Fold thickness of some classes of graphs

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ABSTRACT. A 1-fold of $G$ is the graph $G'$ obtained from a graph $G$ by identifying two nonadjacent vertices in $G$ having at least one common neighbor and reducing the resulting multiple edges to simple edges. A sequence of graphs $G = G_0, G_1, G_2, \ldots, G_k$, where $G_{i+1}$ is a 1-fold of $G_i$ for $i = 0, 1, 2, \ldots, k - 1$ is called a uniform $k$-folding if all the graphs in the sequence are singular or all of them are nonsingular. The largest $k$ for which there exists a uniform $k$-folding of $G$ is called fold thickness of $G$ and it was first introduced in [Campeña, F. J. H.; Gervacio, S. V. On the fold thickness of graphs. Arab. J. Math. (Springer) 9 (2020), no. 2, 345–355]. In this paper, we determine fold thickness of $K_n \circ K_m$, $K_n + K_m$, cone graph and tadpole graph.

1. INTRODUCTION

The motivation for the concept of graph folding as defined by Gervacio et al. [7] is from the situation of folding a meter stick. Let a finite number of unit bars be joined together at ends in such a way that they are free to turn. There are some meter sticks with this structure as in Fig.1. This meter stick can be considered as a physical model of the path $P_n$ on $n$ vertices and can be folded to become a physical model of the complete graph $K_2$. It is a natural question to find the minimum fold of a graph, so the concept of fold thickness was introduced as follows.

Definition 1.1. [2] Let $G$ be a graph that is not isomorphic to a complete graph. If $x$ and $y$ are nonadjacent vertices of $G$ that have atleast one common neighbor, then identify $x$ and $y$ and reduce any resulting multiple edges to simple edges to form a new graph, $G'$. We call $G'$, a 1-fold of $G$.

Definition 1.2. [2] Consider a sequence of graphs $G = G_0, G_1, G_2, \ldots, G_k$ where $G_{i+1}$ is a 1-fold of $G_i$ for $i = 0, 1, 2, \ldots, k - 1$. This sequence is called a $k$-folding of $G = G_0$.

Let $A(G_i)$ be the adjacency matrix of $G_i$. A graph $G_i$ is singular if $A(G_i)$ is singular and nonsingular if $A(G_i)$ is nonsingular.

Definition 1.3. [2] A graph $G$ is said to have a uniform $k$-folding if there is a $k$-folding in which all graphs in the sequence are singular or all of them are nonsingular. The largest integer $k$ for which there exists a uniform $k$-folding of $G$ is called fold thickness of $G$, and is denoted by fold$(G)$.

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If \( G = G_0, G_1, G_2, \ldots, G_k \) is a \( k \)-folding of \( G \), the graph \( G_k \) is referred as a \( k \)-fold of \( G \). The fold thickness of a graph was first defined by F. J. H. Campeña and S.V. Gervacio in [2] and evaluated fold thickness of some special classes of graphs such as wheel graph, cycle graph, bipartite graphs etc.

2. Preliminary results

In this paper \( K_n, P_n \) and \( C_n \) denotes the complete graph, path and cycle graph on \( n \) vertices respectively. The empty graph \( \overline{K}_n \) is the graph with \( n \) vertices and zero edges or it is the complement of complete graph, \( K_n \). \( V(G) \) and \( E(G) \) denotes the vertex set and edge set respectively of a graph \( G \). \( \chi(G) \) denotes the vertex chromatic number of \( G \). For any vertex \( x \) in a graph \( G \), \( N(x) \) is the set of all vertices \( y \) in \( G \) that are adjacent to \( x \) and is called the neighbor set of \( x \). Let \( G_1, G_2, \ldots, G_n \) be the components of \( G \). Label the vertices of \( G \) by labelling the vertices of \( G_1 \), then the vertices of \( G_2 \) and so on. The adjacency matrix of \( G \), \( A(G) \) is a block diagonal matrix,

\[
A(G) = \begin{bmatrix}
A(G_1) & 0 & \cdots & 0 \\
0 & A(G_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A(G_n)
\end{bmatrix}
\]

Thus, the determinant of the adjacency matrix, \( \det A(G) = \prod_{i=1}^{n} \det A(G_i) \).

