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Balcobalancing Numbers and Balcobalancers II

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ABSTRACT. In this work, we derived some new algebraic results on continued fraction expansion of the ratio of the two consecutive balcobalancing numbers, circulant matrices and spectral norms, Pythagorean triples, characteristic polynomials and eigenvalues of the n^{th} power of the companion matrices, Cassini and Catalan identities, cross-ratios and Heisenberg groups related to balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers.

1. INTRODUCTION

A positive integer n is called a balancing number ([1]) if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

holds for some positive integer r which is called balancer corresponding to n. Let B_n denote the n^{th} balancing number. Then $B_0=0, B_1=1, B_2=6$ and $B_{n+1}=6B_n-B_{n-1}$ for $n\geq 2$.

Later Panda and Ray ([10]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{1.2}$$

holds for some positive integer r which is called cobalancer corresponding to n. Let b_n denote the n^{th} cobalancing number. Then $b_0=b_1=0, b_2=2$ and $b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$.

It is clear from (1.1) and (1.2) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n=r_{n+1}$ and $R_n=b_n$ for $n\geq 1$, where R_n is the n^{th} balancer and r_n is the n^{th} cobalancer. Since $R_n=b_n$, we get from (1.1) that

$$b_n=\frac{-2B_n-1+\sqrt{8B_n^2+1}}{2} \text{ and } B_n=\frac{2b_n+1+\sqrt{8b_n^2+8b_n+1}}{2}. \tag{1.3}$$

Thus from (1.3), we see that B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. So $C_n=\sqrt{8B_n^2+1}$ and $c_n=\sqrt{8b_n^2+8b_n+1}$ are integers which are called the n^{th} Lucas-balancing number and the n^{th} Lucas-cobalancing number (see [3, 8, 9, 13, 14]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [6], Liptai proved that there is no Fibonacci balancing number except 1 and in [7] he proved that there is no Lucas-balancing number. In [16], Szalay considered the same problem and obtained some nice results by a different method. In [4], Kovács, Liptai and Olajos extended the concept of balancing numbers to the (a,b)-balancing numbers defined as follows: Let $a>0$ and $b\geq 0$ be coprime integers. If

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$$(a+b)+\cdots+(a(n-1)+b)=(a(n+1)+b)+\cdots+(a(n+r)+b)$$

for some integers $n,r \geq 1$, then $an+b$ is an (a,b)-balancing number. The sequence of (a,b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [5], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y,k,l \in \mathbb{Z}^+$ with $y \geq 4$. A positive integer x such that $x \leq y-2$ is called a (k,l)-power numerical center for y if

$$1^k+\cdots+(x-1)^k=(x+1)^l+\cdots+(y-1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k,l)-power numerical centers. For positive integers k,x, let

$$\Pi_k(x)=x(x+1)\dots(x+k-1).$$

Then it was proved in [4] that the equation $B_m=\Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2,3,4\}$ all solutions were determined. In [20] Tengely, considered the case $k=5$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [12], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers and in [15], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods. In [11], Panda and Panda defined the almost balancing number and its balancer. In [18], the first author considered amost balancing numbers, triangular numbers and square triangular numbers and in [17], he considered the sums and spectral norms of all almost balancing numbers.

2. Results.

In [19], we defined three new integer sequences called balcobalancing numbers, balcobalancers and Lucas-balcobalancing numbers and derived some results on them.

Similarly in this paper, we will deduce some new results on continued fraction expansion of the ratio of the two consecutive balcobalancing numbers, circulant matrices and spectral norms, Pythagorean triples, characteristic polynomials and eigenvalues of the n^{th} power of the companion matrices, Cassini and Catalan identities, cross-ratios and Heisenberg groups related to balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers.

2.1. **Continued Fraction Expansion.** In [3, Theorem 2.17], the authors proved that the continued fraction expansions of two consecutive balancing numbers, cobalancing numbers, Lucas-balancing numbers and Lucas-cobalancing numbers are

$$\frac{B_{n+1}}{B_n}=[5;\ \underbrace{1,4,}_{n-2\text{ times}}\ 1,5] \text{ for } n \geq 2$$

$$\frac{b_{n+1}}{b_n}=\begin{cases}[5;\ \underbrace{1,4,}_{\frac{n-5}{2}\text{ times}}\ 1,5] & \text{for odd } n \geq 5\\[2ex] [5;\ \underbrace{1,4,}_{\frac{n-4}{2}\text{ times}}\ 1,6] & \text{for even } n \geq 4\end{cases}$$

$$\frac{C_{n+1}}{C_n}=[5;\ \underbrace{1,4,}_{n-1\text{ times}}\ 1,2] \text{ for } n \geq 1$$

$$\frac{c_{n+1}}{c_n} = [5; \underbrace{1,4,}_{n-2 \text{ times}} 1,6] \text{ for } n \geq 2.$$

Similarly we can give the following theorem.

