzzy topology contains either hesitant fuzzy maximal clopen or hesitant fuzzy minimal clopen set.

Theorem 3.10. *If J is a collection of distinct hesitant fuzzy maximal clopen sets and $h_1 \in J$, then $\bigwedge_{h_2 \in J-\{h_1\}} h_2 \neq h^0$. Then $\bigwedge_{h_2 \in J-\{h_1\}} h_2$ is a hesitant fuzzy minimal clopen iff $h_1^c = \bigwedge_{h_2 \in J-\{h_1\}} h_2$.*

Proof. Obviously, h_2^c is hesitant fuzzy minimal clopen set as $h_2 \in J - \{h_1\}$ is a hesitant fuzzy maximal clopen set. By deploying theorem 3.4, $h_2^c < h_1$. Hence $\left(\bigwedge_{h_2 \in J-\{h_1\}} h_2\right)^c < h_1$ implies that $h_1 = h^1$ if $\left(\bigwedge_{h_2 \in J-\{h_1\}} h_2\right) = h^0$, a contradiction to the assumption that h_1 is a hesitant fuzzy maximal clopen set. Hence $\bigwedge_{h_2 \in J-\{h_1\}} h_2 \neq h^0$.

Conversly let us assume that $\bigwedge_{h_2 \in J-\{h_1\}} h_2$ is hesitant fuzzy minimal clopen. Consider $\bigwedge_{h_2 \in J-\{h_1\}} h_2$ is a hesitant fuzzy clopen set. As J is finite family of hesitant fuzzy clopen set, then $\bigwedge_{h_2 \in J-\{h_1\}} h_2$ is a hesitant fuzzy clopen set. Since $h^1 - \bigwedge_{h_2 \in J-\{h_1\}} h_2 < h_1$, we have $h^1 - h_1 < \bigwedge_{h_2 \in J-\{h_1\}} h_2$. As h_1 is hesitant fuzzy maximal clopen set, then $h^1 - h_1$ is hesitant fuzzy minimal clopen. If $\bigwedge_{h_2 \in J-\{h_1\}} h_2$ is a hesitant fuzzy minimal clopen distinct from $h^1 - h_1$, then by corollary 3.1 we have $\left(\bigwedge_{h_2 \in J-\{h_1\}} h_2\right) \bigwedge \left(h^1 - h_1\right) = h^0$. This gives that $\bigwedge_{h_2 \in J-\{h_1\}} h_2 < h_1$. Therefore $h^1 - h_1 < \bigwedge_{h_2 \in J-\{h_1\}} h_2 < h_1$ which is wrong. Hence we obtain $h^1 - h_1 < \bigwedge_{h_2 \in J-\{h_1\}} h_2$. □

Theorem 3.11. *If J is a collection of distinct hesitant fuzzy maximal clopen sets and $h_1 \in J$, then $\bigvee_{h_2 \in J-\{h_1\}} h_2 \neq h^1$. Then $\bigvee_{h_2 \in J-\{h_1\}} h_2$ is a hesitant fuzzy minimal clopen iff $h_1^c = \bigvee_{h_2 \in J-\{h_1\}} h_2$.*

Proof. Proof is similar to theorem 3.10. □

If h_3 is a hesitant fuzzy clopen in hesitant fuzzy topology (X, τ), then $h_1 \wedge h_3$ is hesitant fuzzy clopen in (h_1, τ_{h_1}). Also, if h_1 is hesitant fuzzy clopen in (X, τ) then a hesitant fuzzy clopen set in (h_1, τ_{h_1}) is also hesitant fuzzy clopen in (X, τ).

Theorem 3.12. *Let h_1, h_4 be hesitant fuzzy clopen sets in X such that $h_1 \wedge h_4 \neq h^0$. Then $h_1 \wedge h_4$ is a hesitant fuzzy minimal clopen in (h_1, τ_{h_1}) if h_4 is a hesitant fuzzy minimal clopen in (X, τ).*

Proof. Proof is similar to Theorem 4.11 in [8]. □

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DEPARTMENT OF MATHEMATICS
GOVERNMENT ARTS COLLEGE(AUTONOMOUS)
KUMBAKONAM, TAMIL NADU-612 002, INDIA
Email address: asnathanway@gmail.com

DEPARTMENT OF MATHEMATICS
ANNAMALAI UNIVERSITY
ANNAMALAINAGAR, TAMIL NADU-608 002, INDIA
Email address: sivarajamathematics@gmail.com