The corona product [4] \( G \odot H \) of two graphs \( G \) and \( H \) is defined as the graph obtained by taking one copy of \( G \) and \( |V(G)| \) copies of \( H \) and joining by an edge each vertex from the \( i \)-th copy of \( H \) with the \( i \)-th vertex of \( G \).

The sum of two vertex disjoint graphs \( G \) and \( H \) denoted by \( G+H \) is the graph consisting of \( G \) and \( H \) and all edges of the form \( xy \), where \( x \) is a vertex of \( G \) and \( y \) is a vertex of \( H \).

An \( m \)-gonal \( n \)-cone graph, \( C_{m,n} \) is the graph join \( C_m + \overline{K}_n \), where \( C_m \) is a cycle graph and \( \overline{K}_n \) is an empty graph (the graph complement of the complete graph \( K_n \)).

The \((m, n)\)-tadpole graph, also called a dragon graph or kite graph is the graph obtained by joining a cycle \( C_m \) to a path \( P_n \) with a bridge.

**Theorem 2.1.** [3] Let \( G \) be a simple connected graph. The smallest complete graph that \( G \) folds into is the complete graph with order \( \chi(G) \), where \( \chi(G) \) denotes the chromatic number of \( G \).

Thus, a maximum folding of a graph \( G \) on \( n \) vertices or simply a max fold of \( G \) is defined to be a \( k \)-folding of \( G \), where \( k = n - \chi(G) \).

**Theorem 2.2.** [6] If \( x \) and \( y \) are vertices in a graph \( G \) such that \( N(x) = N(y) \), then \( G \) is singular.

**Theorem 2.3.** [6] For each \( n \geq 1 \), \( \det A(K_n) = (-1)^{n-1}(n-1) \).

**Theorem 2.4.** [6] Let \( x \) and \( y \) be vertices in a graph \( G \) such that \( N(x) \subseteq N(y) \). If \( G' \) is the graph obtained from \( G \) by deleting all the edges of the form \( yz \), where \( z \) is a neighbor of \( x \), then \( \det A(G) = \det A(G') \).

The following theorem gives an upper bound for the fold thickness of graphs.

**Theorem 2.5.** [2] For any connected graph \( G \) of order \( n \),

\[
\text{fold}(G) \leq \begin{cases} 
\chi(G) & \text{if } G \text{ is nonsingular,} \\
\chi(G) - 1 & \text{if } G \text{ is singular.}
\end{cases}
\]
Remark 2.1. In view of the above theorem, if there exists a uniform \( k \)-folding of a connected graph \( G \) where \( k \) is equal to the upper bound in the theorem, then \( k \) must be the fold thickness of the graph. This observation will be used to obtain the fold thickness of most of the graphs.

Theorem 2.6. [2] For each integer \( n \geq 1 \),

\[
\det A(P_n) = \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
(-1)^{n/2} & \text{if } n \text{ is even}.
\end{cases}
\]

Theorem 2.7. [2] For each integer \( n \geq 3 \),

\[
\det A(C_n) = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{4}, \\
2 & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\
-4 & \text{if } n \equiv 2 \pmod{4}.
\end{cases}
\]

Theorem 2.8. [2] The path \( P_n \) has fold thickness given by,

\[
\text{fold} (P_n) = \begin{cases} 
0 & \text{if } n \text{ is even} \\
\max\{0, n-3\} & \text{if } n \text{ is odd}.
\end{cases}
\]

Theorem 2.9. [2] The cycle \( C_n \), has fold thickness given by

\[
\text{fold} (C_n) = \begin{cases} 
0 & \text{if } n \equiv 2 \pmod{4}, \\
n-3 & \text{otherwise}.
\end{cases}
\]