Theorem 2.1. *The continued fraction expansion of* $\frac{B_n^{bc}}{B_{n-1}^{bc}}$ *is*

$$\frac{B_n^{bc}}{B_{n-1}^{bc}} = [33; \underbrace{1,32,}_{3k-2 \text{ times}} 1,33,1,3,1,5,\underbrace{1,168,1,5,1,3,1,5,}_{k-1 \text{ times}}1,169]$$

for $n = 6k$;

$$\frac{B_n^{bc}}{B_{n-1}^{bc}} = [33; \underbrace{1,32,}_{3k-1 \text{ times}} 1,172,1,5,1,3,\underbrace{1,5,1,168,1,5,1,3,}_{k-1 \text{ times}}1,6]$$

for $n = 6k+1$;

$$\frac{B_n^{bc}}{B_{n-1}^{bc}} = [33; \underbrace{1,32,}_{3k-1 \text{ times}} 1,33,\underbrace{1,3,1,5,1,168,1,5,}_{k \text{ times}}1,4]$$

for $n = 6k+2$;

$$\frac{B_n^{bc}}{B_{n-1}^{bc}} = [33; \underbrace{1,32,}_{3k \text{ times}}1,172,1,5,1,3,1,5,\underbrace{1,168,1,5,1,3,1,5,}_{k-1 \text{ times}}1,169]$$

for $n = 6k+3$;

$$\frac{B_n^{bc}}{B_{n-1}^{bc}} = [33; \underbrace{1,32,}_{3k \text{ times}}1,33,1,3,\underbrace{1,5,1,168,1,5,1,3,}_{k \text{ times}}1,6]$$

for $n = 6k+4$ *and*

$$\frac{B_n^{bc}}{B_{n-1}^{bc}} = [33; \underbrace{1,32,}_{3k+1 \text{ times}} 1,172,1,5,1,3,1,5,\underbrace{1,168,1,5,1,3,1,5,}_{k-1 \text{ times}}1,168,1,5,1,4]$$

for $n = 6k+5$*, where* $k \geq 1$ *is an integer. The continued fraction expansion of* $\frac{C_n^{bc}}{C_{n-1}^{bc}}$ *is*

$$\frac{C_n^{bc}}{C_{n-1}^{bc}} = [33; \underbrace{1,32,}_{n-2 \text{ times}} 1,28]$$

for $n \geq 2$ *and continued fraction expansion of* $\frac{R_n^{bc}}{R_{n-1}^{bc}}$ *is*

$$\frac{R_n^{bc}}{R_{n-1}^{bc}} = [33; \underbrace{1,32,}_{\frac{n-4}{2} \text{ times}} 1,35,33, \underbrace{1,32,}_{\frac{n-6}{2} \text{ times}} 1,33]$$

for even $n \geq 6$ *and*

$$\frac{R_n^{bc}}{R_{n-1}^{bc}} = [33; \underbrace{1,32,}_{\frac{n-5}{2} \text{ times}} 1,33,35, \underbrace{1,32,}_{\frac{n-5}{2} \text{ times}} 1,33]$$

for odd $n \geq 5$.

Proof. It can be proved by induction on n. $\square$

2.2. **Circulant Matrix and Spectral Norm.** A circulant matrix (see [2]) is a matrix

$$M=\begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-2} & m_{n-1} \\ m_{n-1} & m_0 & m_1 & \cdots & m_{n-3} & m_{n-2} \\ m_{n-2} & m_{n-1} & m_0 & \cdots & m_{n-4} & m_{n-3} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ m_2 & m_3 & m_4 & \cdots & m_0 & m_1 \\ m_1 & m_2 & m_3 & \cdots & m_{n-1} & m_0 \end{bmatrix},$$

where m_i are constant. In this case the eigenvalues of M are

$$\lambda_j(M)=\sum_{u=0}^{n-1} m_u w^{-ju}, \tag{2.4}$$

where $w=e^{\frac{2\pi i}{n}}, i=\sqrt{-1}, j=0,1,\cdots,n-1$. The spectral norm of a matrix $X=[x_{ij}]_{n\times n}$ is

$$||X||_{spec}=\max_{0\leq j\leq n-1}\{\sqrt{\lambda_j}\},$$

where λ_j are the eigenvalues of X^*X and X^* denotes the conjugate transpose of X.

Theorem 2.2. *Let $M(B_n^{bc}), M(C_n^{bc})$ and $M(R_n^{bc})$ denote the circulant matrices of balcobalancing numbers, Lucas-balcobalancing numbers and balcobalancers, respectively. Then*

(1) The eigenvalues of $M(B_n^{bc}), M(C_n^{bc})$ and $M(R_n^{bc})$ are

$$\begin{aligned}\lambda_j(M(B_n^{bc})) &= \frac{(B_{n-1}^{bc}+2)w^{-j}-B_n^{bc}}{w^{-2j}-34w^{-j}+1}\\ \lambda_j(M(C_n^{bc})) &= \frac{(C_{n-1}^{bc}-5)w^{-j}-C_n^{bc}+1}{w^{-2j}-34w^{-j}+1}\\ \lambda_j(M(R_n^{bc})) &= \frac{(R_{n-1}^{bc}-4)w^{-j}-R_n^{bc}}{w^{-2j}-34w^{-j}+1}\end{aligned}$$

for $j=0,1,2,\cdots,n-1$.

(2) The spectral norms of $M(B_n^{bc}), M(C_n^{bc})$ and $M(R_n^{bc})$ are

$$\begin{aligned}||M(B_n^{bc})||_{spec} &= \frac{33B_{n-1}^{bc}-B_{n-2}^{bc}-8n+6}{32}\\ ||M(C_n^{bc})||_{spec} &= \frac{33C_{n-1}^{bc}-C_{n-2}^{bc}+4}{32}\\ ||M(R_n^{bc})||_{spec} &= \frac{33R_{n-1}^{bc}-R_{n-2}^{bc}-8n+12}{32}.\end{aligned}$$

Proof. **(1)** Recall that $B_n^{bc}=\frac{\alpha^{4n+1}+\beta^{4n+1}}{8}-\frac{1}{4}$ by [19, Theorem 3.6]. So we get from (2.4) that

$$\begin{aligned}\lambda_j(B_n^{bc}) &= \sum_{u=0}^{n-1} B_u^{bc} w^{-ju}\\ &= \sum_{u=0}^{n-1}\left(\frac{\alpha^{4u+1}+\beta^{4u+1}}{8}-\frac{1}{4}\right)w^{-ju}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left[\alpha\sum_{u=0}^{n-1}(\alpha^4 w^{-j})^u + \beta\sum_{u=0}^{n-1}(\beta^4 w^{-j})^u\right] - \frac{1}{4}\sum_{u=0}^{n-1} w^{-ju} \\
&= \frac{1}{8}\left[\frac{\alpha-\alpha^{4n+1}}{1-\alpha^4 w^{-j}} + \frac{\beta-\beta^{4n+1}}{1-\beta^4 w^{-j}}\right] \\
&= \frac{1}{8}\left[\frac{w^{-j}(-\alpha\beta^4+\alpha^{4n+1}\beta^4-\alpha^4\beta+\alpha^4\beta^{4n+1})+\alpha+\beta-\alpha^{4n+1}-\beta^{4n+1}}{w^{-2j}-34w^{-j}+1}\right] \\
&= \frac{1}{8}\left[\frac{w^{-j}(\alpha^{4n-3}+\beta^{4n-3}+14)-(\alpha^{4n+1}+\beta^{4n+1}-2)}{w^{-2j}-34w^{-j}+1}\right] \\
&= \frac{w^{-j}(\frac{\alpha^{4n-3}+\beta^{4n-3}-2+16}{8}) - \frac{\alpha^{4n+1}+\beta^{4n+1}-2}{8}}{w^{-2j}-34w^{-j}+1} \\
&= \frac{(B^{bc}_{n-1}+2)w^{-j} - B^{bc}_n}{w^{-2j}-34w^{-j}+1}.
\end{aligned}
$$

(2) For the circulant matrix

$$
B^{bc}_n = \begin{bmatrix}
B^{bc}_0 & B^{bc}_1 & B^{bc}_2 & \cdots & B^{bc}_{n-2} & B^{bc}_{n-1} \\
B^{bc}_{n-1} & B^{bc}_0 & B^{bc}_1 & \cdots & B^{bc}_{n-3} & B^{bc}_{n-2} \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
B^{bc}_2 & B^{bc}_3 & B^{bc}_4 & \cdots & B^{bc}_0 & B^{bc}_1 \\
B^{bc}_1 & B^{bc}_2 & B^{bc}_3 & \cdots & B^{bc}_{n-1} & B^{bc}_0
\end{bmatrix},
$$

for balcobalancing numbers, we have

$$
(B^{bc}_n)^* B^{bc}_n = \begin{bmatrix}
B^{bc}_{11} & B^{bc}_{12} & \cdots & B^{bc}_{1(n-1)} & B^{bc}_{1n} \\
B^{bc}_{21} & B^{bc}_{22} & \cdots & B^{bc}_{2(n-1)} & B^{bc}_{2n} \\
. & . & & . & . \\
. & . & \cdots & . & . \\
. & . & & . & . \\
B^{bc}_{(n-1)1} & B^{bc}_{(n-1)2} & \cdots & B^{bc}_{(n-1)(n-1)} & B^{bc}_{(n-1)n} \\
B^{bc}_{n1} & B^{bc}_{n2} & \cdots & B^{bc}_{n(n-1)} & B^{bc}_{nn}
\end{bmatrix},
$$