3. Fold Thickness of Some Classes of Graphs

3.1. Corona Product, \( K_n \odot \overline{K_m} \). In this section we evaluate the fold thickness of corona product, \( K_n \odot \overline{K_m} \) of complete graph \( K_n \) and an empty graph \( \overline{K_m} \), \( m \geq 2 \). The vertices of the graph \( K_n \odot \overline{K_m} \) is labelled as follows: let \( v_1, v_2, \ldots v_n \) be the vertices of \( K_n \) and let \( u_{i1}, u_{i2}, \ldots u_{im} \) be the pendant vertices adjacent to the \( i \)-th vertex \( v_i \) of \( K_n \) for \( i = 1, 2, \ldots n \).

Theorem 3.10. If \( m \geq 2 \), then the fold thickness of \( K_n \odot \overline{K_m} \) is given by,

\[
\text{fold} (K_n \odot \overline{K_m}) = mn - 2.
\]

Proof. The graph \( K_n \odot \overline{K_m}, m \geq 2 \) is singular, since the vertices \( u_{ij} \) and \( u_{ik} \), where \( i \in \{1, 2, \ldots n\}, j, k \in \{1, 2, \ldots m\} \) has common neighbor \( v_i \). Therefore, by Theorem 2.5,

\[
\text{fold} (K_n \odot \overline{K_m}) \leq (m + 1)n - \chi(K_n \odot \overline{K_m}) - 1 = (m + 1)n - \chi(K_n) - 1 = mn - 1.
\]

For \( i = 1, 2, \ldots n - 1 \), first identify the pendant vertices \( u_{i1}, u_{i2}, \ldots u_{im} \) to a single vertex and then identify it with an eligible vertex of \( K_n \). Thus, a uniform \( m(n - 1) \)-folding \( G_0 = K_n \odot \overline{K_m}, G_1, \ldots G_{m(n-1)} \) is obtained in which every graph in the sequence is singular and \( G_{m(n-1)} \) is the graph \( K_n \) plus \( m - 1 \) pendant vertices \( u_{n1}, u_{n2}, \ldots u_{nm} \) adjacent to the vertex \( v_n \). Next, identify the vertices \( u_{n2}, u_{n3}, \ldots u_{nm} \) of \( G_{m(n-1)} \) one by one to obtain a graph \( G' \) which is \( K_n \) plus two pendant vertices adjacent to the vertex \( v_n \). Thus, a uniform \( (m - 2) \)-folding of \( G_{m(n-1)} \) is obtained in which each graph is singular. If the two pendant vertices of \( G' \) are identified, then we obtain a nonsingular graph which is \( K_n \) plus one pendant vertex adjacent to one of its vertices. Thus, \( \text{fold} (K_n \odot \overline{K_m}) = m(n - 1) + m - 2 = mn - 2 \). \( \square \)
3.2. Sum graph, \(K_n + \overline{K_m}\). In this section, we evaluate the fold thickness of sum of \(K_n\) and \(\overline{K_m}\). Let \(u_1, u_2, \ldots, u_n\) be the vertices of \(K_n\) and \(v_1, v_2, \ldots, v_m\) be the vertices of \(\overline{K_m}\).

**Theorem 3.11.** If \(m \geq 2\), then the fold thickness of \(K_n + \overline{K_m}\) is given by,

\[
\text{fold}(K_n + \overline{K_m}) = m - 2.
\]

**Proof.** The graph \(K_n + \overline{K_m}\), \(m \geq 2\) is singular since, for any two vertices \(x\) and \(y\) in \(V(\overline{K_m})\), \(N(x) = N(y) = V(K_n)\). Note that \(\chi(K_n + \overline{K_m}) = \chi(K_n) + 1 = n + 1\). By Theorem 2.5, \(\text{fold}(K_n + \overline{K_m}) \leq m + n - \chi(K_n + \overline{K_m}) - 1 = m + n - (n + 1) - 1 = m - 2\). Identify the vertices \(v_2, v_3, \ldots, v_m\) to a single vertex. So, a uniform \(m - 2\)-folding \(G_0 = K_n + \overline{K_m}, G_1, \ldots, G_{m-2}\) is obtained in which the last graph \(G_{m-2}\) is isomorphic to the graph \(K_n + \overline{K_2}\). If the vertices of \(\overline{K_2}\) are identified, then a complete graph on \(n + 1\) vertices is obtained and is nonsingular. Hence, \(\text{fold}(K_n + \overline{K_m}) = m - 2\).

3.3. Cone graph, \(C_{m,n}\). In this section, we determine the fold thickness of cone graph \(C_{m,n}\). \(C_{m,1}\) is the wheel graph and its fold thickness is studied in [2]. Hence, fold thickness for the case \(n \geq 2\) is evaluated in this section.

**Theorem 3.12.** For \(m \geq 3\) and \(n \geq 2\), the fold thickness of cone graph \(C_{m,n}\) is given by,

\[
\text{fold}(C_{m,n}) = \begin{cases} 
m + n - 5 & \text{if } m \text{ is odd} \\
m + n - 4 & \text{if } m \text{ is even}. \end{cases}
\]

**Proof.** By definition of cone graph, if \(x, y \in V(\overline{K_n})\), \(N(x) = N(y) = V(C_m)\). Hence, by Theorem 2.2, \(C_{m,n}, n \geq 2\) is singular. Clearly, \(\chi(C_{m,n}) = \chi(C_m) + 1\). So, \(\chi(C_{m,n}) = 3\), if \(m\) is even and \(\chi(C_{m,n}) = 4\), if \(m\) is odd. Since \(N(x) = N(y)\) for any pair of vertices \(x\) and \(y\) in \(V(\overline{K_n})\), folding \(C_m\) in any manner keeps \(C_{m,n}\) singular. So, we have to find the maximum folding of \(C_m\).

**Case 1:** \(m\) is odd.

By Theorem 2.5, \(\text{fold}(C_{m,n}) \leq m + n - 4 - 1 = m + n - 5\). Since \(\chi(C_m) = 3\), there exists an \((m - 3)\)-maximum folding of \(C_m\), by Theorem 2.1. After these \((m - 3)\) foldings, \(C_m\) becomes \(K_3\) and \(C_{m,n}\) becomes \(K_3 + \overline{K_n}\), a singular graph. Fold \(K_3 + \overline{K_n}\) by identifying any two vertices in \(\overline{K_n}\). Continue this process \((n - 2)\) times until there remains exactly two vertices in \(\overline{K_n}\). The graph now obtained is \(K_3 + \overline{K_2}\). In this \((n - 2)\)-folding, all graphs are singular. If the remaining two vertices of \(\overline{K_2}\) in \(K_3 + \overline{K_2}\) are identified, we obtain the graph \(K_3 + K_1\), which is the complete graph \(K_4\) and is nonsingular. So, the maximum folding of \(K_3 + \overline{K_n}\) in which all graphs are singular is \((n - 2)\). Hence, an \((m - 3) + (n - 2) = (m + n - 5)\)-uniform folding of \(C_{m,n}\) is obtained. Consequently, \(\text{fold}(C_{m,n}) = m + n - 5\).