where

$$
\begin{aligned}
B^{bc}_{11} &= (B^{bc}_0)^2 + (B^{bc}_{n-1})^2 + \cdots + (B^{bc}_2)^2 + (B^{bc}_1)^2 \\
B^{bc}_{12} &= B^{bc}_0 B^{bc}_1 + B^{bc}_{n-1} B^{bc}_0 + \cdots + B^{bc}_2 B^{bc}_3 + B^{bc}_1 B^{bc}_2 \\
&\cdots \\
B^{bc}_{1(n-1)} &= B^{bc}_0 B^{bc}_{n-2} + B^{bc}_{n-1} B^{bc}_{n-3} + \cdots + B^{bc}_2 B^{bc}_0 + B^{bc}_1 B^{bc}_{n-1} \\
B^{bc}_{1n} &= B^{bc}_0 B^{bc}_{n-1} + B^{bc}_{n-1} B^{bc}_{n-2} + \cdots + B^{bc}_2 B^{bc}_1 + B^{bc}_1 B^{bc}_0 \\
B^{bc}_{21} &= B^{bc}_1 B^{bc}_0 + B^{bc}_0 B^{bc}_{n-1} + \cdots + B^{bc}_3 B^{bc}_2 + B^{bc}_2 B^{bc}_1 \\
B^{bc}_{22} &= (B^{bc}_1)^2 + (B^{bc}_0)^2 + \cdots + (B^{bc}_3)^2 + (B^{bc}_2)^2 \\
&\cdots \\
B^{bc}_{2(n-1)} &= B^{bc}_1 B^{bc}_{n-2} + B^{bc}_0 B^{bc}_{n-3} + \cdots + B^{bc}_3 B^{bc}_0 + B^{bc}_2 B^{bc}_{n-1} \\
B^{bc}_{2n} &= B^{bc}_1 B^{bc}_{n-1} + B^{bc}_0 B^{bc}_{n-2} + \cdots + B^{bc}_3 B^{bc}_1 + B^{bc}_2 B^{bc}_0
\end{aligned}
$$

$$\begin{aligned}
&\cdots\\
B^{bc}_{n1} &= B^{bc}_{n-1}B^{bc}_0 + B^{bc}_{n-2}B^{bc}_{n-1} + \cdots + B^{bc}_1B^{bc}_2 + B^{bc}_0B^{bc}_1\\
B^{bc}_{n2} &= B^{bc}_{n-1}B^{bc}_1 + B^{bc}_{n-2}B^{bc}_0 + \cdots + B^{bc}_1B^{bc}_3 + B^{bc}_0B^{bc}_2\\
&\cdots\\
B^{bc}_{n(n-1)} &= B^{bc}_{n-1}B^{bc}_{n-2} + B^{bc}_{n-2}B^{bc}_{n-3} + \cdots + B^{bc}_1B^{bc}_0 + B^{bc}_0B^{bc}_{n-1}\\
B^{bc}_{nn} &= (B^{bc}_{n-1})^2 + (B^{bc}_{n-2})^2 + \cdots + (B^{bc}_1)^2 + (B^{bc}_0)^2.
\end{aligned}$$

The eigenvalues of $(B^{bc}_n)^*B^{bc}_n$ are $\lambda_0, \lambda_1, \cdots, \lambda_{n-1}$. Here λ_0 is the maximum and is

$$\begin{aligned}
\lambda_0 &= (B^{bc}_0)^2 + (B^{bc}_1)^2 + \cdots + (B^{bc}_{n-2})^2 + (B^{bc}_{n-1})^2\\
&\quad + 2\begin{bmatrix} B^{bc}_0(B^{bc}_1 + B^{bc}_2 + \cdots + B^{bc}_{n-2} + B^{bc}_{n-1})\\ + B^{bc}_1(B^{bc}_2 + \cdots + B^{bc}_{n-2} + B^{bc}_{n-1})\\ + \cdots\\ + B^{bc}_{n-2}B^{bc}_{n-1}\end{bmatrix}\\
&= (B^{bc}_0 + B^{bc}_1 + \cdots + B^{bc}_{n-1})^2.
\end{aligned}$$

Thus the spectral norm of $M(B^{bc}_n)$ is

$$||M(B^{bc}_n)||_{spec} = \sqrt{\lambda_0} = B^{bc}_0 + B^{bc}_1 + \cdots + B^{bc}_{n-1}.$$

Since $\sum\limits_{i=1}^{n} B^{bc}_i = \frac{33B^{bc}_n - B^{bc}_{n-1} - 8n-2}{32}$ by [19, Theorem 6.18], we conclude that the spectral norm of $M(B^{bc}_n)$ is

$$||M(B^{bc}_n)||_{spec} = \frac{33B^{bc}_{n-1} - B^{bc}_{n-2} - 8n + 6}{32}$$

as we wanted. □

2.3. **Pythagorean Triples.** Notice that a Pythagorean triple consists of three positive integers a, b, c such that $a^2 + b^2 = c^2$ and commonly written (a, b, c). For instance, for Pell numbers P_n, it is known that

$$(2P_nP_{n+1}, P^2_{n+1} - P^2_n, P^2_{n+1} + P^2_n)$$

is a Pythagorean triple. Also for balancing numbers, it was proved in [3, Theorem 2.14] that

$$(B_{n+1} - b_{n+1}, B_{n+1} - b_{n+1} - 1, 2b_{n+1} + 1)$$

is a Pythagorean triple. Similarly we can give the following result.

Theorem 2.3. *For balcobalancing numbers, balcobalancers and Lucas-balcobalancing numbers,*

(1) $(C^{bc}_n - 2R^{bc}_n, 2B^{bc}_n, C^{bc}_n)$ *is a Pythagorean triple.*
(2) $(2B^{bc}_{n+1} - 4R^{bc}_{n+1}, 2B^{bc}_{n+1} - 4R^{bc}_{n+1} - 1, 6R^{bc}_{n+1} - 2B^{bc}_{n+1} + 1)$ *is a Pythagorean triple.*
(3) $(4C^{bc}_n(B^{bc}_n - R^{bc}_n), (C^{bc}_n)^2 - 4(B^{bc}_n - R^{bc}_n)^2, (C^{bc}_n)^2 + 4(B^{bc}_n - R^{bc}_n)^2)$ *is a Pythagorean triple.*

Proof. **(1)** Since $B^{bc}_n = \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}$, $C^{bc}_n = \frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}}$ and $R^{bc}_n = \frac{\alpha^{4n}+\beta^{4n}}{8} - \frac{1}{4}$ by [19, Theorem 3.6], we deduce that

$$\begin{aligned}
&(C^{bc}_n - 2R^{bc}_n)^2 + (2B^{bc}_n)^2\\
&= \left(\frac{\alpha^{4n+1} - \beta^{4n+1}}{2\sqrt{2}} - 2(\frac{\alpha^{4n} + \beta^{4n} - 2}{8})\right)^2 + \left(2(\frac{\alpha^{4n+1} + \beta^{4n+1} - 2}{8})\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= (\frac{\alpha^{4n+1}+\beta^{4n+1}+2}{4})^2 + (\frac{\alpha^{4n+1}+\beta^{4n+1}-2}{4})^2 \\
&= \frac{\alpha^{8n+2}+\beta^{8n+2}+2}{8} \\
&= (\frac{\alpha^{4n+1}-\beta^{4n+1}}{2\sqrt{2}})^2 \\
&= (C_n^{bc})^2.
\end{aligned}
$$

The others can be proved similarly. □

2.4. **Characteristic Polynomials and Eigenvalues.** We proved in [19, Theorem 3.9], that the n^{th} power of M^{bc} is

$$
(M^{bc})^n = \begin{bmatrix} k_{n+2} & l_n & k_{n+1} \\ k_{n+1} & l_{n-1} & k_n \\ k_n & l_{n-2} & k_{n-1} \end{bmatrix}
$$

for every $n \geq 2$, and the n^{th} power of N^{bc} is

$$
(N^{bc})^n = (-1)^n \begin{cases} \begin{bmatrix} k_{n+2}-k_{n+1} & k_n - k_{n+1} \\ -k_n + k_{n+1} & -k_n + k_{n-1} \end{bmatrix} & \text{for even } n \geq 2 \\ \begin{bmatrix} k_{n+1}-k_{n+2} & k_{n+1} - k_n \\ -k_{n+1} + k_n & -k_{n-1} + k_n \end{bmatrix} & \text{for odd } n \geq 1, \end{cases}
$$

where M^{bc} is the companion matrix for balcobalancing numbers and balcobalancers, N^{bc} is the companion matrix for Lucas-balcobalancing numbers, k_n and l_n are integer sequences defined by $k_n = \frac{-8B_{2n}+3C_{2n}-3}{96}$ and $l_n = \frac{-288B_{2n}-102C_{2n}+102}{96}$ for $n \geq 0$.

For characteristic polynomial and eigenvalues of $(M^{bc})^n$ and $(N^{bc})^n$, we can give the following theorem.

Theorem 2.4. *The characteristic polynomial of $(M^{bc})^n$ is*

$$P_\lambda((M^{bc})^n) = -\lambda^3 + (2C_{2n}+1)\lambda^2 - (2C_{2n}+1)\lambda + 1$$

and the eigenvalues of $(M^{bc})^n$ are

$$\lambda_0 = 1,\ \lambda_1 = C_{2n} + 2\sqrt{2}B_{2n} \ \textit{and}\ \lambda_2 = C_{2n} - 2\sqrt{2}B_{2n}.$$

The characteristic polynomial of $(N^{bc})^n$ is

$$P_\lambda((N^{bc})^n) = \lambda^2 - 2C_{2n}\lambda + 1$$

and the eigenvalues of $(N^{bc})^n$ are

$$\lambda_0 = C_{2n} + 2\sqrt{2}B_{2n} \ \textit{and}\ \lambda_1 = C_{2n} - 2\sqrt{2}B_{2n}.$$

Proof. The characteristic polynomial of $(M^{bc})^n$ is

$$P_\lambda((M^{bc})^n) = \det((M^{bc})^n - \lambda I_3)$$

$$
\begin{aligned}
&= \det\left(\begin{bmatrix} k_{n+2} & l_n & k_{n+1} \\ k_{n+1} & l_{n-1} & k_n \\ k_n & l_{n-2} & k_{n-1} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
&= \begin{vmatrix} k_{n+2}-\lambda & l_n & k_{n+1} \\ k_{n+1} & l_{n-1}-\lambda & k_n \\ k_n & l_{n-2} & k_{n-1}-\lambda \end{vmatrix} \\
&= -\lambda^3 + (k_{n+2}+l_{n-1}+k_{n-1})\lambda^2 + (-k_{n+2}l_{n-1} - k_{n+2}k_{n-1} - l_{n-1}k_{n-1} \\
&\quad + k_n l_{n-2} + k_{n+1}l_n + k_{n+1}k_n)\lambda + k_{n+2}l_{n-1}k_{n-1} - k_{n+2}k_n l_{n-2} + l_{n-2}k_{n+1}^2 \\
&\quad - k_{n+1}l_n k_{n-1} + l_n k_n^2 - k_n k_{n+1} l_{n-1} \\
&= -\lambda^3 + (2C_{2n}+1)\lambda^2 - (2C_{2n}+1)\lambda + 1
\end{aligned}
$$