**Case 2:** \(m\) is even.

By Theorem 2.5, \(\text{fold}(C_{m,n}) \leq m + n - 3 - 1 = m + n - 4\). There exists an \((m - 2)\)-maximum folding of \(C_m\), since \(\chi(C_m) = 2\). After these \((m - 2)\) foldings, \(C_m\) becomes \(K_2\) and \(C_{m,n}\) becomes \(K_2 + \overline{K_n}\), a singular graph. Similarly, fold \(K_2 + \overline{K_n}\) as in the above case, \((n - 2)\) times by identifying vertices of \(\overline{K_n}\) until the singular graph \(K_2 + \overline{K_2}\) is obtained. All graphs in this \((n - 2)\)-folding are singular. If the remaining two vertices in \(\overline{K_2}\) of \(K_2 + \overline{K_2}\) are identified, then we obtain \(K_2 + K_1\), which is the complete graph \(K_3\) and is nonsingular. Thus, maximum folding of \(K_2 + \overline{K_n}\) in which all graphs are singular is \((n - 2)\). Hence, an \((m - 2) + (n - 2) = (m + n - 4)\)-uniform folding of \(C_{m,n}\) is obtained. So, \(\text{fold}(C_m) = m + n - 4\).
An example of the folding process for the case $m = 4$ and $n = 3$ is shown in Fig.2. Two vertices having distinct colors cannot be identified, since they are adjacent. We can observe that \( \text{fold}(C_{4,3}) = 3 \).

### 3.4. Tadpole graph.

Here, we evaluate the fold thickness of the tadpole graph \( T_{m,n} \).

**Lemma 3.1.** For $m \geq 3$ and $n \geq 1$,

\[
\det A(T_{m,n}) = \begin{cases} 
(-1)^{(n-1)/2} \det A(P_{m+1}) & \text{if } n \text{ is odd} \\
(-1)^{n/2} \det A(C_m) & \text{if } n \text{ is even}
\end{cases}
\]

**Proof.** Let $x$, $y$ and $z$ be vertices of $T_{m,n}$, where $x$ is the pendant vertex, $y$ is the unique neighbor of $x$ and $z$ is a vertex adjacent to $y$. In $T_{m,1}$, apply Theorem 2.4 and remove the edge $yz$ to obtain the graph $P_{m+1}$. Hence, $\det A(T_{m,1}) = \det A(P_{m+1})$. Similarly, in $T_{m,2}$, the edge $yz$ can be removed to obtain a disconnected graph with two components, $K_2$ and $C_m$ such that $\det A(T_{m,2}) = \det A(K_2) \det A(C_m) = (-1) \det A(C_m)$. If $n \geq 3$, by Theorem 2.4, we get $\det A(T_{m,n}) = \det A(T_{m,n-2}) \det A(K_2)$. Then, by applying mathematical induction on $n$, we can conclude that $\det A(T_{m,n}) = (-1)^{(n-1)/2} \det A(P_{m+1})$ if $n$ is odd and $\det A(T_{m,n}) = (-1)^{n/2} \det A(C_m)$ if $n$ is even. \(\square\)

**Remark 3.2.** By above lemma, we can observe that, if $m$ is odd, $T_{m,n}$ is nonsingular for all $n \geq 1$. If $m \equiv 0 \pmod{4}$, $T_{m,n}$ is singular for all $n \geq 1$. If $m \equiv 2 \pmod{4}$, $T_{m,n}$ is singular for odd $n$ and nonsingular for even $n$.

**Theorem 3.13.** For $m \geq 3$ and $n \geq 1$,

\[
\text{fold}(T_{m,n}) = \begin{cases} 
0 & \text{if } m \equiv 2 \pmod{4} \text{ and } n \text{ is even} \\
m + n - 3 & \text{otherwise}
\end{cases}
\]
Proof. The chromatic number of $T_{m,n}$ and $C_m$ are the same. So, $\chi(T_{m,n}) = 2$, if $m$ is even and $\chi(T_{m,n}) = 3$, if $m$ is odd.

Case 1: $m$ is odd.

By Lemma 3.1, $T_{m,n}$ is non-singular for $n \geq 1$. Therefore, by Theorem 2.5, $\text{fold}(T_{m,n}) \leq m + n - 3$. Consider the $n$-uniform folding $G_0$, $G_1$, $\ldots$, $G_n$ of $T_{m,n}$, where $G_{i+1}$ is a $1$-fold of $G_i$ for $i = 1, 2, \ldots, n - 1$, obtained by identifying the pendant vertex of $G_i$ with an eligible vertex. Each graph $G_i$, $i = 1, 2, \ldots, n - 1$ is nonsingular by Lemma 3.1 and $G_n$ is the cycle $C_m$. Since $\text{fold}(C_m) = m - 3$, an $n + (m - 3) = (m + n - 3)$-uniform folding of $T_{m,n}$ is obtained in which all graphs are nonsingular. Thus, $\text{fold}(T_{m,n}) = m + n - 3$.