since $k_{n+2}+l_{n-1}+k_{n-1} = 2C_{2n}+1$, $-k_{n+2}l_{n-1} - k_{n+2}k_{n-1} - l_{n-1}k_{n-1} + k_n l_{n-2} + k_{n+1}l_n + k_{n+1}k_n = -2C_{2n} - 1$ and $k_{n+2}l_{n-1}k_{n-1} - k_{n+2}k_n l_{n-2} + l_{n-2}k_{n+1}^2 - k_{n+1}l_n k_{n-1} + l_n k_n^2 - k_n k_{n+1} l_{n-1} = 1$. Note that

$$P_\lambda((M^{bc})^n) = -(\lambda - 1)(\lambda^2 - 2C_{2n}\lambda + 1).$$

So the roots of $P_\lambda((M^{bc})^n)$ are $\lambda_0 = 1$ and

$$\lambda_{1,2} = \frac{2C_{2n} \pm \sqrt{(2C_{2n})^2 - 4}}{2} = C_{2n} \pm \sqrt{C_{2n}^2 - 1} = C_{2n} \pm 2\sqrt{2}B_{2n}.$$

The other case can be proved similarly. □

2.5. **Cassini and Catalan Identities.** Recall that the Cassini identity for Fibonacci numbers F_n is

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$$

for $n \geq 1$ and the Catalan identity for Fibonacci numbers F_n is

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2$$

for $n \geq r \geq 1$. Similarly for all balcobalancing numbers, we can give the following result.

Theorem 2.5. *The Cassini identities for all balcobalancing numbers are*

$$
\begin{aligned}
(B_n^{bc})^2 - B_{n+1}^{bc}B_{n-1}^{bc} &= 8B_n^{bc} + 20 \\
(C_n^{bc})^2 - C_{n+1}^{bc}C_{n-1}^{bc} &= -144 \\
(R_n^{bc})^2 - R_{n+1}^{bc}R_{n-1}^{bc} &= 8R_n^{bc} - 16
\end{aligned}
$$

for $n \geq 1$ and the Catalan identities for all balcobalancing numbers are

$$
\begin{aligned}
(B_n^{bc})^2 - B_{n-r}^{bc}B_{n+r}^{bc} &= (\frac{5B_r^{bc} - B_{r-1}^{bc} + 4}{12})^2 + B_n^{bc}(\frac{5B_r^{bc} - B_{r-1}^{bc} - 2}{6}) - \frac{1}{4} \\
(C_n^{bc})^2 - C_{n-r}^{bc}C_{n+r}^{bc} &= -(\frac{5C_r^{bc} - C_{r-1}^{bc}}{12})^2 \\
(R_n^{bc})^2 - R_{n-r}^{bc}R_{n+r}^{bc} &= R_r^{bc}(2R_n^{bc} - R_r^{bc})
\end{aligned}
$$

for $n \geq r \geq 1$.

Proof. Notice that $B_n^{bc} = \frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}$ by [19, Theorem 3.6]. So we easily get

$$\begin{aligned}
(B_n^{bc})^2 - B_{n+1}^{bc}B_{n-1}^{bc} &= \left(\frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}\right)^2 \\
&\quad - \left(\frac{\alpha^{4n+5}+\beta^{4n+5}}{8} - \frac{1}{4}\right)\left(\frac{\alpha^{4n-3}+\beta^{4n-3}}{8} - \frac{1}{4}\right) \\
&= 8\left(\frac{\alpha^{4n+1}+\beta^{4n+1}}{8} - \frac{1}{4}\right) + 20 \\
&= 8B_n^{bc} + 20.
\end{aligned}$$

The other cases can be proved similarly. □

2.6. **Cross-Ratio.** The cross-ratio of a quadruple of distinct points on the real line with coordinates z_1, z_2, z_3, z_4 is given by

$$(z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}. \tag{2.5}$$

It is known that the cross-ratio for Fibonacci numbers F_n is

$$(F_{n+1}, F_{n+2}; F_{n+3}, F_{n+4}) = \frac{F_{n+3}}{2F_{n+1}}.$$

Similarly we can give the following result.

Theorem 2.6. *The cross-ratios for all balcobalancing numbers are*

$$\begin{aligned}
(B_{n+1}^{bc}, B_{n+2}^{bc}; B_{n+3}^{bc}, B_{n+4}^{bc}) &= \frac{288B_{n+2}^{bc}B_{n+3}^{bc} + 72B_{n+2}^{bc} + 72B_{n+3}^{bc} + 324}{280B_{n+2}^{bc}B_{n+3}^{bc} + 70B_{n+2}^{bc} + 70B_{n+3}^{bc} + 175} \\
(C_{n+1}^{bc}, C_{n+2}^{bc}; C_{n+3}^{bc}, C_{n+4}^{bc}) &= \frac{72C_{n+2}^{bc}C_{n+3}^{bc} - 612}{70C_{n+2}^{bc}C_{n+3}^{bc} - 315} \\
(R_{n+1}^{bc}, R_{n+2}^{bc}; R_{n+3}^{bc}, R_{n+4}^{bc}) &= \frac{144R_{n+2}^{bc}R_{n+3}^{bc} + 36R_{n+2}^{bc} + 36R_{n+3}^{bc} - 144}{140R_{n+2}^{bc}R_{n+3}^{bc} + 35R_{n+2}^{bc} + 35R_{n+3}^{bc} - 70}.
\end{aligned}$$

Proof. Recall that $B_n^{bc} = \frac{\alpha^{4n+1}+\beta^{4n+1}-2}{8}$. So we get from (2.5) that

$$\begin{aligned}
&(B_{n+1}^{bc}, B_{n+2}^{bc}; B_{n+3}^{bc}, B_{n+4}^{bc}) \\
&= \frac{(B_{n+3}^{bc} - B_{n+1}^{bc})(B_{n+4}^{bc} - B_{n+2}^{bc})}{(B_{n+3}^{bc} - B_{n+2}^{bc})(B_{n+4}^{bc} - B_{n+1}^{bc})} \\
&= \frac{(\frac{\alpha^{4n+13}+\beta^{4n+13}-2}{8} - \frac{\alpha^{4n+5}+\beta^{4n+5}-2}{8})(\frac{\alpha^{4n+17}+\beta^{4n+17}-2}{8} - \frac{\alpha^{4n+9}+\beta^{4n+9}-2}{8})}{(\frac{\alpha^{4n+13}+\beta^{4n+13}-2}{8} - \frac{\alpha^{4n+9}+\beta^{4n+9}-2}{8})(\frac{\alpha^{4n+17}+\beta^{4n+17}-2}{8} - \frac{\alpha^{4n+5}+\beta^{4n+5}-2}{8})} \\
&= \frac{[\alpha^{4n+5}(\alpha^8 - 1) + \beta^{4n+5}(\beta^8 - 1)][\alpha^{4n+9}(\alpha^8 - 1) + \beta^{4n+9}(\beta^8 - 1)]}{[\alpha^{4n+9}(\alpha^4 - 1) + \beta^{4n+9}(\beta^4 - 1)][\alpha^{4n+5}(\alpha^{12} - 1) + \beta^{4n+5}(\beta^{12} - 1)]} \\
&= \frac{[24\sqrt{2}(\alpha^{4n+9} - \beta^{4n+9})][24\sqrt{2}(\alpha^{4n+13} - \beta^{4n+13})]}{[4\sqrt{2}(\alpha^{4n+11} - \beta^{4n+11})][140\sqrt{2}(\alpha^{4n+11} - \beta^{4n+11})]} \\
&= \frac{36(\alpha^{8n+22} + \beta^{8n+22} + 34)}{35(\alpha^{8n+22} + \beta^{8n+22} + 2)}.
\end{aligned} \tag{2.6}$$

Here we notice that

$$\begin{aligned}\alpha^{8n+22}+\beta^{8n+22} &= \alpha^{4n+9}\alpha^{4n+13}+\alpha^{4n+9}\beta^{4n+13}+\alpha^{4n+13}\beta^{4n+9}+\beta^{4n+9}\beta^{4n+13}\\ &\quad -2(\alpha^{4n+9}+\beta^{4n+9}+\alpha^{4n+13}+\beta^{4n+13})\\ &\quad +2(\alpha^{4n+9}+\beta^{4n+9}+\alpha^{4n+13}+\beta^{4n+13})+34\\ &= (\alpha^{4n+9}+\beta^{4n+9}-2)(\alpha^{4n+13}+\beta^{4n+13}-2)\\ &\quad +2(\alpha^{4n+9}+\beta^{4n+9}+\alpha^{4n+13}+\beta^{4n+13})+30\\ &= 64(\frac{\alpha^{4n+9}+\beta^{4n+9}-2}{8})(\frac{\alpha^{4n+13}+\beta^{4n+13}-2}{8})\\ &\quad +16(\frac{\alpha^{4n+9}+\beta^{4n+9}-2}{8}+\frac{\alpha^{4n+13}+\beta^{4n+13}-2}{8})+38\\ &= 64B^{bc}_{n+2}B^{bc}_{n+3}+16B^{bc}_{n+2}+16B^{bc}_{n+3}+38.\end{aligned}$$

Thus from (2.6), we get

$$(B^{bc}_{n+1},B^{bc}_{n+2};B^{bc}_{n+3},B^{bc}_{n+4})=\frac{288B^{bc}_{n+2}B^{bc}_{n+3}+72B^{bc}_{n+2}+72B^{bc}_{n+3}+324}{280B^{bc}_{n+2}B^{bc}_{n+3}+70B^{bc}_{n+2}+70B^{bc}_{n+3}+175}.$$

The others can be proved similarly. □

2.7. **Heisenberg Group.** Let x, y, z be real numbers. Then the set of matrices

$$\begin{bmatrix} 1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{bmatrix}$$

is a group under matrix multiplication. This group known as the Heisenberg group which is denoted by $H_3(x,y,z)$.