Case 2: $m \equiv 0 \pmod{4}$

Here, $T_{m,n}$, $n \geq 1$ is singular. So, by Theorem 2.5, $\text{fold}(T_{m,n}) \leq m + n - 2 - 1 = m + n - 3$. Fold this graph $n$ times as in the above case by identifying the pendant vertex with the eligible vertex on each step. Then, we obtain an $n$-uniform folding $G_0$, $G_1$, $\ldots$, $G_n$, where $G_n$ is $C_m$. By Lemma 3.1, each graph in this sequence is singular. Since $\text{fold}(C_m) = m - 3$, we get an $(m + n - 3)$-uniform folding of $T_{m,n}$ and hence $\text{fold}(T_{m,n}) = m + n - 3$.

Case 3: $m \equiv 2 \pmod{4}$

By Lemma 3.1, the graph $T_{m,n}$ is singular if $n$ is odd and nonsingular if $n$ is even.

If $n$ is odd, $\text{fold}(T_{m,n}) \leq m + n - 2 - 1 = m + n - 3$ by Theorem 2.5. Fold the graph $T_{m,n}$ once as in Fig 3 to obtain the graph $T_{m-2,n+1}$. Since, $m - 2 \equiv 0 \pmod{4}$, the graph $T_{m-2,n+1}$ is singular. From Case 2, $\text{fold}(T_{m-2,n+1}) = (m - 2) + (n + 1) - 3 = m + n - 4$. Therefore, we get a $1 + (m + n - 4) = m + n - 3$ - uniform folding of $T_{m,n}$. Hence, $\text{fold}(T_{m,n}) = m + n - 3$.

![Figure 3](image-url) A uniform 1-fold of $T_{m,n}$ when $m \equiv 2 \pmod{4}$ and $n$ is odd.

When $n$ is even, obtain a 1-fold of $T_{m,n}$ by identifying two eligible vertices. There are three possibilities. Both vertices are in $V(P_n)$, one is in $V(P_n)$ and other is in $V(C_m)$ or both are in $V(C_m)$. In any case, one of the following graphs are obtained: (1) A graph which has at least two distinct vertices having a common neighbor (2) A graph, after applying Theorem 2.4 sufficient number of times by taking $x$ as the pendant vertex and $y$ as a vertex adjacent to the unique neighbor of $x$, leads to a disconnected graph in which one component is $T_{m,1}$ (3) A graph, after applying Theorem 2.4 sufficient number of times by taking $x$ as the pendant vertex and $y$ as a vertex adjacent to the unique neighbor of $x$, leads to a disconnected graph in which one component has at least two distinct vertices having a common neighbor. By Theorem 2.2 and Remark 3.2, all these graphs are singular. This implies that, any 1-fold of $T_{m,n}$ is a singular graph. Hence, $\text{fold}(T_{m,n}) = 0$. □
The $m$-pan graph is the graph obtained by joining a cycle graph $C_m$ to $K_1$ with a bridge and hence it is isomorphic to the $(m, 1)$-tadpole graph.

**Corollary 3.1.** The fold thickness of $m$-pan graph is given by $\text{fold}(T_{m,1}) = m - 2$.

**Proof.** The $m$-pan graph is the special case of tadpole graph $T_{m,n}$ when $n = 1$. By Theorem 3.13, the conclusion follows easily. \qed

4. Conclusion

In this paper, we have determined the fold thickness of $K_n \odot \overline{K_m}, K_n + \overline{K_m}$, cone graph and tadpole graph. It can be extended to various classes of graphs such as product graphs, bipartite graphs and in general to any incomplete graph. This also gives a lot of problems for further research such as characterizing graphs with fold thickness as a parameter.

**References**