For balcobalancing numbers B^{bc}_n, C^{bc}_n and R^{bc}_n, we let

$$H^{bc}_3=H^{bc}_3(B^{bc}_i,C^{bc}_i,R^{bc}_i)=\begin{bmatrix} 1 & B^{bc}_i & C^{bc}_i\\ 0 & 1 & R^{bc}_i\\ 0 & 0 & 1\end{bmatrix}$$

for $i=1,2,\cdots$. Then we can give the following theorem.

Theorem 2.7. *The n^{th} power of H^{bc}_3 is*

$$(H^{bc}_3)^n=\begin{bmatrix} 1 & nB^{bc}_i & nC^{bc}_i+\frac{n(n-1)}{2}B^{bc}_iR^{bc}_i\\ 0 & 1 & nR^{bc}_i\\ 0 & 0 & 1\end{bmatrix}$$

for $n, i\geq 1$.

Proof. We prove it by induction on n. Let $n=1$. Then

$$(H^{bc}_3)^1=\begin{bmatrix} 1 & B^{bc}_i & C^{bc}_i\\ 0 & 1 & R^{bc}_i\\ 0 & 0 & 1\end{bmatrix}=H^{bc}_3.$$

So it is true for $n=1$. Let us assume that it is satisfied for $n-1$, that is,

$$(H^{bc}_3)^{n-1}=\begin{bmatrix} 1 & (n-1)B^{bc}_i & (n-1)C^{bc}_i+\frac{(n-1)(n-2)}{2}B^{bc}_iR^{bc}_i\\ 0 & 1 & (n-1)R^{bc}_i\\ 0 & 0 & 1\end{bmatrix}.$$

Then we easily deduce that

$$\begin{aligned}
(H_3^{bc})^n &= H_3^{bc}(H_3^{bc})^{n-1} \\
&= \begin{bmatrix} 1 & B_i^{bc} & C_i^{bc} \\ 0 & 1 & R_i^{bc} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (n-1)B_i^{bc} & (n-1)C_i^{bc} + \frac{(n-1)(n-2)}{2} B_i^{bc} R_i^{bc} \\ 0 & 1 & (n-1)R_i^{bc} \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & (n-1)B_i^{bc} + B_i^{bc} & (n-1)C_i^{bc} + \frac{(n-1)(n-2)}{2} B_i^{bc} R_i^{bc} + (n-1)B_i^{bc} R_i^{bc} + C_i^{bc} \\ 0 & 1 & (n-1)R_i^{bc} + R_i^{bc} \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & nB_i^{bc} & nC_i^{bc} + \frac{n(n-1)}{2} B_i^{bc} R_i^{bc} \\ 0 & 1 & nR_i^{bc} \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

So it is true for every n. □

3. Conclusion.

In [19], we defined three integer sequences called balcobalancing numbers, balcobalancers and Lucas-balcobalancing numbers. We said that a positive integer n is called a balcobalancing number if the Diophantine equation

$$1 + 2 + \cdots + (n-1) + 1 + 2 + \cdots + (n-1) + n = 2[(n+1) + (n+2) + \cdots + (n+r)]$$

(which is the sum of (1.1) and (1.2)) verified for some positive integer r which is called balcobalancer. From above equation, we get

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 4n + 1}}{2}. \tag{3.7}$$

Let B_n^{bc} denote the n^{th} balcobalancing number. Then from (3.7), B_n^{bc} is a balcobalancing number if and only if $8(B_n^{bc})^2 + 4B_n^{bc} + 1$ is a perfect square. Thus

$$C_n^{bc} = \sqrt{8(B_n^{bc})^2 + 4B_n^{bc} + 1}$$

is an integer which is called the n^{th} Lucas-balcobancing number. We proved in [19, Theorem 2.2] that the general terms of B_n^{bc}, C_n^{bc} and R_n^{bc} (which is the n^{th} balcobalancer) are

$$B_n^{bc} = \frac{c_{2n+1} - 1}{4}, \ C_n^{bc} = 2b_{2n+1} + 1 \ \text{ and } \ R_n^{bc} = \frac{4b_{2n+1} - c_{2n+1} + 1}{4}$$

for $n \geq 1$. Further we deduced some algebraic relations on binet formulas, recurrence relations, companion matrices, relationship with Pell, Pell-Lucas, triangular and square triangular numbers and sums of them including sums of balancing numbers and Pell numbers.

In the present paper, we again consider the balcobalancing numbers and derived some new algebraic results on continued fraction expansion of the ratio of the two consecutive balcobalancing numbers, circulant matrices and spectral norms, Pythagorean triples, characteristic polynomials and eigenvalues of the n^{th} power of the companion matrices, Cassini and Catalan identities, cross-ratios and Heisenberg groups.

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